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# Coarser connected metrizable topologies

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## Abstract

We show that every metric space,  $X$ , with  $w(X) \geq \mathfrak{c}$  has a coarser connected metrizable topology.

*Key words:* Coarser connected topology, extent, metrizable

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## 1. Introduction, Definitions, Notation

The study of coarser connected topologies was started by Tkačenko, Tkachuk, and Uspenskij who developed necessary and sufficient conditions for a topological space to have coarser connected Hausdorff or regular topology [5]. Continuing the development, Gruenhage, Tkachuk, and Wilson showed that any noncompact metrizable space has a coarser connected Hausdorff topology [4]. Fleissner, Porter and Roitman showed that any zero-dimensional metrizable space with weight at least  $\mathfrak{c}$  has a coarser connected metrizable topology [3]. We improve the last result by removing the zero-dimensional hypothesis. We show that every metric space with weight at least  $\mathfrak{c}$  has a coarser connected metrizable topology.

We write  $(X, \tau, \mu)$  for a metric space  $X$  with metric  $\mu$  and corresponding topology  $\tau$ . For a set  $A$  and cardinal  $\lambda$ , we write  $[A]^\lambda$  for the collection of subsets of  $A$  of cardinality  $\lambda$  and  $[A]^{<\lambda}$  for the collection on subsets of cardinality less than  $\lambda$ . The extent,  $e(X)$ , of a space  $X$  is the supremum of cardinals,  $\lambda$ , such that there is a closed discrete subset of  $X$  of size  $\lambda$ . If a space,  $X$ , has a closed discrete set of size  $e(X)$  we say the extent of  $X$  is attained in  $X$ .

Suppose  $U$  is a subset of a space  $X$ . If  $C$  is a closed discrete (with respect to  $U$ ) subset of  $U$ , then  $C$  is not necessarily closed in  $X$ . Hence, it is possible that  $e(U)$  is attained in  $U$ , but there is no closed discrete (in  $X$ ) subset of  $U$  of

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size  $e(U)$ . If, on the other hand, there is a closed discrete (in  $X$ ) subset of  $U$  of size  $e(U)$ , we say that  $e(U)$  is attained in  $X$ .

For a cardinal  $\kappa$  and positive real number  $\epsilon$ , the hedgehog with spininess  $\kappa$  and spine length  $\epsilon$  is the space  $Z = \{(\alpha, r) : \alpha \in \kappa, r \in (0, \epsilon]\} \cup \{(0, 0)\}$  with metric  $\rho$  defined as follows:

$$\rho((\alpha, r), (\alpha', r')) = \begin{cases} r + r' & \text{if } \alpha \neq \alpha' \\ |r - r'| & \text{if } \alpha = \alpha'. \end{cases}$$

The hedgehog space is a connected metric space.

## 2. Some Lemmas

If  $X$  is a metric space such that  $w(X) < \mathfrak{c}$ , then  $X$  does not necessarily have a coarser connected metrizable topology. For example the disjoint union of countably many Cantor sets is a separable, metrizable, disconnected space with no coarser connected regular topology [5]. We aim to show that all metric spaces,  $X$ , with  $w(X) \geq \mathfrak{c}$  have a coarser connected metrizable topology. This result has already been proven if  $e(X)$  is attained (Druzhinina [1], Fleissner, Porter, Roitman [3]). If  $e(X) = \mathfrak{c}$ , then we will see later that  $e(X)$  is attained. So, we consider metric spaces,  $X$ , such that  $e(X) > \mathfrak{c}$  is not attained.

In the proof of the main theorem, we partition the space  $(X, \tau, \mu)$  into subsets  $W_i$  for  $i \in \omega$  such that  $e(W_i)$  is attained in  $X$ . Lemma 2.1 will be used to define a coarser topology on  $X$  in which  $W_i$  is connected. We show that  $\mu$  can be adjusted so that  $W_i$  is connected without changing  $\mu$  on  $X \setminus W_i$ .

Lemmas 2.2–2.5 discuss the consequences of  $e(X)$  not being attained and provide a tool for the partition of  $X$  into  $W'_i$ s. We apply Lemma 2.6 to a particular compact subset of  $X$  for the purpose of defining the partition of  $X$  into  $W'_i$ s.

**Lemma 2.1.** *Suppose that  $Y$  is a subset of a metric space  $(X, \tau, \mu)$ ,  $e(\text{cl}Y) \geq \mathfrak{c}$  is attained in  $X$  by  $C \subset \text{int}(Y)$  and  $\text{diam}_\mu(\text{cl}Y) \leq \epsilon$ . If  $U \subset Y$  is an open set containing  $C$ , then there is a coarser topology  $\tau'$  on  $X$  generated by a corresponding metric  $\mu'$  such that*

- i)  $\mu' \leq \mu + 2\epsilon$ ,
- ii)  $\text{cl}_{\tau'}Y = \text{cl}_\tau Y$  and  $\tau'|_{\text{cl}Y}$  is connected, and
- iii)  $\mu|_{(X \setminus U)^2} = \mu'|_{(X \setminus U)^2}$ .

*Proof.* Let  $e(\text{cl}Y) = \kappa$ . So,  $C$  is closed discrete in  $X$  and  $|C| = \kappa$ . By replacing  $C$  with a subset, we may assume that  $|Y \setminus C| \geq \kappa$ . Let  $\mathcal{U} = \{U_c \subset U : c \in C\}$  be a discrete in  $X$  collection of open sets in  $X$  such that  $c \in$

$U_c \subset \text{cl}(U_c) \subset U$ . For each  $c \in C$  define a continuous function  $f_c : X \rightarrow [0, \epsilon]$  such that  $f_c(c) = \epsilon$  and  $f_c[X \setminus U_c] = \{0\}$ . For each  $x, y \in X$  define  $\mu^*(x, y) = \sum_{c \in C} |f_c(x) - f_c(y)| + \mu(x, y)$ . Because  $C$  is closed discrete in  $X$ , it is easy to check that  $\mu^*$  generates  $\tau$ ,  $\mu^*|_{(X \setminus U)^2} = \mu|_{(X \setminus U)^2}$  and  $\mu^* \leq \mu + 2\epsilon$ .

Let  $(Z, \rho)$  be a hedgehog space with spininess  $\kappa$  and spine length  $\epsilon$ . Let  $T = \{(\alpha, \epsilon) : \alpha \in \kappa\}$  and let  $S = Z \setminus T$ . The set  $T$  is the set of tips of the spines of the hedgehog which is a closed discrete set of size  $\kappa$ . Let  $D \subset Y \setminus C$  be a dense subset of  $Y \setminus C$  of size  $d(Y) = e(Y) = \kappa$ . Define a one-to-one map  $f : Z \rightarrow Y$  such that  $f[T] = D$  and  $f[S] = C$ . For  $x, y$  in the image of  $f$ , let  $\lambda(x, y) = \min\{\mu^*(x, y), \rho(f^{-1}(x), f^{-1}(y))\}$ . For all other  $x, y \in X$ , let  $\lambda(x, y) = \mu^*(x, y)$ .

Define a function  $\mu'$  on  $X \times X$  as follows:

$$\mu'(x, y) = \inf\{\lambda(x, x_1) + \lambda(x_1, x_2) + \cdots + \lambda(x_{n-1}, x_n) + \lambda(x_n, y)\}$$

where  $x_1, x_2, \dots, x_n$  ranges over all finite sequences (including the empty sequence) of distinct elements of  $X \setminus \{x, y\}$ . Since  $\rho$  and  $\mu^*$  satisfy the triangle inequality, in defining  $\mu'$  it suffices to consider sums of the form

$$\begin{aligned} \mu^*(x, x_1) + \rho(f^{-1}(x_1), f^{-1}(x_2)) + \cdots + \mu^*(x_{n-1}, x_n) \\ + \rho(f^{-1}(x_n), f^{-1}(y)) \end{aligned} \quad (1)$$

where the sum may start or end with either a  $\rho$  term or  $\mu^*$  term and the terms of the sum alternate between  $\rho$  and  $\mu^*$ . Since  $x_1, x_2, \dots, x_n$  are in the image of  $f$ , note that  $x_1, x_2, \dots, x_n \in D \cup C$ . Moreover, if  $x \notin Y$  and  $y \in Y$  each sum begins with a  $\mu^*$  term and  $x_1 \in Y$ . Hence,  $\mu'(x, y) \geq \mu^*(x, Y) \geq \mu(x, Y)$ . Letting  $y \in Y$  vary, we have  $\mu'(x, Y) \geq \mu(x, Y)$ .

Now we verify that  $\mu'$  is a metric. Since  $\mu'$  is an infimum over all finite sequences in  $X$ , it satisfies the triangle inequality. It is clear that  $\mu'(x, y) = \mu'(y, x)$  and  $\mu'(x, x) = 0$  for all  $x, y \in X$ . It remains to show that  $\mu'(x, y) = 0$  implies that  $x = y$ .

**Claim 1.** *Suppose that for a particular sum in the form of (1) and some  $m < n$ ,  $\lambda(x_i, x_{i+1}) < \epsilon$  for each  $i \in \{m, m+1, \dots, n-1\}$ . Then, if  $\lambda(x_m, x_{m+1}) = \rho(f^{-1}(x_m), f^{-1}(x_{m+1}))$  and  $x_m \in D$ , then  $x_{m+i} \in C$  for all  $i$  odd and  $x_{m+i} \in D$  for all  $i$  even in  $\{0, 1, \dots, n-m\}$ .*

*Proof.* Suppose that  $\lambda(x_m, x_{m+1}) = \rho(f^{-1}(x_m), f^{-1}(x_{m+1}))$  and  $x_m \in D$ . Also suppose that  $\lambda(x_i, x_{i+1}) < \epsilon$  for each  $i \in \{m, m+1, \dots, n-1\}$ . Fix  $i$  even such that  $0 \leq i < n-m$ , and suppose that  $x_{m+i} \in D$ . Since  $x_{m+i} \in D$ ,  $f^{-1}(x_{m+i}) \in T$ . But since the  $\rho$  and  $\mu$  terms of the sum alternate,

$$\lambda(x_{m+i}, x_{m+i+1}) = \rho(f^{-1}(x_{m+i}), f^{-1}(x_{m+i+1})) < \epsilon$$

and hence  $f^{-1}(x_{m+i+1}) \in S$ . By the definition of  $f$ ,  $x_{m+i+1} \in C$ . Now, since  $\lambda(x_{m+i+1}, x_{m+i+2}) = \mu^*(x_{m+i+1}, x_{m+i+2}) < \epsilon$  and  $x_{m+i+1} \in C$ , it must be that  $x_{m+i+2} \in D$ . So, since  $x_m \in D$ , by induction,  $x_{m+i} \in C$  for all  $i$  odd and  $x_{m+i} \in D$  for all  $i$  even in  $\{0, 1, \dots, n-m\}$ .  $\square$

**Claim 2.**  $\mu'(x, y) = 0$  implies that  $x = y$ .

*Proof.* Suppose there were  $x, y \in X$  such that  $\mu'(x, y) = 0$  but  $x \neq y$ . If defined, let  $x' = f^{-1}(x)$  and  $y' = f^{-1}(y)$ . Let  $x_1, \dots, x_n \in D \cup C$  be a sequence that yields an alternating  $\mu^*, \rho$  sum between  $x$  and  $y$  that is less than

$$\delta(x, y) = \min\{\epsilon, \mu^*(x, C \setminus \{x\}), \mu^*(y, C \setminus \{y\}), \rho(x', T \setminus \{x'\}), \rho(y', T \setminus \{y'\})\}.$$

Since  $\mu^*$  and  $\rho$  are metrics, we see that  $n \geq 1$ . Since  $C$  is closed discrete in  $X$  and  $T$  is closed in  $Z$ ,  $\delta(x, y)$  is a positive real number.

*Case 1.* The alternating sum begins and ends with  $\mu^*$  terms.

If  $x_1 \in C$  then  $\mu^*(x, x_1) \geq \mu^*(x, C \setminus \{x\}) \geq \delta(x, y)$ , which is a contradiction.

Note that  $n \geq 2$  is even. If  $x_1 \in D$ , then since  $\lambda(x_i, x_{i+1}) < \epsilon$  for each  $1 \leq i < n$  and  $\lambda(x_1, x_2) = \rho(f^{-1}(x_1), f^{-1}(x_2))$ , by Claim 1,  $x_n \in C$ . Hence  $\mu^*(x_n, y) \geq \mu^*(y, C \setminus \{y\}) \geq \delta(x, y)$ , which is a contradiction.

*Case 2.* The sum begins with a  $\mu^*$  term and ends with a  $\rho$  term.

If  $x_1 \in C$  then  $\mu^*(x, x_1) \geq \mu^*(x, C \setminus \{x\}) \geq \delta(x, y)$ , which is a contradiction.

Note that  $n$  is odd. If  $x_1 \in D$ , then since  $\lambda(x_i, x_{i+1}) < \epsilon$  for each  $1 \leq i < n$  and  $\lambda(x_1, x_2) = \rho(f^{-1}(x_1), f^{-1}(x_2))$ , by Claim 1,  $x_n \in D$ . Then  $\rho(f^{-1}(x_n), y') \geq \rho(y', T \setminus \{y'\}) \geq \delta(x, y)$ , which is a contradiction.

A symmetric argument shows that the sum beginning with  $\rho$  and ending with  $\mu^*$  yields a contradiction.

*Case 3.* The sum begins and ends with  $\rho$  terms.

Suppose  $x_1 \in D$ . Then  $\rho(f^{-1}(x), f^{-1}(x_1)) = \rho(x', f^{-1}(x_1)) \geq \rho(x', T \setminus \{x'\}) \geq \delta(x, y)$ , which is a contradiction.

Suppose  $x_1 \in C$ . Since  $\lambda(x_1, x_2) = \mu^*(x_1, x_2) < \epsilon$ ,  $x_2 \in D$ . Note that  $n \geq 2$  is even. If  $n = 2$ , then since  $x_2 \in D$ ,  $\rho(f^{-1}(x_n), y') \geq \rho(y', T \setminus \{y'\}) \geq \delta(x, y)$ , which is a contradiction. If  $n > 2$ ,  $\lambda(x_i, x_{i+1}) < \epsilon$  for each  $i$  with  $2 \leq i < n$  and  $\lambda(x_2, x_3) = \rho(f^{-1}(x_2), f^{-1}(x_3))$ . Therefore,  $x_n \in D$  by Claim 1. Hence  $\rho(f^{-1}(x_n), y') \geq \rho(y', T \setminus \{y'\}) \geq \delta(x, y)$ , which is a contradiction.  $\square$

So,  $\mu'$  defines a metric on  $X$ . Let  $\tau'$  be the topology on  $X$  generated by  $\mu'$ . Since  $\mu^*$  generates  $\tau$  and  $\mu' \leq \mu^*$ ,  $\tau'$  is coarser than  $\tau$ . Since  $\mu' \leq \mu^*$

and  $\mu^* \leq \mu + 2\epsilon$ , we have that  $\mu' \leq \mu + 2\epsilon$  showing i) of the lemma.

**Note 1.** In the proof of Claim 2, we have actually shown that for all  $x, y \in X$

$$\min\{\mu^*(x, y), \delta(x, y)\} \leq \mu'(x, y) \leq \mu^*(x, y).$$

**Claim 3.** *The map  $f : (Z, \rho) \rightarrow (Y, \mu'|_Y)$  is continuous.*

*Proof.* Let  $x \in Y$ ,  $\delta > 0$  and let  $U = B_{\mu'}(x, \delta)$  be the  $\mu'$ -ball of radius  $\delta$  about  $x$ . Suppose  $z \in f^{-1}[U]$ . Then  $\mu'(f(z), x) = \delta' < \delta$ . Let  $\xi = (\delta - \delta')/2$ . Suppose  $z' \in Z$  satisfies  $\rho(z, z') < \xi$ . We wish to show that  $\mu'(f(z'), x) < \delta$  so that  $B_\rho(z, \xi) \subset f^{-1}[U]$ . Since  $\mu'(x, f(z)) = \delta'$ , there is a sequence  $x_1, \dots, x_n \in C \cup D \subset Y$  such that

$$\lambda(x, x_1) + \dots + \lambda(x_n, f(z)) < \delta' + \xi.$$

Adding the term  $\rho(z, z') = \rho(f^{-1}(f(z)), f^{-1}(f(z')))$  to this sum,

$$\lambda(x, x_1) + \dots + \lambda(x_n, f(z)) + \rho(z, z') < \delta' + 2\xi < \delta$$

illustrates that the sequence  $x_1, \dots, x_n, f(z)$  yields a sum between  $x$  and  $f(z')$  that is less than  $\delta$ . Hence,  $\mu'(x, f(z')) < \delta$  as desired. So, the map  $f$  is continuous and the claim is proven.  $\square$

Since  $(Z, \rho)$  is connected,  $\tau \subset \tau'$  and  $C \cup D$  is dense in  $(Y, \tau|_Y)$ , by Claim 3,  $(Y, \mu'|_Y)$  is connected. For  $x \notin Y$ , we argued before that  $\mu'(x, Y) \geq \mu(x, Y)$ , so  $\text{cl}_{\tau'} Y \subset \text{cl}_\tau Y$ . However  $\text{cl}_\tau Y \subset \text{cl}_{\tau'} Y$ , since  $\tau'$  is coarser than  $\tau$ . Hence  $\text{cl}_{\tau'} Y = \text{cl}_\tau Y = \text{cl}_\tau(D \cup C)$  and therefore  $(\text{cl}Y, \tau'|_{\text{cl}Y})$  is connected. Therefore, ii) of the lemma is shown.

We now show iii). Let  $x, y \notin U$ . We verify that  $\mu'(x, y) = \mu(x, y)$ . By definition,  $\mu'(x, y) \leq \mu^*(x, y) = \mu(x, y)$  so we need only show that  $\mu'(x, y) \geq \mu^*(x, y)$ .

Suppose for a contradiction that  $\mu'(x, y) < \mu^*(x, y)$ . In other words, there exist  $x_1, \dots, x_n \in D \cup C$  such that

$$\mu'(x, y) \leq \lambda(x, x_1) + \dots + \lambda(x_n, y) < \mu^*(x, y).$$

Since  $x, y \notin U$ ,  $\mu^*(x, y) = \mu(x, y)$ . If  $f^{-1}(x)$  and  $f^{-1}(y)$  are defined, then since  $x, y \notin C$ ,  $\rho(f^{-1}(x), f^{-1}(y)) = 2\epsilon > \mu(x, y) = \mu^*(x, y)$ . Hence, the sequence  $x_1, \dots, x_n$  is not empty. Now, since  $\text{diam}_\mu(Y) \leq \epsilon$ , for any  $z_1, z_2 \in Y$ ,

$$\mu(x, y) \leq \mu(x, z_1) + \epsilon + \mu(z_2, y).$$

Hence,

$$\mu(x, y) \leq \mu(x, Y) + \epsilon + \mu(y, Y).$$

Combining these inequalities we have

$$\mu'(x, y) \leq \lambda(x, x_1) + \cdots + \lambda(x_n, y) < \mu(x, Y) + \epsilon + \mu(y, Y). \quad (2)$$

*Case 1.*  $x, y \in Y$ .

Then inequality (2) becomes  $\mu'(x, y) \leq \lambda(x, x_1) + \cdots + \lambda(x_n, y) < \epsilon$ . Since  $x, y \notin U$ ,  $\mu^*(x, C \setminus \{x\}) = \mu^*(x, C) \geq \epsilon$  and  $\mu^*(y, C \setminus \{y\}) = \mu^*(y, C) \geq \epsilon$ . Since  $x \notin C$ , if  $x' = f^{-1}(x)$  is defined, then  $x \in D$ . Hence  $\rho(x', T \setminus \{x'\}) \geq \epsilon$ . Similarly, if  $y' = f^{-1}(y)$  is defined,  $\rho(y', T \setminus \{y'\}) \geq \epsilon$ . So,  $\delta(x, y) = \min\{\epsilon, \mu^*(x, C \setminus \{x\}), \mu^*(y, C \setminus \{y\}), \rho(x', T \setminus \{x'\}), \rho(y', T \setminus \{y'\})\} \geq \epsilon$ . By Note 1 after Claim 2,  $\mu'(x, y) \geq \min\{\mu^*(x, y), \epsilon\}$ . This contradicts the assumption that  $\mu'(x, y) < \mu^*(x, y)$ .

*Case 2.*  $x \notin Y$  and  $y \in Y$ .

Since  $y \in Y$ ,  $\mu(y, Y) = 0$ , hence (2) becomes  $\mu'(x, y) \leq \lambda(x, x_1) + \lambda(x_1, x_2) + \cdots + \lambda(x_n, y) < \mu(x, Y) + \epsilon$ . Suppose  $x_1 \in C$ . Since  $x \notin Y$ ,  $\lambda(x, x_1) = \mu^*(x, x_1) = \mu(x, x_1) + \epsilon \geq \mu(x, Y) + \epsilon$ , a contradiction. Suppose that  $x_1 \in D$ . Since  $\lambda(x, x_1) = \mu^*(x, x_1) \geq \mu(x, Y)$  we have that  $\lambda(x_1, x_2) + \cdots + \lambda(x_n, y) < \epsilon$ . If  $n$  is even,  $\lambda(x_n, y) = \mu^*(x_n, y)$  and by Claim 1,  $x_n \in C$ . Therefore  $\lambda(x_n, y) = \mu^*(x_n, y) = \mu(x_n, y) + \epsilon > \epsilon$ , a contradiction. If  $n$  is odd,  $\lambda(x_n, y) = \rho(f^{-1}(x_n), f^{-1}(y))$  and so  $y \in C \cup D$ . Since  $y \notin U$ , we have that  $y \in D$  and therefore  $\rho(f^{-1}(x_n), f^{-1}(y)) = 2\epsilon > \epsilon$ , a contradiction.

A symmetric argument yields a contradiction in the case that  $x \in Y$  and  $y \notin Y$ .

*Case 3.*  $x, y \notin Y$ .

Since  $x, y \notin Y$ , they are not in the image of  $f$  and therefore the alternating sum must start and end with  $\mu^*$  terms implying  $n$  is even and  $n \geq 2$ . Since  $x_1, x_n \in C \cup D \subset Y$ ,  $\mu^*(x, x_1) \geq \mu(x, x_1) \geq \mu(x, Y)$  and  $\mu^*(x_n, y) \geq \mu(x_n, y) \geq \mu(y, Y)$ . Combining this with (2) gives:

$$\begin{aligned} \rho(f^{-1}(x_1), f^{-1}(x_2)) + \mu^*(x_2, x_3) + \cdots + \rho(f^{-1}(x_{n-1}), f^{-1}(x_n)) < \epsilon, \\ \mu^*(x, x_1) < \mu(x, Y) + \epsilon \end{aligned} \quad (3)$$

and

$$\mu^*(x_n, y) < \mu(y, Y) + \epsilon. \quad (4)$$

Suppose  $x_1 \in C$ . Then  $\mu^*(x, x_1) = \mu(x, x_1) + \epsilon \geq \mu(x, Y) + \epsilon$  contradicting (3). So,  $x_1 \notin C$ . Similarly,  $x_n \in C$  contradicts (4). So,  $x_n \notin C$ . Hence we have shown that  $x_1, x_n \in D$ . Now,  $\lambda(x_i, x_{i+1}) < \epsilon$  for each  $i$  with  $1 \leq i < n$  and  $\lambda(x_1, x_2) = \rho(f^{-1}(x_1), f^{-1}(x_2))$ . So, since  $x_1 \in D$  and  $n$  is even, by Claim 1,  $x_n \in C$ , which is a contradiction. Therefore,  $\mu'(x, y) = \mu^*(x, y) = \mu(x, y)$ , and iii) of the lemma follows.  $\square$

If  $X$  is a metric space with  $e(X)$  not attained, then  $X$  must have a certain form. This form is described in the next lemma.

**Lemma 2.2.** (Fitzpatrick, Gruenhagen, Ott [2]) *Let  $(X, \tau)$  be a metric space with metric  $\mu$  in which  $e(X) = \kappa$  is not attained. Let  $K$  be the set of points  $x$  of  $X$  such that every neighborhood of  $x$  has extent  $\kappa$ . Then*

- (1)  $\kappa$  is a singular cardinal of cofinality  $\omega$ .
- (2)  $K$  is a compact, nowhere dense subset of  $X$ .
- (3) If  $U$  is an open subset of  $X$  such that  $\text{cl}_\tau U \cap K = \emptyset$ , then  $e(U) < \kappa$ .
- (4)  $K$  is nonempty.

Recall that Konig's Lemma implies that  $\text{cf}(\mathfrak{c}) > \omega$ . So, if  $(X, \tau)$  is a metric space and  $e(X) = \mathfrak{c}$ , by Lemma 2.2, the extent of  $X$  must be attained.

**Definition 2.3.** (see [3]) An open set  $V$  is called *e-homogeneous* if for every nonempty open subset  $V'$  of  $V$ ,  $e(V') = e(V)$ .

Suppose  $U$  is a nonempty open subset of a metric space. We argue that there is a nonempty open e-homogeneous set  $V$  such that  $V \subset U$ . If  $U$  is e-homogeneous then set  $V = U$ . If  $U$  is not e-homogeneous, let  $V_0 = U$  and let  $V_1$  be a nonempty open subset of  $U$  such that  $e(V_1) < e(U)$ . Continue by defining  $V_n \subset V_{n-1}$  such that  $e(V_n) < e(V_{n-1})$  if  $V_{n-1}$  is not homogeneous. Since  $e(V_1), e(V_2), \dots$  is a decreasing sequence of cardinal numbers, there is  $n \in \omega$  such that  $V = V_n$  is e-homogeneous.

Suppose the extent of an open subset,  $U$ , of a metric space is not attained in  $U$ . Apply Lemma 2.2 with  $X = U$ . Then, by property (2), there is  $x \in U \setminus K$  and an open set  $V \subset U$  with  $e(V) < e(U)$  and  $x \in V$ . Therefore, if  $U$  is an e-homogeneous subset of a metric space,  $e(U)$  is attained in  $U$ . However, as mentioned before, this does not mean that  $e(U)$  is attained in  $X$ .

**Lemma 2.4.** *Let  $U$  be an e-homogeneous subset of a metric space  $(X, \tau, \mu)$  with metric  $\mu$  and  $e(U) > \aleph_0$ . Then  $e(U)$  is attained in  $X$ .*

*Proof.* Since  $U$  is e-homogeneous,  $e(U) = \lambda$  is attained in  $U$ . Suppose that  $\text{cf}(\lambda) > \aleph_0$ . Then,  $e(U)$  is attained by some closed discrete (in  $U$ ) subset  $C' \subset U$  of cardinality  $\lambda$ . However, it may not be the case that  $C'$  is closed discrete in  $X$ . We will construct a set  $C \subset U$  such that  $C$  is closed discrete in  $\text{cl}_\tau U$ , hence closed discrete in  $X$ , and  $|C| = e(U) = \lambda$ . Let  $\mathcal{U}$  be a discrete collection of open subsets of  $U$  that witnesses that  $C'$  is closed discrete in  $U$ . Let

$$L = \text{cl}_\tau \left( \bigcup \mathcal{U} \right) \setminus \bigcup_{U \in \mathcal{U}} \text{cl}_\tau U.$$

Note that  $L \subset \text{cl}_\tau U \setminus U$ . If  $L = \emptyset$  then  $\mathcal{U}$  is discrete in  $X$  and  $C'$  is closed discrete in  $X$ . Therefore, suppose that  $L \neq \emptyset$ . Let  $C_n = \{c \in C' : \mu(c, L) \geq$



$1/n\}$  and note that  $\bigcup C_n = C'$ . Since  $cf(\lambda) > \aleph_0$  there is  $n \in \omega$  so that  $|C_n| = \lambda$ . Set  $C = C_n$ . By construction,  $C$  is closed and  $\text{cl}_\tau C \cap L = \emptyset$ , so  $C$  is closed discrete in  $X$ ,  $C \subset U$  and  $|C| = \lambda = e(U)$  as desired.

Now suppose that  $cf(\lambda) = \aleph_0$ . Let  $\{\lambda_n : n \in \omega\}$  be an increasing sequence of cardinals such that  $\lambda = \sup_{n \in \omega} \lambda_n$ . Let  $W$  be an open subset of  $U$  such that  $\text{cl}_\tau W \subset U$ . Note that since  $U$  is e-homogeneous,  $e(W) = e(\text{cl}_\tau W) = e(U) = \lambda$ . Since  $e(\text{cl}_\tau W) = \lambda > \aleph_0$  there is  $A \subset \text{cl}_\tau W \subset U$  a countable closed discrete set in  $\text{cl}_\tau W$ , hence closed discrete in  $X$ . Let  $\mathcal{U} = \{U_n : n \in \omega\}$  be a discrete collection of open subsets of  $U$  that witnesses  $A$  is discrete in  $X$ . For each  $n \in \omega$ ,  $e(\text{cl}_\tau U_n) = \lambda > \lambda_n$ , so there is  $C_n \subset \text{cl}_\tau U_n$  a closed discrete in  $X$  subset of cardinality  $\lambda_n$ . Let  $C = \bigcup C_n$ . Since  $C_n$  is closed discrete in  $\text{cl}_\tau U_n$ , it is closed discrete in  $X$ . Moreover, since  $\mathcal{U}$  is discrete,  $C$  is closed discrete in  $X$ . Finally,  $|C| = \lambda = e(\text{cl}_\tau U)$  by construction and since  $U_n \subset U$  for each  $n$ ,  $C \subset U$ . In either case,  $|C| = \lambda = e(U)$ ,  $C \subset U$  and  $C$  is closed and discrete in  $X$ . Hence,  $e(U)$  is attained in  $X$ .  $\square$

**Lemma 2.5.** *Let  $(X, \tau)$  be a metric space with metric  $\mu$  in which  $e(X) = \kappa$  is not attained. Let  $K$  be the set of points  $x$  of  $X$  such that every neighborhood of  $x$  has extent  $\kappa$ . Then, for every open set  $U$  meeting  $K$  and every  $\theta < \kappa$  there is an open subset  $V$  of  $U$  such that  $\text{cl}_\tau V \subset U$ ,  $e(V) > \theta$  is attained in  $X$  and  $K \cap \text{cl}_\tau V = \emptyset$ .*

*Proof.* Let  $\mathcal{V}$  be a maximal pairwise disjoint collection of e-homogeneous subsets  $V$  of  $U \setminus K$  such that  $\text{cl}_\tau V \cap K = \emptyset$ . For each  $V \in \mathcal{V}$ , since  $\text{cl}_\tau V \cap K = \emptyset$ , by item (3) of Lemma 2.2,  $e(V) < \kappa$ . Suppose that for some  $V \in \mathcal{V}$ ,  $e(V) > \theta$ . By Lemma 2.4,  $e(V)$  is attained in  $X$  and we are done. Suppose on the other hand that  $e(V') \leq \theta$  for all  $V' \in \mathcal{V}$ . Since  $\mathcal{V}$  is maximal,  $K$  is nowhere dense and every open subset of  $X$  has an e-homogeneous subset,  $\bigcup \mathcal{V}$  is dense in  $U$ . Since  $X$  is metric,  $e(W) = d(W)$  for any open subset  $W$ . Suppose that  $|\mathcal{V}| = \lambda < \kappa$ . Then,  $d(U) = d(\bigcup \mathcal{V}) \leq \lambda \cdot \theta < \kappa$  which is a contradiction. So,  $|\mathcal{V}| = \kappa$ . Since  $\mu(V', K) > 0$  for all  $V' \in \mathcal{V}$  and  $cf(\kappa) = \omega$ , there is  $n \in \omega$  such that  $|\{V' \in \mathcal{V} : \mu(K, V') > 1/n\}| > \theta$ . Set  $\mathcal{V}' = \{V' \in \mathcal{V} : \mu(K, V') > 1/n\}$ . We now refine  $\mathcal{V}'$  to a discrete collection of size  $> \theta$ . Let

$$L = \text{cl}_\tau \left( \bigcup \mathcal{V}' \right) \setminus \bigcup_{V \in \mathcal{V}'} \text{cl}_\tau V$$

and let  $L_\epsilon = \{x \in U : \mu(x, L) \leq \epsilon\}$ . Since  $\bigcap_{\epsilon > 0} L_\epsilon = L$  and  $V' \cap L = \emptyset$  for

all  $V' \in \mathcal{V}'$ , there is  $m \in \omega$  such that  $|\{V' \in \mathcal{V}' : V' \setminus L_{1/m} \neq \emptyset\}| > \theta$ . Set  $\mathcal{V}'' = \{V' \in \mathcal{V}' : V' \setminus L_{1/m} \neq \emptyset\}$ . For each  $V' \in \mathcal{V}''$ , let  $W(V') = V' \setminus L_{1/m}$ . The collection  $\mathcal{W} = \{W(V') : V' \in \mathcal{V}''\}$  by construction is discrete and has

cardinality  $> \theta$ . Set  $V = \bigcup \mathcal{W}$ . For each  $W \in \mathcal{W}$ , choose  $x_W \in W$ . Let  $C = \{x_W : W \in \mathcal{W}\}$ . Then  $C$  is closed discrete in  $X$  since  $\mathcal{W}$  is discrete in  $X$ , and  $|C| = |\mathcal{W}| > \theta$ . Since  $e(V) \leq \sup\{e(W) : W \in \mathcal{W}\} \cdot |\mathcal{W}| \leq \theta \cdot |\mathcal{W}| = |C|$ , we have that  $e(V) > \theta$  is attained in  $X$ .  $\square$

**Lemma 2.6.** *Let  $K$  be a compact metric space with topology  $\sigma$  and let  $\mathcal{U} \in [\sigma]^{<\omega}$  be a pairwise disjoint collection such that  $\bigcup \mathcal{U}$  is dense in  $K$ . If  $\epsilon > 0$  and  $\mathcal{V}$  is the collection of open subsets of  $K$  with diameter less than  $\epsilon$ , then there exists a pairwise disjoint  $\mathcal{V}' \in [\mathcal{V}]^{<\omega}$  such that  $\mathcal{V}'$  refines  $\mathcal{U}$  and  $\bigcup \mathcal{V}'$  is dense in  $K$ .*

*Proof.* Since  $\mathcal{V}$  covers  $K$ , compact, there are  $n \in \omega$  and  $V_1, V_2, \dots, V_n \in \mathcal{V}$  such that  $K = \bigcup \{V_i : 1 \leq i \leq n\}$ . Define  $\hat{V}_1 = V_1$  and for  $1 < i \leq n$  let  $\hat{V}_i = V_i \setminus \text{cl}(\bigcup \{V_j : 1 \leq j < i\})$ . Let  $\hat{\mathcal{V}} = \{\hat{V}_i : 1 \leq i \leq n\}$ . Note that  $\hat{\mathcal{V}}$  is pairwise disjoint and  $\bigcup \hat{\mathcal{V}}$  is dense in  $X$ . Now define  $\mathcal{V}' = \{V \cap U : V \in \hat{\mathcal{V}}, U \in \mathcal{U}\}$ . Since  $V \cap U \subset U$  for each  $V \in \hat{\mathcal{V}}$  and  $U \in \mathcal{U}$ ,  $\mathcal{V}'$  refines  $\mathcal{U}$ . Since  $\text{diam}(V \cap U) \leq \text{diam}(V) \leq V_i$  for some  $1 \leq i \leq n$ ,  $\text{diam}(V') < \epsilon$  for all  $V' \in \mathcal{V}'$ . Hence  $\mathcal{V}' \subset \mathcal{V}$ . Since  $\hat{\mathcal{V}}$  is pairwise disjoint,  $\mathcal{V}'$  is as well. Finally,  $\bigcup \mathcal{V}' = \bigcup \hat{\mathcal{V}} \cap \bigcup \mathcal{U}$  and therefore  $\bigcup \mathcal{V}'$  is dense in  $K$ , since  $\bigcup \mathcal{V}'$  and  $\bigcup \mathcal{U}$  are open and dense in  $K$ .  $\square$

### 3. The Main Theorem

**Theorem 3.1.** *If  $(X, \tau)$  is a metric space with metric  $\mu$  and  $w(X) = \kappa \geq \mathfrak{c}$ , then there is  $\sigma$ , a topology on  $X$  coarser than  $\tau$ , such that  $(X, \sigma)$  is connected and metrizable.*

*Proof.* As mentioned in the beginning of Section 2, the theorem has been proven in the case when  $e(X)$  is attained. Therefore, we assume  $e(X) > \mathfrak{c}$  is not attained. Re-scale  $\mu$  so that  $\text{diam}_\mu(X) < 1/2$  by replacing it with  $\frac{\mu}{2(1+\mu)}$ . Let  $K$  be the set of points  $x$  of  $X$  such that every neighborhood of  $x$  has extent  $\kappa$ . By Lemma 2.2,  $K$  is compact.

Let  $\mathcal{C}_0^* = \{K\}$ . For each  $n \in \omega \setminus \{0\}$ , define  $\mathcal{C}_n^* \subset \tau|_K$ , a pairwise disjoint finite collection with the following properties.

- $\text{cl}(\bigcup \mathcal{C}_n^*) = K$ .
- $\mathcal{C}_n^*$  refines  $\mathcal{C}_{n-1}^*$ .
- $B \in \mathcal{C}_n^*$  implies  $\text{diam}(B) < 1/2^n$ .

Let  $n \in \omega \setminus \{0\}$ . Apply Lemma 2.6 with  $\sigma = \tau|_K$ ,  $\mathcal{U} = \mathcal{C}_{n-1}^*$  and  $\epsilon = 1/2^n$  to get  $\mathcal{V}'$ , a pairwise disjoint collection of open sets in  $K$  with diameter less than  $1/2^n$  that refines  $\mathcal{C}_{n-1}^*$  and whose union is dense in  $K$ . Set  $\mathcal{C}_n^* = \mathcal{V}'$ .

*Definition of  $B_i$ 's*

For  $n \in \omega$  enumerate the elements of  $\mathcal{C}_n^*$  as  $\mathcal{C}_n^* = \{B_i^* : i_n \leq i < i_{n+1}\}$  with an increasing sequence of integers,  $i_n$ , where  $i_0 = 0$ . For each  $i \in \omega$ , let  $L_i = \{x \in X : \mu(x, B_i^*) \leq \mu(x, K \setminus B_i^*)\}$ , the set of points in  $X$  that are closer to  $B_i^*$  than they are to its complement. Fix  $n \in \omega$  and let  $B_{i_n} = L_{i_n}$  and for each  $i$  with  $i_n < i < i_{n+1}$ , let  $B_i = L_i \setminus \bigcup\{\text{cl}(B_j) : i_n \leq j < i\}$ . Note that  $B_0^* = K$ ,  $B_0 = X$  and that  $B_j^* \subset B_i^*$  implies  $L_j \subset L_i$ .

We define  $\mathcal{C}_n = \{B_i : i_n \leq i < i_{n+1}\}$  and verify the following.

- (i) For each  $n \in \omega$ ,  $\mathcal{C}_n$  is pairwise disjoint.
- (ii) For each  $n \in \omega$ ,  $\bigcup \mathcal{C}_n$  is dense in  $X$ .
- (iii) For all  $i \in \omega$ ,  $\text{int}(B_i) \cap K \neq \emptyset$ .

From the definition of  $B_i$  and  $\mathcal{C}_n$ , (i) is clear. Towards (ii), let  $n \in \omega$  and let  $x \in X$ . Since  $\bigcup \mathcal{C}_n^*$  is dense in  $K$ ,  $\mu(x, K) = \mu(x, \bigcup \mathcal{C}_n^*)$ . Since  $\mathcal{C}_n^*$  is finite, there is  $i$  such that  $i_n \leq i < i_{n+1}$  and  $\mu(x, K) = \mu(x, B_i^*)$ . Therefore,  $\mu(x, B_i^*) \leq \mu(x, K \setminus B_i^*)$  and either  $x \in B_i$  or  $x \in \text{cl}(\bigcup\{B_j : i_n \leq j < i\})$ . In either case,  $x \in \text{cl}(\bigcup \mathcal{C}_n)$ . For (iii), note that  $\text{int}(B_i) \cap K = B_i^* \setminus \text{cl}(\bigcup\{B_j^* : i_n \leq j < i\}) = B_i^* \neq \emptyset$  since  $\mathcal{C}_n^*$  is pairwise disjoint.

Let  $U_i = \{x \in X : \mu(x, K) < 1/2^{i+1}\}$ .

**Claim 1.** For  $n \in \omega$ ,  $i_n \leq i < i_{n+1}$ ,  $\text{diam}_\mu(L_i \cap U_m) \leq 3/2^n$  for any  $m \geq n - 1$ .

*Proof.* Note that if  $x \in L_i$ ,  $\mu(x, B_i^*) \leq \mu(x, K \setminus B_i^*)$  which implies  $\mu(x, K) = \mu(x, B_i^*)$ . For  $i$  with  $i_n \leq i < i_{n+1}$ ,  $m \geq n - 1$ ,  $x, y \in L_i \cap U_m$  and  $\epsilon > 0$  there exist  $x_0, y_0 \in B_i^*$  such that  $\mu(x, x_0), \mu(y, y_0) < 1/2^{m+1} + \epsilon/2 \leq 1/2^n + \epsilon/2$ . So,

$$\mu(x, y) \leq \mu(x, x_0) + \mu(y, y_0) + \mu(x_0, y_0) \leq 2/2^n + \epsilon + 1/2^n \leq 3/2^n + \epsilon.$$

Hence,  $\text{diam}_\mu(L_i \cap U_m) \leq 3/2^n$ .  $\square$

*Definition of  $W_i$ 's*

We define  $\mathcal{W} = \{W_i : i \in \omega\}$ , a pairwise disjoint collection of open subsets of  $X$  such that

- (i)  $\text{cl}(W_i) \cap K = \emptyset$ ,
- (ii)  $e(W_i) > c$  is attained in  $X$  by  $C_i \subsetneq W_i$ ,
- (iii) for  $i_n \leq i < i_{n+1}$ ,  $\text{diam}_\mu(W_i) \leq 3/2^n$  and
- (iv)  $\bigcup \mathcal{W}$  is dense in  $X$ .

Let  $\hat{W}_0 = \emptyset$ . Apply Lemma 2.5 with  $\theta = c$  and  $U = \text{int}(B_0) = X$  to get an open subset  $V$  of  $X$  such that  $e(V) > \theta = c$  is attained in  $X$  by  $C_0 \subset V$  and  $\text{cl}_\tau V \cap K = \emptyset$ . By replacing  $C_0$  with a subset of the same cardinality, we may assume that  $C_0 \subsetneq V$ . Set  $W_0 = V$  and  $S_0 = V$ . Let  $k_0 = \min\{k \geq 1 : S_0 \subset X \setminus U_k\} \geq 1$ . By definition  $W_0$  is open, and since  $\text{cl}V \cap K = \text{cl}(W_0) \cap K = \emptyset$ , (i) holds. By the choice of  $W_0$ , (ii) holds and (iii) is trivial since  $\text{diam}_\mu(X) \leq 1/2$ .

Suppose we have defined  $W_i$  for all  $i$  with  $0 \leq i < i_n$  so that (i), (ii), and (iii) are satisfied. Also suppose that  $S_m = \bigcup\{W_i : i < i_{m+1}\}$  is dense in  $X \setminus \text{cl}(U_{k_{m-1}+1})$  and that  $k_m = \min\{k \geq m : S_m \subset X \setminus U_k\}$  whenever  $0 < m < n$ .

Let  $i$  be such that  $i_n \leq i < i_{n+1}$ . Let  $\hat{W}_i = B_i \setminus (\text{cl}(U_{k_{n-1}+1} \cup S_{n-1}))$ . Since  $\text{int}(B_i) \cap K \neq \emptyset$  and  $K \subset X \setminus \text{cl}(S_{n-1})$ ,  $\text{int}(B_i) \setminus \text{cl}(S_{n-1})$  is an open set meeting  $K$ . So, apply Lemma 2.5 with  $\theta = \max\{e(\hat{W}_i), c\}$  and  $U = \text{int}(B_i) \setminus \text{cl}(S_{n-1})$  to get an open subset  $V$  of  $U$  such that  $\text{cl}_\tau V \subset U$ ,  $e(V) > \theta$  is attained in  $X$  by  $C_i \subset V$  and  $\text{cl}_\tau V \cap K = \emptyset$ . As before, we may assume  $C_i \subsetneq V$ . Set  $W_i = \hat{W}_i \cup V$ . According to our choice,  $W_i$  satisfies (i) and (ii) for each  $i$  such that  $i_n \leq i < i_{n+1}$ . Set  $S_n = \bigcup\{W_i : i_n \leq i < i_{n+1}\} \cup S_{n-1}$ . Let  $k_n = \min\{k \geq n : S_n \subset X \setminus U_k\}$ . By Claim 1,  $\text{diam}(B_i \cap U_{n-1}) \leq 3/2^n$  since  $B_i \subset L_i$  and  $i_n \leq i < i_{n+1}$ . Also,  $S_{n-1}$  is dense in  $X \setminus U_{k_{n-2}+1}$ . Therefore, for any  $i$  as above,  $W_i = \hat{W}_i \cup V \subset B_i \setminus \text{cl}(S_{n-1}) \subset B_i \cap U_{k_{n-2}+1} \subset B_i \cap U_{n-1}$ , since  $k_{n-2} \geq n-2$ . Hence  $\text{diam}_\mu(W_i) \leq 3/2^n$  and (iii) is satisfied.

Towards (iv), since  $K$  is nowhere dense in  $X$ , we only show that  $\bigcup \mathcal{W}$  is dense in  $X \setminus K$ . Let  $x \in X \setminus K$  and let  $n \in \omega \setminus \{0\}$  be such that  $x \in X \setminus \text{cl}(U_{k_{n-1}+1})$ . Then either  $x \in \text{cl}(S_{n-1}) \subset \text{cl}(S_n)$  or  $x \in X \setminus \text{cl}(U_{k_{n-1}+1} \cup S_{n-1})$ . If  $x \in X \setminus \text{cl}(U_{k_{n-1}+1} \cup S_{n-1})$  then since  $\{B_i : i_n \leq i < i_{n+1}\}$  is dense in  $X$ ,  $x \in \text{cl}(B_i \setminus (\text{cl}(U_{k_{n-1}+1} \cup S_{n-1}))) \subset \text{cl}(W_i) \subset \text{cl}(S_n)$  for some  $i$ . In either case  $x \in \text{cl}(S_n)$ . So,

$$x \in \text{cl}\left(\bigcup\{W_i : i \in \omega\}\right) = \text{cl}\left(\bigcup\{S_n : n \in \omega\}\right).$$

Hence  $\bigcup\{W_i : i \in \omega\}$  is dense in  $X \setminus K$  and in  $X$ .

#### Linking the $W_i$ 's

Suppose that  $n, i, j \in \omega$  are such that  $i_n \leq i < i_{n+1} \leq j < i_{n+2}$  and  $B_j^* \subset B_i^*$ . Then  $W_i \subset B_i \cap U_{n-1} \subset L_i \cap U_{n-1}$  and  $W_j \subset B_j \cap U_n \subset L_j \cap U_n \subset L_i \cap U_{n-1}$ . Hence by Claim 1,  $x \in W_i$  and  $y \in W_j$  implies  $\mu(x, y) \leq 3/2^n$ . Since for each  $j \in \omega$ ,  $C_j \subsetneq W_j$ , we may choose  $x_j \in W_j \setminus C_j$  arbitrarily. For every  $i$  with  $i_n \leq i < i_{n+1}$ , let  $J_i = \{j : i_{n+1} \leq j < i_{n+2}, B_j^* \subset B_i^*\}$  and let  $X_i = \{x_j : j \in J_i\}$ . Notice that  $\text{diam}(W_i \cup X_i) \leq 3/2^n$ ,  $C_i \subset W_i \subset \text{int}(W_i \cup X_i)$  and that by the definition of  $B^*$ ,  $\bigcup_{i \in \omega} J_i = \omega \setminus \{0\}$ .

*Defining the connected topology on  $X$*

We define a sequence of metrics  $\mu_n$  on  $X$  such that  $\nu = \lim \mu_n$  is a well defined metric that generates a coarser connected topology on  $X$ . We define  $\mu_n$  by induction.

*Stage 0*

Apply Lemma 2.1 with  $Y = W_0 \cup X_0$ , metric  $\mu$ , topology  $\tau$ , closed discrete set  $C_0$ ,  $\epsilon = 3/2$  and  $U = W_0$  to get a topology  $\tau_0$  on  $X$  generated by the metric  $\mu_0$  on  $X$  such that

- (i)  $\mu_0 \leq \mu + 6/2$ ,
- (ii)  $\text{cl}_\tau(W_0 \cup X_0) = \text{cl}_{\tau_0}(W_0 \cup X_0)$  and  $\tau_0|_{W_0 \cup X_0}$  is connected, and
- (iii)  $\mu|_{(X \setminus W_0)^2} = \mu_0|_{(X \setminus W_0)^2}$ .

Note that since  $X_0$  is finite, property (ii) implies that  $\text{cl}_{\tau_0}(W_0) = \text{cl}_{\tau_0}(W_0 \cup X_0)$ . So, for all  $i \neq 0$ ,  $\text{int}_{\tau_0}(W_i) = W_i \setminus X_0 \supset W_i \setminus \{x_i\}$  and therefore since  $x_i \notin C_i$ ,  $C_i \subset \text{int}_{\tau_0}(W_i)$ .

*Stage  $n \geq 1$*

Set  $\sigma_0 = \tau_{n-1}$  and  $\rho_0 = \mu_{n-1}$ . For each  $i$  such that  $i_n \leq i < i_{n+1}$ , let  $j = i - i_n$  and apply Lemma 2.1 with  $Y = W_i \cup X_i$ ,  $C = C_i$ ,  $\tau = \sigma_j$ ,  $\mu = \rho_j$ ,  $U = \text{int}_{\sigma_j}(W_i)$  and  $\epsilon = 3/2^n$  to get a topology  $\sigma_{j+1}$  on  $X$  generated by the metric  $\rho_{j+1}$  on  $X$  such that

- (i)  $\rho_{j+1} \leq \rho_j + 6/2^n$ ,
- (ii)  $\text{cl}_{\sigma_j}(W_i \cup X_i) = \text{cl}_{\sigma_{j+1}}(W_i \cup X_i)$  and  $\sigma_{j+1}|_{W_i \cup X_i}$  is connected, and
- (iii)  $\rho_j|_{(X \setminus \text{int}_{\sigma_j}(W_i))^2} = \rho_{j+1}|_{(X \setminus \text{int}_{\sigma_j}(W_i))^2}$ .

Let  $\mu_n = \rho_{m_n}$  and  $\tau_n = \sigma_{m_n}$ . Note that, as in Stage 0, property (ii) implies  $C_i \subset \text{int}_{\tau_n}(W_i)$  for all  $i \geq i_{n+1}$ . It is a consequence of (i) and (iii), that if  $x \in X \setminus \bigcup\{W_i : i \leq i_{n+1}\}$  and  $y \in \bigcup\{W_i : i < i_{n+1}\}$  then  $\mu_n(x, y) \leq \mu(x, y) + 6/2^n$ . Furthermore,  $\mu_n = \mu_{n-1}$  on  $X \setminus \bigcup\{W_i : i_n \leq i < i_{n+1}\}$  and  $\tau_n|_{W_i \cup X_i}$  is connected for each  $i$  such that  $i_n \leq i < i_{n+1}$ .

Define  $\nu(x, y) = \lim \mu_n(x, y)$ . This map is a well defined metric since for any  $x, y \in X$ ,  $\nu(x, y) = \mu_m(x, y)$  for all

$$m \geq \max\{n : i_n \leq i < i_{n+1} \text{ and } x \in W_i \text{ or } y \in W_i\} \cup \{0\}.$$

Let  $\sigma$  be the topology generated by  $\nu$ . To show that  $\sigma \subset \tau$  we show that  $\nu$  preserves convergent  $\mu$  sequences. Suppose that  $\{y_j : j \in \omega\}$  and  $y$  are

such that  $\lim_{j \rightarrow \infty} \mu(y, y_j) = 0$ . If  $y \notin K$  there are  $n, m \in \omega$  such that  $y, y_j \in X \setminus \text{cl}(U_{k_n+1})$  for all  $j \geq m$ . Then,  $\nu(y, y_j) = \mu_n(y, y_j)$  for all  $j \geq m$ , hence  $\lim_{j \rightarrow \infty} \nu(y, y_j) = \lim_{j \rightarrow \infty} \mu_n(y, y_j) = 0$ , since  $\mu_n$  preserves  $\mu$  convergent sequences. Now suppose that  $y \in K$ . If  $y_j \in U_m$  for all  $m$ , let  $m_j = j$ . Otherwise, let  $m_j = \max\{m : y_j \in U_m\}$ . Since  $\mu(y, y_j) \rightarrow 0$ ,  $m_j \rightarrow \infty$ . If there is  $i \in \omega$  such that  $y_j \in W_i$  then there is  $n$  such that  $i \geq i_n \geq n = m_j$ , since  $y_j \in W_i \subset U_{m_j}$ . In this case, by the consequence of (i) and (iii),  $\nu(y, y_j) \leq \mu(y, y_j) + 6/2^n = \mu(y, y_j) + 6/2^{m_j}$ . If  $y_j \notin W_i$  for all  $i \in \omega$  then  $\nu(y, y_j) = \mu(y, y_j)$ . In either case  $\nu(y, y_j) \leq \mu(y, y_j) + 6/2^{m_j}$  and therefore  $\nu(y, y_j) \rightarrow 0$ . Note that  $\nu \Big|_{(W_i \cup X_i)^2} = \mu_n \Big|_{(W_i \cup X_i)^2}$  for  $i_n \leq i < i_{n+1}$ . Hence  $W_i \cup X_i$  is connected in  $\sigma$ . Furthermore,  $X_i \cap W_j \neq \emptyset$  for each  $j \in J_i$  and hence  $W_i \cup \bigcup_{j \in X_i} W_j$  is connected as well. This means that  $W_i$  is ‘linked’ to  $W_0$  for every  $i$  such that  $1 = i_1 \leq i < i_2$  and since  $\bigcup_{i \in \omega} J_i = \omega \setminus \{0\}$ , any later  $W_i$  is ‘linked’ to  $W_0$ . Therefore any  $\sigma$ -clopen subset,  $Z$ , of  $X$  would have to be empty, or contain  $W_i$  for all  $i$ . Since  $\bigcup_{i \in \omega} W_i$  is dense in  $X$ ,  $Z$  is trivial. Hence  $\sigma$  is connected.  $\square$

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