Virtual Seifert Surfaces and Slice Obstructions For Knots in Thickened Surfaces

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Virtual Seifert Surfaces and Slice Obstructions For Knots in Thickened Surfaces

Hans U. Boden ¹  Micah Chrisman ²  Robin Gaudreau ¹

¹McMaster University
²Monmouth University

30 June 2017
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Smooth Concordance on Knots

Let $\Sigma$ be a c.c.o. surface with two boundary components that is embedded in $S^3 \times \mathbb{I}$.

1. $\partial \Sigma = K_1 \sqcup -K_0$.
2. $K_i = \Sigma \cap (S^3 \times \{i\}), i = 0, 1$.
3. **Concordant**: $\Sigma \simeq S^1 \times \mathbb{I}$.
4. **Notation**: $K_1 \simeq K_0$.

Smoothness is required because of Morse theory.
Critical Points of Morse Functions

“death”

“saddle”

“birth”
Critical Points of Morse Functions

“death”

“saddle”

“birth”

\[ K \sqcup \quad \text{birth} \uparrow \downarrow \text{death} \]

\text{saddle move}
Critical Points of Morse Functions

A concordance is a “connected” sequence of isotopies, births $b$ (local minima), deaths $d$ (local maxima), and saddle moves $s$ such that the Euler characteristic $\# b - \# s + \# d = 0$. 
Slice & Ribbon Knots

Do a saddle move.
Slice & Ribbon Knots

Do a saddle move.  One death gives an unknot.
**Slice & Ribbon Knots**

Do a saddle move.  

One death gives an unknot.

**Definition (Slice, Ribbon Knot)**

$K$ is **slice** if $K \cong \bigcirc$. A slice knot is **ribbon** if it has a concordance to $\bigcirc$ with no births.
Concordance of Knots in $\Sigma \times \mathbb{I}$

$(K, \Sigma) := \Sigma$ a c.c.o. surface, $K$ a knot in $\Sigma \times \mathbb{I}$. 
Concordance of Knots in $\Sigma \times \mathbb{I}$

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Definition (Turaev ‘08)

$(K_0, \Sigma_0), (K_1, \Sigma_1)$ are concordant if there is an oriented 3-manifold $M$, an embedding $-\Sigma_0 \sqcup \Sigma_1 \hookrightarrow \partial M$ and an oriented annulus $A$ embedded in $M \times \mathbb{I}$ such that $\partial A = -K_0 \sqcup K_1$.

$(K, \Sigma)$ is slice if it is concordant to the unknot in $S^2 \times \mathbb{I}$. 
Concordance of Knots in $\Sigma \times \mathbb{I}$

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$(K, \Sigma)$ is *slice* if it is concordant to the unknot in $S^2 \times \mathbb{I}$.

Knots in $S^3$ can be considered as knot in $S^2 \times \mathbb{I}$. These are called *classical knots*. 
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Knots in $S^3$ can be considered as knot in $S^2 \times \mathbb{I}$. These are called *classical knots*.

**Theorem (Boden-Nagel, ‘17)**

A classical knot $K$ in $S^3$ is slice iff $(K, S^2)$ is slice.
Virtual Knots

The four pictures below all represent the same virtual knot.
Virtual Knots

The four pictures below all represent the same virtual knot.
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(1)  (2)  (3)  (4)

A virtual knot is a classical knot if and only if it can be represented as a diagram on a planar surface.
### Extended Reidemeister Moves:

**Definition (Kauffman ‘99)**

Two virtual knot diagrams are *equivalent* if a finite sequence of extended Reidemeister moves takes one to the other.

<table>
<thead>
<tr>
<th>R1:</th>
<th>R2:</th>
<th>R3:</th>
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<tbody>
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Virtual Knot Concordance

Definition (Concordance of Virtual Knots, Kauffman ≈ ’14)

Two oriented virtual knot are concordant if they are obtained from one another by a finite sequence of extended Reidemeister moves, births $b$, deaths $d$, and saddle moves $s$ satisfying $\#b - \#s + \#d = 0$. 
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\(^1\) It follows from Carter-Kamada-Saito (‘01) that concordance of knots in thickened surfaces is equivalent to concordance of virtual knots.
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1. It follows from Carter-Kamada-Saito (‘01) that concordance of knots in thickened surfaces is equivalent to concordance of virtual knots.

2. Up to symmetry, there are 92800 virtual knots having classical crossing number $\leq 6$ (Green ‘04).

3. The Henrich-Turaev polynomial and Turaev’s graded genus are inconclusive slice obstructions on some knots.
Virtual Rasmussen Invariant

Theorem (Dye-Kaestner-Kauffman ‘17)

The slice genus of a virtual knot having all positively signed crossings is \((-r + n + 1)/2\), where \(r\) is the number of virtual Seifert circles and \(n\) the number of classical crossings.
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**Theorem (Boden-C.-Gaudreau)**

If \(v\) is a non-classical slice virtual knot, then every diagram of \(v\) must have at least one positive crossing and at least one negative crossing.
Goal: Slice obstructions from signatures

- (Im-Lee-Lee ‘10) defined a signature for checkerboard colorable virtual knots via Goeritz matrices. This generalizes Gordon-Litherland signature.
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- (Cimasoni-Turaev ‘07) give signatures of knots in quasi-cylinders \( M \) with \( H_2(M) = 0 \), but did not consider the effect of stabilization (i.e. virtual knots).
Most Important Case: Almost Classical Knots

Definition (Silver-Williams ‘06)

An *almost classical knot* is a virtual knot possessing an Alexander numerable knot diagram.

- \(\iff\) bounds a Seifert surface in some \(\Sigma \times I\) representation.
- \(\iff\) is homologically trivial in some \(\Sigma \times I\) representation.
- \(\iff\) has a diagram s.t. index of every crossing is 0.

Only 76 AC knots of \(\leq 6\) crossings!!!
Linking numbers & Alexander polynomials.

- \( H_1(\Sigma \times \mathbb{I} - J, \Sigma \times 1; \mathbb{Z}) \cong \mathbb{Z} \) generated by a meridian \( \mu \) of \( J \).
Linking numbers & Alexander polynomials.

- $H_1(\Sigma \times I - J, \Sigma \times 1; \mathbb{Z}) \cong \mathbb{Z}$ generated by a meridian $\mu$ of $J$.

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  [K] = m \cdot [\mu] \in H_1(\Sigma \times \mathbb{I} - J, \Sigma \times 1).
  \]

- **Linking number**: \( \text{lk}(J, K) = m \). (Note: not symmetric)
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- $\alpha_1, \ldots, \alpha_{2g}$ a canonical system of curves on $F$ forming a basis of $H_1(F)$.

- **Directed Seifert matrix**: $V^\pm := (\text{lk}(\alpha_i^\pm, \alpha_j))$.

- $\Delta_K(t) = \det(\sqrt{t} \cdot V^- - 1/\sqrt{t} \cdot V^+) \ (\text{Boden et al '16})$
Directed signature functions

- $V^\pm + (V^\pm)^\tau$ symmetric, possibly singular.
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- $\sigma^\pm(K, F) := \text{signature}(V^\pm + (V^\pm)\tau)$. 
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- Directed Alexander-Conway poly: (controls singularity)

$$\nabla_{K,F}^{\pm}(t) = \det(t^{1/2} V^{\pm} - t^{-1/2}(V^{\pm})^\tau).$$
Directed signature functions

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- Directed Alexander-Conway poly: (controls singularity)
  $$\nabla_{K,F}^\pm(t) = \det(t^{1/2}V^\pm - t^{-1/2}(V^\pm)^\tau).$$

- Directed Tristam-Levine signature functions: $\omega \in S^1 - \{1\}$
  $$\hat{\sigma}_\omega^\pm(K, F) = \text{signature}((1 - \omega)V^\pm + (1 - \overline{\omega})(V^\pm)^\tau).$$
Theorem (Boden-C.-Gaudreau)

Suppose that $K$ is a knot in $\Sigma \times I$ bounding a Seifert surface $F$. Suppose that there is an oriented 3-manifold $W$ such that $\partial W = \Sigma$ and that $K$ bounds a slice disk in $W \times I$. Then:

1. there are polynomials $f^\pm(t) \in \mathbb{Z}[t]$ such that $\nabla^\pm_{K,F}(t) = f^\pm(t)f^\pm(t^{-1})$, and

2. if $\omega \in \mathbb{S}^1 - \{1\}$ and $\nabla^\pm_{K,F}(\omega) \neq 0$, then $\sigma^\pm(\omega, K, F) = 0$. 
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1. Upper (+) and lower (−) are not always the same. They are always the same for classical knots and agree with the classical signature.
Theorem (Boden-C.-Gaudreau)

Suppose that $K$ is a knot in $\Sigma \times \mathbb{I}$ bounding a Seifert surface $F$. Suppose that there is an oriented 3-manifold $W$ such that $\partial W = \Sigma$ and that $K$ bounds a slice disk in $W \times I$. Then:

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   and

2. if $\omega \in \mathbb{S}^1 - \{1\}$ and $\nabla_{K,F}^\pm(\omega) \neq 0$, then $\sigma_{\omega}^\pm(K, F) = 0$.

1. Upper (+) and lower (−) are not always the same. They are always the same for classical knots and agree with the classical signature.

2. Different Seifert surfaces can produce different directed signature functions.
Classical Seifert surfaces

Three twisted bands.
(canonical)
Classical Seifert surfaces

Three twisted bands.
(canonical)

Ideally: No twists!.
(band projection)
Classical Seifert surfaces

Three twisted bands. (canonical)

Ideally: No twists!. (band projection)

Two twisted bands. (typical intermediate)
Classical Seifert surfaces

Three twisted bands. (canonical)

Ideally: No twists!. (band projection)

Two twisted bands. (typical intermediate)

Cutting along each twisted band gives a local diffeomorphism.
Idea of virtual Seifert surfaces

Arrows of the Gauss diagram are partitioned into two types:

“twists” (solid) and “non-twists” (dashed).
Idea of virtual Seifert surfaces

A spanning surface generalizes Gauss diagrams to Seifert surfaces.

Embed the solid parts as a 1-complex in a cco surface $S$. 
Idea of virtual Seifert surfaces

Map to $\mathbb{R}^2$ so that $\partial S$ goes to the virtual knot diagram.

Cutting out the twist arrows restricts to a local diffeomorphism.
Definition of virtual Seifert surfaces

Definition (Virtual Seifert Surface)

Let $K$ be a virtual knot diagram and $D$ its Gauss diagram. A virtual Seifert surface for $K$ is a pair $(S, \phi)$ where $S$ is a two-sided spanning surface of $D$ with twist partition $P_D$ and $\phi: S \to \mathbb{R}^2$ is a smooth map such that $\phi(D) = K$, $\phi|_{\text{Sub}(S)}$ is a local diffeomorphism, and $\phi$ is at most two-to-one on every subsurface of $S$. 
Definition of virtual Seifert surfaces

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Theorem (Boden-C.-Gaudreau)
For every AC Gauss diagram $D$, there is an AC knot diagram $K$ with Gauss diagram $D$ and a canonical virtual Seifert surface $(S, \phi)$ of $K$ such that the boundaries of the subsurfaces of $S$ are mapped to the Seifert circles of $K$. Moreover, every AC knot has a virtual Seifert surface that is a band projection.
Virtual Seifert Surface Algorithm: Let $K$ be an AC knot diagram with Gauss diagram $D$ and Carter surface $\Sigma$. Let $C_0, C_1, C_2$ be the standard cellular decomposition of $\Sigma$.

1. Compute $\partial_2 : C_2 \rightarrow C_1$ and Seifert cycles from $D$. 
Virtual Seifert Surface Algorithm: Let $K$ be an AC knot diagram with Gauss diagram $D$ and Carter surface $\Sigma$. Let $C_0, C_1, C_2$ be the standard cellular decomposition of $\Sigma$.

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4. Deform $(S, \phi)$ to a band projection.
Example (6.87548):

\[
(\partial_2)^t = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 1 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 1 & -1 & 0 & -1 & 1 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]
Example (6.87548):
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\[ v^+ = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ v^- = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \]

- For all \( \omega \), \( \hat{\sigma}_\omega^\pm(K, F) = 0 \).
- \( \nabla_{K,F}^-(t) \neq f(t)f(t^{-1}) \).
- \( \therefore \) 6.87548 is not slice.
Relation to Im-Lee-Lee signatures

Theorem (Boden-C.-Gaudreau)

For a minimal diagram $K$ on $\Sigma$ and a Seifert surface $F$ of $K$, $\sigma(K, F)$ is equal to one of the two ILL-signatures of $K$.

- In practice, you can get from one surface to the other by taking the connect sum of $F$ with a parallel copy of $\Sigma$. 
A Technique For Slicing Gauss Diagrams

6.24131 is slice (Note: not AC).
Results for AC knots

We used all the following tools to determine the slice status of all non-classical AC knots having ≤ 6 crossings.

1. The GD slicing technique.
2. The graded genus (computed with Mathematica).
3. The directed signature functions (checked with ILL-signatures).
4. Factorization of directed Alexander-Conway polynomial.

Together with the known classical (u) results, we obtain the following complete list of 19 slice AC knots. The other 57 are non-slice.

<p>| | | | | |</p>
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Parity Projection

The Manturov parity projection (‘10) is a map $P_n^\infty$ from virtual knots to AC knots. Roughly: take a knot diagram on a surface and keep lifting it by certain finite cyclic covers of the surface until it is AC.
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**Theorem (Boden-C.-Gaudreau)**

If $K_0$ and $K_1$ are concordant virtual knots, then $P_n^\infty(K_0)$ and $P_n^\infty(K_1)$ are concordant for any $n \geq 2$ and $n = 0$.

\[
\therefore \text{Slice obstructions for AC knots yield obstructions for virtual knots.}
\]
Virtual knots of \(\leq 6\) crossings

We used all the following tools to determine the slice status of all non-classical virtual knots having \(\leq 6\) crossings.

1. The GD slicing technique.
2. The graded genus (computed with Mathematica)
3. The polynomial \(H_K(s, t, q)\) (a conjectured slice obstruction).

Up to the conjecture, we have the following summary results:

- 13 slice knots with 4 crossings (out of 108).
- 45 slice knots with 5 crossings (out of 2448)
- slice status of 3 of 5 crossing v-knots remains unknown
- 1866 slice knots with 6 crossings (out of 90235)
- slice status of 44 of 6 crossing v-knots remains unknown
Open Questions & Problems

1. Is $H_K(s, t, q)$ a slice obstruction?

2. Long virtual knots form a group $\mathcal{VC}$ under concatenation (C. ‘17). Is $\mathcal{VC}$ abelian?

3. Construct an analog of the algebraic concordance group for AC knots.

4. Are there any finite order elements in $\mathcal{VC}$ that are not concordant to any classical knot?

5. If a classical knot is slice, is it a virtual ribbon knot?

6. Classify the finite-type concordance invariants of virtual knots.

7. Extend signatures to oriented AC links.

8. Determine the slice genus of all virtual knots having $\leq 6$ crossings.
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