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Generic Approximation and Interpolation by Entire Functions via Restriction of the Values of the Derivatives

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Controlling values on a countable set

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3. If A and B are countable dense subsets of $\mathbb{R}$, is there a smooth order-isomorphism $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(A) = B$?

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2. Fact: if A and B are dense subsets of $\mathbb{R}$ and $f\colon A \to B$ is an order-isomorphism, then f extends uniquely to an order-isomorphism $\mathbb{R} \to \mathbb{R}$.
3. If A and B are countable dense subsets of $\mathbb{R}$, is there a smooth order-isomorphism $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(A) = B$?
4. (Stäckel 1899) Are there transcendental analytic functions y of x such that all values of y obtained from rational values of x are rational and vice versa?

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5. (Franklin, TAMS 1925) If A and B are countable dense subsets of $\mathbb{R}$, there is a real-analytic order-isomorphism $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(A) = B$.

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5. (Franklin, TAMS 1925) If A and B are countable dense subsets of $\mathbb{R}$, there is a real-analytic order-isomorphism $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(A) = B$.
6. (Erdős, Mich. Math. J. 1957) Does there exist an entire function f, not of the form $f(x) = a_0 + a_1 x$, such that the number $f(x)$ is rational or irrational according as x is rational or irrational? More generally, if A and B are two denumerable, dense sets, does there exist an entire function which maps A onto B?

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7. (Barth-Schneider, J. Lond. Math. Soc. 1970) If A and B are countable dense subsets of $\mathbb{R}$, there is an entire transcendental order-isomorphism $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(A) = B$.

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8. (Shelah, JSL 1980) A universal linear order of size $\aleph_1$ in a model where $\aleph_1 < 2^{\aleph_0}$.
   Part of the strategy: build a model in which any two everywhere nonmeager sets $A, B \subseteq \mathbb{R}$ of cardinality $\aleph_1$ are order-isomorphic.

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8. (Shelah, JSL 1980) A universal linear order of size $\aleph_1$ in a model where $\aleph_1 < 2^{\aleph_0}$.
   Part of the strategy: build a model in which any two everywhere nonmeager sets $A, B \subseteq \mathbb{R}$ of cardinality $\aleph_1$ are order-isomorphic.
9. (Burke, TAMS, 2009) A model in which for disjoint sequences of sets $(A_i : i < \omega_1)$, $(B_i : i < \omega_1)$ such that
   - A_i and B_i are countable dense sets for $i < \omega$
   - A_i and B_i are everywhere nonmeager, size $\aleph_1$, for $\omega \leq i < \omega_1$

   there is an entire order-isomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f(A_i) = B_i$ for all $i < \omega_1$.

A theorem of Hoischen

Theorem (Hoischen 1973)

Let $g : \mathbb{R}^t \to \mathbb{R}$. Let $\varepsilon : \mathbb{R}^t \to \mathbb{R}$ be a positive continuous function.

1. *If g is a C^N function for some nonnegative integer N, then there exists an entire function f such that $f(\mathbb{R}^t) \subseteq \mathbb{R}$ and $|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$ for $x \in \mathbb{R}^t$, $|\alpha| \leq N$.*
2. *If g is a C^∞ function, then for each open cover $U_0 \subseteq U_1 \subseteq \dots$ of $\mathbb{R}^t$, there exists an entire function f such that $f(\mathbb{R}^t) \subseteq \mathbb{R}$ and for all $k = 0, 1, 2, \dots$, $|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$ for $x \in \mathbb{R}^t \setminus U_k$, $|\alpha| \leq k$.*

A theorem of Hoischen

Theorem (Hoischen 1975 ($t = 1$))

Let $g : \mathbb{R}^t \to \mathbb{C}$. Let $\varepsilon : \mathbb{R}^t \to \mathbb{R}$ be a positive continuous function. Let $T \subseteq \mathbb{R}^t$ be a closed discrete set.

1. *If g is a C^N function for some nonnegative integer N, then there exists an entire function f such that $f(\mathbb{R}^t) \subseteq \mathbb{R}$ and*
   1. $|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$ *for* $x \in \mathbb{R}^t$, $|\alpha| \leq N$, *and*
   2. $(D^\alpha f)(x) = (D^\alpha g)(x)$ *for* $x \in T$, $|\alpha| \leq N$.
2. *If g is a C^∞ function, then for each open cover $U_0 \subseteq U_1 \subseteq \dots$ of $\mathbb{R}^t$, there exists an entire function f such that $f(\mathbb{R}^t) \subseteq \mathbb{R}$ and for all $k = 0, 1, 2, \dots$*
   1. $|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$ *for* $x \in \mathbb{R}^t \setminus U_k$, $|\alpha| \leq k$, *and*
   2. $(D^\alpha f)(x) = (D^\alpha g)(x)$ *for* $x \in T \setminus U_k$, $|\alpha| \leq k$.

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B. Transcendental number theory.

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3. (Faber, Math. Ann. 1904) There exists an entire transcendental $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{Q}$, such that f and all its derivatives are algebraic at all algebraic points.

Controlling values on a countable set

4. (W. Rudin, Amer. Math. Monthly 1977) Suppose that A is a countable subset of $\mathbb{R}^t$, and for each multi-index α, B_α is a dense subset of $\mathbb{R}$. Then there exists an $f \in C^\infty(\mathbb{R}^t)$ such that $D^\alpha f$ maps A into B_α, for every α.

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5. (Huang-Marques-Mereb, Bull. Aust. Math. Soc. 2010) For each countable set $A \subseteq \mathbb{C}$ and dense subsets $B_{p,n} \subseteq \mathbb{C}$, for $n \geq 0$ and $p \in A$, there exists a transcendental entire function $f\colon \mathbb{C} \to \mathbb{C}$ such that $D^n f(p) \in B_{p,n}$, for all $\alpha \in A$ and all $n \geq 0$.

The Fubini and Kuratowski-Ulam theorems

C. The Fubini theorem for null sets and the Kuratowski-Ulam theorem.

1. (Fubini, 1907) Let $A \subseteq \mathbb{R}^2$ have (Lebesgue) measure zero. Then for almost all $y \in \mathbb{R}$, $\{x \in \mathbb{R} : (x, y) \in A\}$ has measure zero in $\mathbb{R}$.

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2. (Kuratowski-Ulam, Fund. Math. 1932) Let $A \subseteq \mathbb{R}^2$ be meager. Then for all but a meager set of $y \in \mathbb{R}$, $\{x \in \mathbb{R} : (x, y) \in A\}$ is meager in $\mathbb{R}$.

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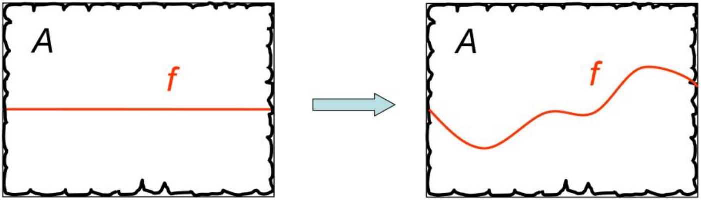

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3. (Ciesielski-Shelah, J. Appl. Anal. 2000, Ciesielski-Natkaniec, Fund. Math. 2003) A model in which for each everywhere nonmeager set $A \subseteq \mathbb{R}^2$, there is a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $\{x \in \mathbb{R} : (x, f(x)) \in A\}$ is everywhere nonmeager.

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4. (Rosłanowski-Shelah, Israel J. Math. 2006) A model in which for each $A \subseteq \mathbb{R}^2$ of full outer measure, there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $\{x \in \mathbb{R} : (x, f(x)) \in A\}$ has full outer measure.

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5. (Burke, Topology Appl. 2007) A model in which for each everywhere nonmeager set $A \subseteq \mathbb{R}^{t+1}$, there is an entire function $f\colon \mathbb{R}^t \to \mathbb{R}$ such that $\{x \in \mathbb{R}^t : (x, f(x)) \in A\}$ is everywhere nonmeager in $\mathbb{R}^t$.

Main Theorem

Definition

A *fiber-preserving local homeomorphism* on $\mathbb{R}^{t+1} \cong \mathbb{R}^t \times \mathbb{R}$ is a homeomorphism $h\colon G_h^1 \to G_h^2$ between two open sets $G_h^1, G_h^2 \subseteq \mathbb{R}^{t+1}$ such that h has the form $h(x, y) = (x, h^*(x, y))$ for some continuous map $h^*\colon \mathbb{R}^{t+1} \to \mathbb{R}$. We write k_h for the inverse of h.

Main Theorem

Theorem

Let $g : \mathbb{R}^t \to \mathbb{R}$ *be a* C^∞ *function and let* $\varepsilon\colon \mathbb{R}^t \to \mathbb{R}$ *be a positive continuous function. Let* $U_0 \subseteq U_1 \subseteq \dots$ *be an open cover of* $\mathbb{R}^t$*.*

- *Let* $A \subseteq \mathbb{R}^t$ *be a countable set and for each* $p \in A$ *and multi-index* α*, let* $A_{p,\alpha} \subseteq \mathbb{R}$ *be a countable dense set.*
- *Let* $T \subseteq \mathbb{R}^t$ *be a closed discrete set disjoint from A.*
- *Let* $C \subseteq \mathbb{R}^{t+1}$ *be a meager set.*
- *Let* $\mathcal{H}$ *be a countable family of fiber-preserving local homeomorphisms of* $\mathbb{R}^{t+!}$*.*

Main Theorem

Theorem (cont'd)

Then there exists an entire function $f\colon \mathbb{C}^t \to \mathbb{C}$ *such that* $f[\mathbb{R}^t] \subseteq \mathbb{R}$ *and for all* $k = 0, 1, 2, \ldots$

(a) $|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$ *when* $x \in \mathbb{R}^t \setminus U_k$, $|\alpha| \leq k$;

(b) $(D^\alpha f)(x) = (D^\alpha g)(x)$ *when* $x \in T \setminus U_k$, $|\alpha| \leq k$;

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(c) *for each* $p \in A$ *and multi-index* α, $(D^\alpha f)(p) \in A_{p,\alpha}$;

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(b) $(D^\alpha f)(x) = (D^\alpha g)(x)$ *when* $x \in T \setminus U_k$, $|\alpha| \le k$;

(c) *for each* $p \in A$ *and multi-index* α, $(D^\alpha f)(p) \in A_{p,\alpha}$;

(d) *for each multi-index* α, *for any* $q \in \mathbb{R}$, $h \in \mathcal{H}$ *and any open ball* $U \subseteq \mathbb{R}^t \setminus T$, *if* $(x, (D^\alpha f)(x)) \in G_h^1$ *and*

$$q = h^*(x, (D^\alpha f)(x))$$

for some $x \in U \cap \operatorname{cl} Y_{h,q,\alpha}$, *where* $Y_{h,q,\alpha} = \{p \in A : $ *for some* $q' \in A_{p,\alpha}$, $(p, q') \in G_h^1$ *and* $q = h^*(p, q')\}$, *then* $q = h^*(p, (D^\alpha f)(p))$ *for some* $p \in U \cap A$;

Main Theorem

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Then there exists an entire function $f\colon \mathbb{C}^t \to \mathbb{C}$ *such that* $f[\mathbb{R}^t] \subseteq \mathbb{R}$ *and for all* $k = 0, 1, 2, \ldots$

(a) $|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$ *when* $x \in \mathbb{R}^t \setminus U_k$, $|\alpha| \leq k$;

(b) $(D^\alpha f)(x) = (D^\alpha g)(x)$ *when* $x \in T \setminus U_k$, $|\alpha| \leq k$;

(c) *for each* $p \in A$ *and multi-index* α, $(D^\alpha f)(p) \in A_{p,\alpha}$;

(d) *for each multi-index* α, *for any* $q \in \mathbb{R}$, $h \in \mathcal{H}$ *and any open ball* $U \subseteq \mathbb{R}^t \setminus T$, *if* $(x, (D^\alpha f)(x)) \in G_h^1$ *and*

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for some $x \in U \cap \operatorname{cl} Y_{h,q,\alpha}$, *where* $Y_{h,q,\alpha} = \{p \in A : \text{for some } q' \in A_{p,\alpha}, (p, q') \in G_h^1 \text{ and } q = h^*(p, q')\}$, *then* $q = h^*(p, (D^\alpha f)(p))$ *for some* $p \in U \cap A$;

(e) *for every* α, $\{x \in \mathbb{R}^t : (x, (D^\alpha f)(x)) \in C\}$ *is meager in* $\mathbb{R}^t$.

Main Theorem

Theorem (cont'd)

If g is only a C^N function for some nonnegative integer N, then the same theorem holds with the conditions $x \in \mathbb{R}^t \setminus U_k$, $|\alpha| \leq k$, $k = 0, 1, 2, \ldots$ replaced by $x \in \mathbb{R}^t$, $|\alpha| \leq N$.

Simultaneous order-isomorphisms

Example

For each $n = 0, 1, 2, \ldots$, let A_n and B_n be countable dense subsets of $\mathbb{R}$ and $(0, \infty)$, respectively. Let $N \in \mathbb{N}$ and let $U_0 \subseteq U_1 \subseteq \ldots$ be an open cover of $\mathbb{R}$. Then there is a function $f \colon \mathbb{R} \to (0, \infty)$ which is the restriction of an entire function and is such that

1. For $n = 0, \ldots, N$ and all $x \in \mathbb{R}$, $(D^n f)(x) > 0$.
2. For $n = 0, 1, 2, \ldots$ and all $x \in \mathbb{R} \setminus U_n$, $(D^n f)(x) > 0$.
3. For $n = 0, 1, 2, \ldots$, $x \in \mathbb{R}$, $y \in (0, \infty)$ if $(D^n f)(x) = y$ then $x \in A_n$ if and only if $y \in B_n$.

The measure case

Theorem

Under the hypotheses of the Main Theorem where this time $C \subseteq \mathbb{R}^{t+1}$ satisfies $m_{t+1}(C) = 0$, there exists a C^∞ function $f\colon \mathbb{R}^t \to \mathbb{R}$ such that $f[\mathbb{R}^t] \subseteq \mathbb{R}$ and for all $k = 0, 1, 2, \ldots$

(a) $|(D^\alpha f)(x) - (D^\alpha g)(x)| < \varepsilon(x)$ *when* $x \in \mathbb{R}^t \setminus U_k$, $|\alpha| \leq k$;

(b) $(D^\alpha f)(x) = (D^\alpha g)(x)$ *when* $x \in T \setminus U_k$, $|\alpha| \leq k$;

(c) *for each* $p \in A$ *and multi-index* α, $(D^\alpha f)(p) \in A_{p,\alpha}$;

(d) *for each multi-index* α, *for any* $q \in \mathbb{R}$, $h \in \mathcal{H}$ *and any open ball* $U \subseteq \mathbb{R}^t \setminus T$, *if* $(x, (D^\alpha f)(x)) \in G_h^1$ *and* $q = h^*(x, (D^\alpha f)(x))$ *for some* $x \in U \cap \operatorname{cl} Y_{h,q,\alpha}$, *where* $Y_{h,q,\alpha} = \{p \in A : \text{for some } q' \in A_{p,\alpha}, (p, q') \in G_h^1 \text{ and } q = h^*(p, q')\}$, *then* $q = h^*(p, (D^\alpha f)(p))$ *for some* $p \in U \cap A$.

(e) $m_t(\{x \in \mathbb{R}^t : (x, (D^\alpha f)(x)) \in C\}) = 0$ *for all multi-indices* α.

The measure case

Problem

Let C be a set of Lebesgue measure zero in the plane. Is there an entire function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ so that $\{x \in \mathbb{R} : (x, f(x)) \in C\}$ has Lebesgue measure zero in $\mathbb{R}$ if we require

(i) f has rational coefficients?

(ii) f takes rational values on rational numbers?

Further details

Burke, Maxim R., Approximation and interpolation by entire functions with restriction of the values of the derivatives. Topology Appl. 213 (2016) 24–49.