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Every Scattered Space is Subcompact

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Abstract. We prove that every scattered space is hereditarily subcompact and any finite union of subcompact spaces is subcompact. It is a long-standing open problem whether every Čech-complete space is subcompact. Moreover, it is not even known whether the complement of every countable subset of a compact space is subcompact. We prove that this is the case for linearly ordered compact spaces as well as for ω-monolithic compact spaces. We also establish a general result for Tychonoff products of discrete spaces which implies that dense G_δ-subsets of Cantor cubes are subcompact.

Keywords: Čech-complete space, subcompact space, compact space, countable subset, linearly ordered space, double arrow space, scattered space, finite unions, ω-monolithic spaces, discrete spaces, products, Cantor cubes, G_δ-set

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0. Introduction.

There are quite a few properties designed to generalize completeness of a metric space. The main motivation for their discovery was the fact that the topology of a metrizable space X can be generated by a complete metric if and only if X is Čech-complete. Nowadays, Čech-completeness is the most important topological equivalent of completeness of a metric space.

An illustrative example of a weaker property is pseudocompleteness defined by Oxtoby [Ox]; for metric spaces it is equivalent to the existence of a dense Čech-complete subspace. The class of pseudocomplete spaces has nice categorical properties and contains the class of pseudocompact spaces. There are some old open problems about pseudocompleteness: it is still unknown whether it is preserved by open maps and dense G_δ-subspaces (see [AL1]). However, it is known that every Čech-complete space is pseudocomplete.

Another example is subcompactness, the weakest of so called Amsterdam properties defined by de Groot (see [dG]). A metrizable space is subcompact if and only if it is Čech-complete; subcompactness is preserved by open subspaces, free unions and arbitrary products but it is an open question whether it is preserved by dense G_δ-subspaces (see [BL2]). In particular, it is not known whether every Čech-complete space is subcompact. Even if we assume that K is a compact space and $A \subset K$ is a countable set, it is not clear whether $X = K\backslash A$ is subcompact.

The purpose of this paper is to study subcompact spaces; we prove, among other things, that any finite union of subcompact spaces is subcompact and every scattered space is hereditarily subcompact. We also establish that $K\backslash A$ is subcompact if K is a linearly ordered compact space and $A \subset K$ is countable. It would be nice to find a general class of compact spaces $\mathcal{K}$ such that every dense G_δ-subset of each $K \in \mathcal{K}$ is subcompact but this seems to be a difficult problem. So far we proved this for some concrete spaces K like the double arrow space and the Cantor cubes.

1. Notation and terminology.

All spaces are assumed to be Tychonoff. If X is a space then $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$. The set $\mathbb{R}$ is the real line with its usual topology and $\mathbb{D} = \{0, 1\}$ is the doubleton with the discrete topology.

A space Y is called *pseudocomplete* if it has a sequence $\{\mathcal{B}_n : n \in \omega\}$ of π-bases such that for any family $\{B_n : n \in \omega\}$ with $B_n \in \mathcal{B}_n$ and $\overline{B}_{n+1} \subset B_n$ for each $n \in \omega$, we have $\bigcap_{n\in\omega} B_n \neq \emptyset$. For any cardinal κ, the set $\{x \in \mathbb{R}^\kappa : |x^{-1}(\mathbb{R}\backslash\{0\})| \leqslant \omega\}$ is called *the* Σ*-product of* $\mathbb{R}^\kappa$; compact subsets of Σ-products of real lines are called *Corson compact.* Recall that a family $\mathcal{N}$ of subsets of a space X is *a network* in X if every open subset of X is a union of a subfamily of $\mathcal{N}$. A family $\mathcal{E}$ is *an outer network (base)* for a set $F \subset X$ if (every $E \in \mathcal{E}$ is open and) $F \subset \bigcap \mathcal{E}$ and for any $U \in \tau(F, X)$ there exists $E \in \mathcal{E}$ with $E \subset U$.

A space X is ω-monolithic if $\overline{A}$ has a countable network for any countable set $A \subset X$. The space X is called *perfectly normal* if every closed $F \subset X$ is a G_δ-set.

Given a space Y, a family $\mathcal{U} \subset \tau^*(Y)$ is called *a regular filterbase* if, for any $U, V \in \mathcal{U}$ there is $W \in \mathcal{U}$ such that $\overline{W} \subset U \cap V$. The space Y is *subcompact* if it has a base $\mathcal{B} \subset \tau^*(Y)$ such that every regular filterbase $\mathcal{U} \subset \mathcal{B}$ has non-empty intersection; such a base is also called *subcompact.* A space X is *scattered* if every non-empty subspace of X has an isolated point. A space is *Čech-complete* if it is homeomorphic to a dense G_δ-subset of a compact space. A space X has countable tightness (this is also denoted by $t(X) \leqslant \omega$) if $\overline{A} = \bigcup\{\overline{B} : B \subset A \text{ and } |B| \leqslant \omega\}$ for any set $A \subset X$.

The rest of our notation is standard and follows [En].

2. Scattered spaces, subcompactness and finite unions.

Every scattered space has a dense set of isolated points so it is pseudocomplete. However, easy second countable examples show that having a dense set of isolated points need not imply subcompactness. Any second countable scattered space is Čech-complete (see [KU]), but a stationary set $A \subset \omega_1$ such that $\omega_1\backslash A$ is also stationary, is an example of a scattered space which is not Čech-complete. The following theorem exhibits one more completeness property in scattered spaces.

2.1. Theorem. *Every scattered space is hereditarily subcompact.*

Proof. It suffices just to show subcompactness of a scattered space because the property of being scattered is hereditary. So, assume that X is a scattered space and let X_0 be the set of all isolated points of X. If α is an ordinal and we have sets $\{X_\beta : \beta < \alpha\}$ then let X_α be the set of isolated points of $X\backslash(\bigcup_{\beta<\alpha} X_\beta)$. The space X being scattered, there exists an ordinal ξ such that $X_\xi = \emptyset$; let μ be the least such ξ. Then $X = \bigcup\{X_\alpha : \alpha < \mu\}$ and the decomposition $\mathcal{D} = \{X_\alpha : \alpha < \mu\}$ (called the Cantor–Bendixson decomposition) has the following properties:

(1) the family $\mathcal{D}$ is disjoint;
(2) the set $\bigcup\{X_\beta : \beta < \alpha\}$ is open in X for any $\alpha \leqslant \mu$;
(3) for any $x \in X_\alpha$ there exists a set $O_x \in \tau(x, X)$ such that $O_x \cap X_\alpha = \{x\}$ and $O_x \subset \bigcup\{X_\beta : \beta \leqslant \alpha\}$.

For each $x \in X$ fix a local base $\mathcal{B}_x$ at the point x such that $\bigcup \mathcal{B}_x \subset O_x$. We claim that $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in X\}$ is a subcompact base of X. To prove this, fix a filterbase $\mathcal{F} \subset \mathcal{B}$. For every $U \in \mathcal{F}$ there exists $x \in X$ such that $U \in \mathcal{B}_x$; there is a unique $\alpha < \mu$ with $x \in X_\alpha$ so we can let $\xi(U) = \alpha$.

Consider the ordinal $\beta = \min\{\xi(U) : U \in \mathcal{F}\}$ and choose a set $U \in \mathcal{F}$ such that $\xi(U) = \beta$. By the definition of the ordinal $\xi(U)$ there exists $x \in X_\beta$ such that $U \cap X_\beta = \{x\}$ and $U \subset \bigcup\{X_\alpha : \alpha \leqslant \beta\}$. If $V \in \mathcal{F}$ and $x \notin V$ then there exists a set $W \subset U \cap V \subset \bigcup\{X_\alpha : \alpha < \beta\}$, which shows that $\xi(W) < \beta$, which is a contradiction. Therefore $x \in \bigcap\{V : V \in \mathcal{F}\}$, i.e., $\bigcap \mathcal{F} \neq \emptyset$ which proves that the space X is subcompact. □

The following corollary is probably known but we could not find a reference.

2.2. Corollary. *Every scattered metrizable space is Čech-complete.*

Lutzer asked in [BL2, Question 3.15] whether every hereditarily subcompact space is scattered. The following statement gives a positive answer for countably tight spaces.

2.3. Corollary. *A space X of countable tightness is hereditarily subcompact if and only if X is scattered.*

Proof. Any scattered space is hereditarily subcompact by Theorem 2.1. Now assume that X is hereditarily subcompact, $t(X) \leqslant \omega$ and there exists $Y \subset X$ which has no isolated points. Take any point $y \in Y$ and let $Z_0 = \{y\}$. Proceeding inductively assume that we have countable subsets $Z_0, Z_1, \ldots, Z_n$ of the set Y such that

(4) $x \in \overline{Z_{i+1} \backslash \{x\}}$ for every $i < n$ and $x \in Z_i$.

For any point $x \in Z_n$ it follows from $x \in \overline{Y \backslash \{x\}}$ that we can find a countable set $A_x \subset Y \backslash \{x\}$ with $x \in \overline{A_x}$. Letting $Z_{n+1} = \bigcup\{A_x : x \in Z_n\}$ we obtain a sequence $Z_0, \ldots, Z_{n+1}$ which satisfies (4) for all $i < n+1$ so our inductive construction can be continued to construct a family $\{Z_i : i \in \omega\}$ for which (4) holds for all $n < \omega$.

It is straightforward that $Z = \bigcup_{n \in \omega} Z_n$ is a countable subset of Y without isolated points so Z does not have the Baire property and hence it is not subcompact. This contradiction shows that X must be scattered. □

Recall that every countable Čech-complete space is scattered [KU]. The following corollary strengthens this result.

2.4. Corollary. *For any countable space X, the following properties are equivalent:*
(a) X is hereditarily subcompact;
(b) X is subcompact;
(c) X is scattered.

Proof. The implication (a)$\Longrightarrow$(b) is trivial and (c)$\Longrightarrow$(a) is a consequence of Theorem 2.1. To prove (b)$\Longrightarrow$(c) assume that X is subcompact and $Y \subset X$ is dense-in-itself. Choose a faithful enumeration $\{x_n : n \in \omega\}$ of the space X and suppose that $\mathcal{B}$ is a subcompact base of X. The set Y being infinite, we can pick a point $y_0 \in Y$ and $B_0 \in \mathcal{B}$ such that $y_0 \in B_0$ and $x_0 \notin B_0$.

Proceeding inductively, assume that we have elements $B_0, \ldots, B_n$ of the base $\mathcal{B}$ such that

(5) $\overline{B}_{i+1} \subset B_i$ for any $i < n$;
(6) $B_i \cap Y \neq \emptyset$ and $B_i \cap \{x_0, \ldots, x_i\} = \emptyset$ for each $i \leqslant n$.

Since Y has no isolated points, the set $B_n \cap Y$ has to be infinite so we can find a point $y_{n+1} \in (Y \cap B_n)\backslash\{x_0, \ldots, x_{n+1}\}$. Choose a set $B_{n+1} \in \mathcal{B}$ for which $y_{n+1} \in B_{n+1} \subset \overline{B}_{n+1} \subset B_n$ and $B_{n+1} \cap \{x_0, \ldots, x_{n+1}\} = \emptyset$. It is immediate that properties (5) and (6) hold if we replace n with $n+1$ so our inductive procedure can be continued to construct a sequence $\{B_n : n \in \omega\} \subset \mathcal{B}$ such that the conditions (5) and (6) are satisfied for all $n \in \omega$.

It follows from (5) that $\mathcal{F} = \{B_n : n \in \omega\}$ is a regular filterbase in $\mathcal{B}$ so $\bigcap \mathcal{F} \neq \emptyset$. However, an immediate consequence of (6) is that $\bigcap \mathcal{F} = \emptyset$ which is a contradiction. Therefore the space X has to be scattered. □

2.5. Theorem. *Any finite union of subcompact spaces is subcompact.*

Proof. Evidently, it suffices to prove this theorem for the union of two spaces, so assume that $X = Y \cup Z$ where Y and Z are subcompact subspaces of X. Fix subcompact bases $\mathcal{B}_Y$ and $\mathcal{B}_Z$ in the spaces Y and Z respectively and consider the family $\mathcal{E}_Y = \{U \in \tau(X) : U \cap Y \in \mathcal{B}_Y\}$. We claim that $\mathcal{E}_Y$ contains a local base in X at every point $x \in Y$. Indeed, if $x \in Y$ and $x \in U \in \tau(X)$ then there exists $B \in \mathcal{B}_Y$ such that $x \in B \subset U \cap Y$. Choose a set $B' \in \tau(X)$ with $B' \cap Y = B$; then $V = B' \cap U \in \mathcal{E}_Y$ and $x \in V \subset U$. Analogously, the family $\mathcal{E}_Z = \{U \in \tau(X) : U \cap Z \in \mathcal{B}_Z\}$ contains a local base in X at every point $x \in Z$ so $\mathcal{B} = \mathcal{E}_Y \cup \mathcal{E}_Z$ is a base in X.

To see that $\mathcal{B}$ is subcompact assume that $\mathcal{F}$ is a regular filterbase in $\mathcal{B}$ and $\bigcap \mathcal{F} = \emptyset$. We claim that both families $\mathcal{F}_Y = \mathcal{F} \cap \mathcal{E}_Y$ and $\mathcal{F}_Z = \mathcal{F} \cap \mathcal{E}_Z$ are regular filterbases.

Striving for contradiction, assume first that neither $\mathcal{F}_Y$ nor $\mathcal{F}_Z$ is a regular filterbase. Then there exist $U_0, V_0 \in \mathcal{F}_Y$ such that the closure of any element of $\mathcal{F}_Y$ is not contained in $U_0 \cap V_0$. Analogously, there exist $U_1, V_1 \in \mathcal{F}_Z$ such that the closure of any element of $\mathcal{F}_Z$ is not contained in $U_1 \cap V_1$. The family $\mathcal{F}$ being a regular filterbase, there exists $W \in \mathcal{F}$ such that $\overline{W} \subset U_0 \cap V_0 \cap U_1 \cap V_1$. If $W \in \mathcal{F}_Y$ then it follows from $\overline{W} \subset U_0 \cap V_0$ that we obtained a contradiction with the choice of the sets U_0 and V_0. If $W \in \mathcal{F}_Z$ then it follows from $\overline{W} \subset U_1 \cap V_1$ that we have a contradiction with the choice of the sets U_1 and V_1.

Therefore we can assume, without loss of generality, that $\mathcal{F}_Y$ is a regular filterbase. It is straightforward that the family $\mathcal{G}_Y = \{B \cap Y : B \in \mathcal{F}_Y\} \subset \mathcal{B}_Y$ is a regular filterbase in Y so $\bigcap \mathcal{F}_Y \supset P = \bigcap \mathcal{G}_Y \neq \emptyset$. If every element of $\mathcal{F}_Z$ contains P, then $P \subset \bigcap \mathcal{F}$ so $\bigcap \mathcal{F} \neq \emptyset$, which is a contradiction. Hence we can choose a set $V \in \mathcal{F}_Z$ such that P is not contained in V. Given any two elements $G, H \in \mathcal{F}_Z$ there exists $W \in \mathcal{F}$ with $\overline{W} \subset V \cap G \cap H$. If $W \in \mathcal{F}_Y$ then $P \subset W \subset V$ which is a contradiction. Therefore $W \in \mathcal{F}_Z$ and $\overline{W} \subset G \cap H$, i.e., we proved that $\mathcal{F}_Z$ is also a regular filterbase.

The family $\mathcal{G}_Z = \{B \cap Z : B \in \mathcal{F}_Z\} \subset \mathcal{B}_Z$ is easily seen to be a regular filterbase in Z so $\bigcap \mathcal{F}_Z \supset Q = \bigcap \mathcal{G}_Z \neq \emptyset$. If every element of $\mathcal{F}_Y$ contains Q then $\emptyset \neq Q \subset \bigcap \mathcal{F}$ which is a contradiction, so we can choose a set $B \in \mathcal{F}_Y$ such that Q is not contained in B. There exists a set $W \in \mathcal{F}$ with $W \subset V \cap B$; if $W \in \mathcal{F}_Y$ then $P \subset W \subset V$ gives a contradiction. If $W \in \mathcal{F}_Z$ then $Q \subset W \subset B$ contradicts the choice of B so $\bigcap \mathcal{F} \neq \emptyset$, i.e., $\mathcal{E}$ is, indeed, a subcompact base in X. □

2.6. Corollary. *Any G_δ-subset of the double arrow space is subcompact.*

Proof. If X is the double arrow space then $X = X_0 \cup X_1$ where X_0 and X_1 are subspaces of X homeomorphic to the Sorgenfrey line. If Y is a G_δ-subspace of X then $Y_0 = Y \cap X_0$ and $Y_1 = Y \cap X_1$ are homeomorphic to G_δ-subspaces of the Sorgenfrey line and hence they are both subcompact by Theorem 3.3 of [BL1]. Since $Y = Y_0 \cup Y_1$, Theorem 2.5 does the rest. □

If we have a compact space K and a set $A \subset K$, then it turns out that subcompactness of $K \setminus A$ is determined by subcompactness of $\overline{A} \setminus A$. This shows that if we are trying to prove that some spaces X with a countable remainder in a compact space are subcompact, there is no loss of generality to assume that X is relatively small, i.e., it can be considered to be a subspace of a separable space.

2.7. Corollary. *If X is a subcompact space and $\overline{A} \setminus A$ is subcompact for some $A \subset X$ then $X \setminus A$ is also subcompact.*

Proof. Follows from Theorem 2.5, the equality $X \setminus A = (X \setminus \overline{A}) \cup (\overline{A} \setminus A)$ and the fact that $X \setminus \overline{A}$ is subcompact being an open subset of a subcompact space. □

2.8. Corollary. *If X is an ω-monolithic locally compact space and A is a countable subset of X then $X \setminus A$ is subcompact.*

Proof. The set $\overline{A}$ has a countable network; being locally compact, it has a countable base so $\overline{A} \setminus A$ is subcompact being metrizable and Čech-complete which shows that we can apply Corollary 2.7 to conclude that $X \setminus A$ is subcompact. □

2.9. Corollary. *If X is a Corson compact space and $A \subset X$ is countable then $X \setminus A$ is subcompact.*

2.10. Observation. *If X is a (linearly ordered) compact space, $A \subset X$ and we want to prove that $Y = X \setminus A$ is subcompact then it follows from the equality*

$X\backslash A = \overline{Y}\backslash(A \cap \overline{Y})$ and the fact that $\overline{Y}$ is a (linearly ordered) compact space that we can pass from X to $\overline{Y}$ if necessary and consider, without loss of generality, that Y is dense in X.

We are going to prove that the complement of any countable subset of a compact linearly ordered space is subcompact. This is not easy and requires several auxiliary statements.

2.11. Proposition. *If X is a first countable space then for any countable set $A \subset X$ there exists a continuous map $f : X \to M$ of X onto a second countable space M such that $f^{-1}f(x) = \{x\}$ for any $x \in A$.*

Proof. For every point $x \in A$ we can find a continuous function $f_x : X \to \mathbb{R}$ such that $\{x\} = f_x^{-1}f_x(x)$. If f is the diagonal product of the family $\{f_x : x \in A\}$ and $M = f(X)$ then f is as promised. $\square$

2.12. Observation. *If X is a space and $\mathcal{F}$ is a decomposition of X then $\mathcal{F}$ is called continuous if for any $F \in \mathcal{F}$ and any $U \in \tau(F, X)$ there exists $V \in \tau(F, X)$ such that $V \subset U$ and V is saturated, i.e., $G \in \mathcal{F}$ and $G \cap V \neq \emptyset$ implies $G \subset V$. A closed decomposition of a compact space X generates a quotient map $\varphi : X \to X/\mathcal{F}$ (by collapsing every element of $\mathcal{F}$ to a point). The space $X/\mathcal{F}$ can in general fail to be Hausdorff but it is a well-known fact that if $\mathcal{F}$ is continuous then X is Hausdorff (see [AP, Ch. II, Problem 322]).*

2.13. Proposition. *Suppose that X is a linearly ordered space, $A \subset X$ and the set $Y = X\backslash A$ is dense in X. A pair of distinct points $x, y \in A$ is called an* A-jump *if $x < y$ and the interval (x, y) is empty. For any point $x \in A$, let $Q_x = \{x\}$ if x is not contained in an A-jump and let $Q_x = \{x, y\}$ if either $\{x, y\}$ or $\{y, x\}$ is an A-jump for some $y \in A$. Then the family $\mathcal{H} = \{Q_x : x \in A\}$ is well-defined and disjoint. We will call $\mathcal{H}$ the canonical decomposition of A.*

Proof. If $\{x, y\}$ is an A-jump then for any $z \in A\backslash\{x, y\}$ the set $\{z, x\}$ is not a jump. Indeed, if $z < x < y$ then there are no points of Y in the non-empty interval (z, y) which is a contradiction with density of Y in X. The other cases of the inequalities between x, y and z are considered analogously. Therefore distinct A-jumps are disjoint. $\square$

2.14. Proposition. *Suppose that X is a perfectly normal linearly ordered compact space and $A \subset X$ is a countable subset of X such that $Y = X\backslash A$ is dense in X. Let $\mathcal{H}$ be the canonical decomposition of A. If $\mathcal{F} = \mathcal{H} \cup \{\{x\} : x \in X\backslash A\}$ then $\mathcal{F}$ is a continuous decomposition of X.*

Proof. It is immediate that every interval in X whose endpoints do not belong to an A-jump is a saturated set. Assume first that a set $Q = \{x, y\} \in \mathcal{H}$ is an A-jump with $x < y$ and take any $U \in \tau(Q, X)$. It follows from density of Y in X that there are $a, b \in Y$ such that $a < x$ and $y < b$ while $(a, b) \subset U$. Therefore $V = (a, b) \supset Q$ is a saturated set. If $F = \{x\} \in \mathcal{F}$ is a singleton and $F \subset X\backslash\overline{A}$

then, for any $U \in \tau(F, X)$ the set $V = U \backslash \overline{A}$ is saturated so assume that $x \in \overline{A}$ and $x \in U \in \tau(X)$; we can consider that U is an interval. Let $U_l = \{y \in U : y < x\}$ and $U_r = \{y \in U : x < y\}$. If $U_l = U_r = \emptyset$ then x is isolated in X which is impossible for the points of $\overline{A}$.

If $U_l \neq \emptyset$ and $U_r \neq \emptyset$ then it follows from density of Y in X that there exist points $a \in Y \cap U_l$ and $b \in U_r \cap Y$. Then $V = (a, b)$ is saturated open set and $F \subset V \subset U$. If $U_r \neq \emptyset$ and $U_l = \emptyset$ then choose a point $b \in U_r \cap Y$ and observe that $V = [x, b)$ is a saturated open set such that $F \subset V \subset U$. The case when $U_r = \emptyset$ and $U_l \neq \emptyset$ is analogous so $\mathcal{F}$ is a continuous decomposition of X. □

2.15. Proposition. *Suppose that X is a perfectly normal linearly ordered compact space and $A \subset X$ is a countable subset of X such that $Y = X \backslash A$ is dense in X. Let $\mathcal{H}$ be the canonical decomposition of A. Then there exists a continuous map $\xi : X \to M$ of X onto a second countable space M such that for each $Q \in \mathcal{H}$ there exists a point $z \in M$ such that $Q = \xi^{-1}(z)$.*

Proof. Proposition 2.14 implies that there exists a continuous onto map $\varphi : X \to K$ of X onto a Hausdorff compact space K such that $\varphi^{-1}\varphi(x) = \{x\}$ for any point $x \in X$ which does not belong to an A-jump and for every A-jump F there exists $y \in K$ with $F = \varphi^{-1}(y)$. If B is the set of all images of A-jumps in K then we can apply Proposition 2.11 to find a second countable space M and a continuous onto map $\mu : K \to M$ such that $\mu^{-1}\mu(y) = \{y\}$ for any $y \in B$. The map $\xi = \mu \circ \varphi$ is as promised. □

2.16. Proposition. *Suppose that X is a perfectly normal linearly ordered compact space and $A \subset X$ is a countable subset of X such that $Y = X \backslash A$ is dense in X. Then there exists a continuous pseudometric d on the space X with the following properties:*

(a) if $x \in X \backslash A$ and $a \in A$ then $d(x, a) > 0$;
(b) if $a, b \in A$ are distinct points and $\{a, b\}$ is not an A-jump then $d(a, b) > 0$;
(c) if $\{a, b\}$ is an A-jump then $d(a, b) = 0$.

Proof. Let $\mathcal{H}$ be the canonical decomposition of A. Apply Proposition 2.15 to find a continuous map $\xi : X \to M$ of X onto a second countable space M such that for each $Q \in \mathcal{H}$ there exists a point $z \in M$ such that $Q = \xi^{-1}(z)$. If ρ is a metric on M which generates its topology, then let $d(x, y) = \rho(\xi(x), \xi(y))$ for any $x, y \in X$. It is immediate that d is as promised. □

2.17. Theorem. *If X is a linearly ordered compact space, A is a countable subset of X then $Y = X \backslash A$ is subcompact.*

Proof. Observe first that we can consider that X is perfectly normal and Y is dense in X. Indeed, By Corollary 2.7 it suffices to show that $\overline{A} \backslash A$ is subcompact. But the space $\overline{A}$ is separable and every separable linearly ordered space is hereditarily Lindelöf so we can pass from X to $\overline{A}$ if necessary to be able to consider that X is hereditarily Lindelöf and hence perfectly normal. Now, Observation 2.10 makes it possible to consider that Y is dense in X. Of course, Y is perfectly normal being a subspace of a perfectly normal space X.

Say that a set $B \subset X$ is saturated if for any A-jump F it follows from $F \cap B \neq \emptyset$ that $F \subset B$. For any $x \in X$ let $L_x = \{y \in X : y < x\}$ and $R_x = \{y \in X : x < y\}$; fix a continuous pseudometric d on X as in Proposition 2.16 and choose a faithful enumeration $\{a_n : n \in \omega\}$ of the set A. It is easy to find an increasing sequence $\{A_n : n \in \omega\}$ of finite subsets of A such that $\{a_0, \ldots, a_n\} \subset A_n$ and A_n is saturated for every $n \in \omega$.

We will construct a countable local base $\mathcal{B}_x$ at the point x for every $x \in X \backslash A$. If $x \in X \backslash \overline{A}$ then take any countable local base $\mathcal{B}_x$ at the point x such that every $B \in \mathcal{B}_x$ is an interval and $\overline{B} \cap A = \emptyset$. If $x \in \overline{A}$ then we have three possible mutually exclusive cases:

1) $x \in \overline{A \cap L_x}$ and $x \notin \overline{A \cap R_x}$.
2) $x \notin \overline{A \cap L_x}$ and $x \in \overline{A \cap R_x}$.
3) $x \in \overline{A \cap L_x}$ and $x \in \overline{A \cap R_x}$.

Case 1. Let $\{I_n : n \in \omega\}$ be a local base at x such that I_n is an interval, $I_{n+1} \subset I_n$ and $I_n \cap R_x \cap A = \emptyset$ for any $n \in \omega$. Fix any $n \in \omega$ and consider the point $a = \max(A_n \cap L_x)$. It is evident that $A_n \cap L_x$ is saturated with respect to the canonical decomposition of A. Besides, $\{a, x\}$ cannot be a jump because $x \in \overline{A \cap L_x}$. If $\{a, y\}$ is a jump for some $y \in Y$ then let $G(x, n) = (a, x] \cup (I_n \cap R_x)$, say that $G(x, n)$ is of type 1 and let $y = q(x, n)$. If a is a limit point of R_a then it is a limit point of $R_a \cap Y$ so we can find a point $y \in Y \cap (a, x)$ such that $d(y, a) < \frac{1}{2} d(a, x)$ and hence $d(x, y) > d(y, a)$. Take a point $z \in Y \cap (a, y)$ and let $G(x, n) = (z, x] \cup (I_n \cap R_x)$. In this case, we say that $G(x, n)$ is of type 2 and $q(x, n) = y$. It is straightforward that $\{G(x, n) : n \in \omega\}$ is a local base at x.

Case 2. Do the construction as in Case 1, but whatever was done on the left side of x, do it on the right side and vice versa.

Case 3. From the construction in Case 1, apply what was done on the left side of x for both sides of x.

Let $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in Y\}$ and consider the base $\mathcal{C} = \{U \cap Y : U \in \mathcal{B}\}$ in the space Y. To see that $\mathcal{C}$ is subcompact, take a family $\mathcal{F} \subset \mathcal{B}$ such that $\mathcal{F}' = \{U \cap Y : Y \in \mathcal{F}'\}$ is a regular filterbase in Y. Observe that $\overline{U} = \overline{U \cap Y}$ for any $U \in \mathcal{F}$ so the closures of the elements of $\mathcal{F}$ form a filterbase in the compact space X. Therefore it suffices to show that $(\bigcap\{\overline{U} : U \in \mathcal{F}\}) \cap Y \neq \emptyset$. Striving for contradiction, assume that $Q = \bigcap\{\overline{U} : U \in \mathcal{F}\} \subset A$.

The set $\overline{U}$ is an interval for any $U \in \mathcal{F}$ and any intersection of intervals is an interval so Q is an interval. Since Y is dense in X, the set Q cannot have more than two points, i.e., the set Q is either a singleton or an A-jump. By our choice of the pseudometric d, the d-diameter of Q is zero.

The family $\{\overline{U} : U \in \mathcal{F}\}$ is an outer network for Q so we can choose a sequence $\{U_n : n \in \omega\} \subset \mathcal{F}$ such that $\overline{U}_{n+1} \subset \overline{U}_n$ for every $n \in \omega$ while $\operatorname{diam}(\overline{U}_n) \to 0$ and

$\mathcal{N} = \{\overline{U}_n : n \in \omega\}$ is an outer network for Q (the diameter is taken with respect to the pseudometric d; recall that X is perfectly normal so every closed subset of X has a countable outer base). Any infinite subfamily of $\mathcal{N}$ is also an outer network for Q so we can assume that every $U_n = G(x_n, k_n)$ is of the same type (one or two) and comes from the same case of Cases 1–3. Fix $m \in \omega$ such that $Q \cap A_m \neq \emptyset$. If $k_n \geqslant m$ then $G(x_n, k_n)$ cannot intersect A_m so $k_n \leqslant m$ for all $n \in \omega$ which shows that we can assume, without loss of generality, that there is $l \in \omega$ such that $k_n = l$ for all $n \in \omega$. Suppose that every x_n comes from Case 1 and let a be the minimal point of Q. We have $a < x_n$ and hence $b_n = \max(A_l \cap L_{x_n}) < a$; it follows from $b_n = \max(A_l \cap L_a)$ that there is $b \in A_l$ such that $b_n = b$ for all $n \in \omega$.

Let us consider first that each $G(x_n, k_n)$ is of type 1 so the set $\{b, q(x_n, l)\}$ is a jump and hence there is $z \in Y$ such that $q(x_n, l) = z$ for all n which implies that $z \in U_n$ for all $n \in \omega$ which is a contradiction with $\bigcap_{n\in\omega} \overline{U}_n = Q$.

Now suppose that every $G(x_n, k_n)$ is of type 2 and let $\delta = d(a, b) > 0$. There exists $n \in \omega$ such that $\operatorname{diam}(G(x_n, k_n)) < \frac{\delta}{4}$; let $c = q(x_n, l)$. Then $d(a, c) < \frac{\delta}{4}$ and $d(x_n, c) < \frac{\delta}{4}$ and also $d(c, b) \leqslant d(x_n, c) < \frac{\delta}{4}$. Therefore

$$d(a, b) \leqslant d(a, x_n) + d(x_n, c) + d(c, b) < \tfrac{3}{4}\delta$$

which is a contradiction. Now, if x_n comes from Case 2 or Case 3 then the evident modifications of the above proof show that we also obtain a contradiction. □

In March 2008, at the Spring Topology and Dynamics Conference in Milwaukee, Lutzer asked whether $\mathbb{D}^{\mathfrak{c}} \setminus A$ is subcompact for any dense countable set $A \subset \mathbb{D}^{\mathfrak{c}}$. We will prove a general result which implies that the answer to this question is positive.

2.18. Theorem. *Suppose that $D \neq \emptyset$ is a discrete space and I is a non-empty set. Then any dense G_δ-subspace of D^I is subcompact.*

Proof. If I is countable then D^I is a completely metrizable space and hence every dense G_δ-subset X of D^I is also completely metrizable so X is subcompact by [dG]. Thus we can assume, without loss of generality, that I is an uncountable set. Fix a decreasing family $\{U_n : n \in \omega\}$ of dense open subsets of D^I; we must prove that $X = \bigcap_{n\in\omega} U_n$ is subcompact. Observe that D^I is subcompact, being a product of subcompact spaces, so every U_n is also subcompact and therefore there is no loss of generality to assume that $U_0 \neq D^I$ and $U_{n+1} \neq U_n$ for any $n \in \omega$. Any subcompact space has the Baire property so X is a dense subset of D^I.

Denote by $\operatorname{Fn}(I, D)$ the family of all functions from a finite subset of I to the set D; if $s \in \operatorname{Fn}(I, D)$ then $\operatorname{dom}(s)$ is its domain and $[s] = \{f \in D^I : f|\operatorname{dom}(s) = s\}$. It is clear that the family $\{[s] : s \in \operatorname{Fn}(I, D)\}$ is a base of the space D^I. Observe that for any $s, t \in \operatorname{Fn}(I, D)$ with $[s] \cap [t] \neq \emptyset$ we have the equality $s|(\operatorname{dom}(s) \cap \operatorname{dom}(t)) = t|(\operatorname{dom}(s) \cap \operatorname{dom}(t))$. Say that a function $s \in \operatorname{Fn}(I, D)$ is *n-minimal* if $[s] \subset U_n$ but $[s|A]$ is not contained in U_n for any proper subset A of $\operatorname{dom}(s)$. As an immediate consequence of the definition,

(7) if we have distinct n-minimal functions s and t such that $[s] \cap [t] \neq \emptyset$, then $\operatorname{dom}(s) \setminus \operatorname{dom}(t) \neq \emptyset$.

Consider the family $\mathcal{B}_n = \{[s] : s \in \mathrm{Fn}(I, D)$ and there exists a set $A \subset \mathrm{dom}(s)$ such that $|A| \leqslant n$ and $s|(\mathrm{dom}(s)\backslash A)$ is n-minimal$\}$. Let $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \omega\}$; we claim that the family $\mathcal{C} = \{B \cap X : B \in \mathcal{B}\}$ is a subcompact base in X. The elements of $\mathcal{B}$ are clopen in D^I and hence all elements of $\mathcal{C}$ are clopen subsets of X which shows that any filterbase $\mathcal{F} \subset \mathcal{C}$ is regular.

Take any $x \in X$ and a finite set $A \subset I$; let $n = |A|$. There exists a minimal finite set $B \subset I\backslash A$ such that $[x|(A \cup B)] \subset U_n$. Let $E = A \cup B$; for any $b \in B$ the set $[x|(E\backslash\{b\})]$ is not contained in U_n by the choice of B. As a consequence, there exists a set $D \subset A$ such that $x|(E\backslash D)$ is n-minimal. It follows from $|D| \leqslant |A| = n$ that $[x|E] \in \mathcal{B}_n$; since $A \subset E$, we have $x \in [x|E] \subset [x|A]$ so $\mathcal{B}$ contains a local base at every $x \in X$. This proves that $\mathcal{C}$ is a base in X.

To see that $\mathcal{C}$ is subcompact, take an arbitrary filterbase $\mathcal{F} \subset \mathcal{C}$. There exists a family $\mathcal{G} \subset \mathcal{B}$ such that $\mathcal{F} = \{G \cap X : G \in \mathcal{G}\}$; it is straightforward that $\mathcal{G}$ is also a filterbase. Observe first that $\bigcap \mathcal{G} \neq \emptyset$, because letting $x(a) = s(a)$ for any $a \in A = \bigcup\{\mathrm{dom}(s) : [s] \in \mathcal{G}\}$ and any s such that $a \in \mathrm{dom}(s)$ and $[s] \in \mathcal{G}$, we consistently define a function $x : A \to D$. If $y \in D^I$ and $y|A = x$ then $y \in \bigcap \mathcal{G}$.

If there exists a minimal element G in the family $\mathcal{G}$, then $G \subset \bigcap \mathcal{G}$ so any point of $X \cap G$ belongs to $\bigcap \mathcal{F}$.

Therefore we can assume, without loss of generality, that there exists a strictly decreasing sequence $\{G_n : n \in \omega\} \subset \mathcal{G}$. Take $s_n \in \mathrm{Fn}(I, D)$ such that $G_n = [s_n]$ and let $A_n = \mathrm{dom}(s_n)$ for any $n \in \omega$. It follows from $G_n \supset G_{n+1}$ and $G_n \neq G_{n+1}$ that $A_n \subset A_{n+1}$ and $A_n \neq A_{n+1}$ for any $n \in \omega$. There exists a sequence $\{k_n : n \in \omega\} \subset \omega$ such that $s_n|(A_n\backslash B_n)$ is k_n-minimal and $|B_n| \leqslant k_n$; let $E_n = A_n\backslash B_n$ for all $n \in \omega$.

If the sequence $\{k_n : n \in \omega\}$ is unbounded then it follows from $[s_n] \subset U_{k_n}$ for all $n \in \omega$ that every $x \in \bigcap \mathcal{G}$ must belong to $\bigcap\{G_n : n \in \omega\} \subset \bigcap\{U_n : n \in \omega\} = X$ and hence $x \in \bigcap \mathcal{F}$. Therefore, we can assume that there exists $l \in \omega$ such that $k_n \leqslant l$ for all $n \in \omega$. Passing to an appropriate subsequence of $\{G_n : n \in \omega\}$ if necessary, we can assume, without loss of generality, that there exists $k \in \omega$ such that $k_n = k$ for all $n \in \omega$.

If $n_1 < n_2$ then the property (7) shows that E_{n_1} cannot be contained in E_{n_2} and therefore

(8) $E_{n_1} \cap B_{n_2} \neq \emptyset$ whenever $n_1 < n_2$.

The set E_0 being finite, we can use the property (8) to choose an infinite set $Q_0 \subset \omega$ and $a_0 \in E_0$ such that $a_0 \in B_n$ for all $n \in Q_0$. Proceeding inductively, let $q_0 = 0$ and assume that we have integers $q_0 < \ldots < q_r$, infinite sets $Q_0 \supset \ldots \supset Q_r$ and indices $a_0, \ldots, a_r$ such that

(9) $a_i \in B_n \cap E_{q_i}$ for all $n \in Q_i$ and $i \leqslant r$.

Take any number $q_{r+1} \in Q_r$ such that $q_r < q_{r+1}$. The set $E_{q_{r+1}}$ being finite, we can use the property (8) to choose an infinite set $Q_{r+1} \subset Q_r$ and $a_{r+1} \in E_{q_{r+1}}$ such that $a_{r+1} \in B_n$ for all $n \in Q_{r+1}$.

It is immediate that condition (9) is now satisfied for all numbers $i \leqslant r + 1$, so our inductive construction can be continued to construct sequences $\{q_i : i \in \omega\}$,

$\{Q_i : i \in \omega\}$ and $\{a_i : i \in \omega\}$ such that (9) holds for all $i \in \omega$.

If $i < j$ then $a_i \in B_{q_j}$ and $a_j \in E_{q_j}$; it follows from $E_{q_j} \cap B_{q_j} = \emptyset$ that $a_i \neq a_j$. As a consequence, $\{a_0, \ldots, a_k\} \subset B_{q_{k+1}}$ so $|\{a_0, \ldots, a_k\}| = k+1 \leqslant |B_{q_{k+1}}| \leqslant k$ which is a contradiction. Therefore, the sequence $\{k_n : n \in \omega\}$ cannot be bounded; this shows that $\bigcap \mathcal{F} \neq \emptyset$ and hence X is subcompact. □

2.19. Corollary. *For any cardinal κ, every dense G_δ-subset of the Cantor cube $\mathbb{D}^\kappa$ is subcompact.*

3. Open problems.

There are still quite a few natural subclasses of the class of Čech-complete spaces for which we do not know whether their elements are subcompact. The most intriguing question is whether the complement of a countable set in a compact space is subcompact.

3.1. Problem. *Let X be a compact space. Must $X \setminus A$ be subcompact for any countable $A \subset X$?*

3.2. Problem. *Is it true that $\beta\omega \setminus A$ is subcompact for any countable $A \subset \beta\omega$?*

3.3. Problem. *Let X be a monotonically normal compact space. Must $X \setminus A$ be subcompact for any countable $A \subset X$?*

3.4. Problem. *Let X be a dyadic compact space. Must $X \setminus A$ be subcompact for any countable $A \subset X$?*

3.5. Problem. *Let X be a first countable compact space. Must $X \setminus A$ be subcompact for any countable $A \subset X$?*

3.6. Problem. *Let X be a perfectly normal compact space. Must $X \setminus A$ be subcompact for any countable $A \subset X$?*

3.7. Problem. *Let X be a subcompact space with a countable network. Must $X \setminus A$ be subcompact for any countable set $A \subset X$?*

3.8. Problem. *Let X be a subcompact space with a countable network. Must every dense G_δ-subset of X be subcompact?*

3.9. Problem. *Let X be a first countable compact space. Must every dense G_δ-subset of X be subcompact?*

3.10. Problem. *Let X be an ω-monolithic compact space. Must every dense G_δ-subset of X be subcompact?*

3.11. Problem. *Let X be an Eberlein compact space. Must every dense G_δ-subset of X be subcompact?*

3.12. Problem. *Let X be a perfectly normal compact space. Must every dense G_δ-subset of X be subcompact?*

3.13. Problem. *Let X be a dyadic compact space. Must every dense G_δ-subset of X be subcompact?*

3.14. Problem. *Let X be a linearly ordered compact space. Must every dense G_δ-subset of X be subcompact?*

3.15. Problem. *Let X be a monotonically normal compact space. Must every dense G_δ-subset of X be subcompact?*

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