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Domains and Probability Measures: A Topological Retrospective

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A Topological Retrospective

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Overview

I. Review of Domain Theory Basics

II. Classical Approaches to Measures and Probability

III. Domain Theoretic Approach to Probability Measures

IV. Applications and New Results

Informatic partial order

$p \sqsubseteq q$ if q contains more information than p.

Example: Zero finding

$[a, b] \sqsubseteq [c, d] \in \mathbb{IR}$ iff $[c, d] \subseteq [a, b]$.

Directed completeness

$\emptyset \neq D \subseteq P$ *directed* if $x, y \in D \;\Rightarrow\; (\exists z \in D)\, x, y \leq z$.
P *directed complete:* D directed $\;\Rightarrow\;$ $\sup D$ exists.

$D \subseteq \mathbb{IR}$ directed $\Rightarrow\;$ $\sup D = \bigcap D$.

Approximation

$x \ll y$ iff $y \leq \sup D \;\Rightarrow\; (\exists d \in D)\, x \leq d$.
Domain: $\twoheaddownarrow y = \{x \mid x \ll y\}$ directed and $y = \sup \twoheaddownarrow y$

$[a, b] \ll [c, d]$ iff $[c, d] \subseteq (a, b)$;

$[c, d] = \bigcap \{[a, b] \mid [c, d] \subseteq (a, b)\}$.

Morphisms

$f : P \to Q$ *Scott continuous* if :

- f monotone, and
- D directed $\Rightarrow$ $f(\sup D) = \sup f(D)$.

DCPO – Directed complete partial orders and Scott continuous maps

DOM – Domains and Scott-continuous maps

Theorem: Tarski, Knaster, Scott
$D \in$ DCPO with least element, $\bot$, $f : D \to D$ monotone. Then:

- $\mathtt{Fix}\, f = \sup_{\alpha \in Ord} f^{\alpha}(\bot)$ is the least fixed point of f.
- f Scott continuous $\Longrightarrow \mathtt{Fix}\, f = \sup_{n \geq 0} f^{n}(\bot)$.

Least fixed point semantics:

$\mathtt{rec}\, x.p \longrightarrow p[\mathtt{rec}\, x.p/x] \quad \Longrightarrow \quad [\![\mathtt{rec}\, x.p]\!] = \mathtt{Fix}\, [\![p]\!]$.

Morphisms

$f : P \to Q$ *Scott continuous* if :

- f monotone, and
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Properties:

- $f : P \times Q \to R$ jointly Scott continuous iff f is separately Scott continuous.
- $[P \to Q]$ ordered *pointwise:* $f \sqsubseteq g$ iff $f(x) \leq g(x)$ $(\forall x \in P)$.

  $[P \to Q]$ is a DCPO if P, Q are DCPOs.
- Cartesian closed categories of domains: BCD $\subseteq$ RB $\subseteq$ FS:

  BCD – Bounded complete domains – generalize Scott domains

  – essentially, continuous lattices without a top element

Scott Topology

U Scott open if:

- $U = \uparrow U = \{x \in P \mid (\exists u \in U)\, u \leq x\}$ and
- D directed, $\sup D \in U \;\Rightarrow\; D \cap U \neq \emptyset$.

Always T_0, in fact, *sober;* $T_1 \;\Rightarrow$ flat order.

$\lim\, \{x\}_{x \in D} = \sup D$ for D directed.

$f\colon P \to Q$ Scott continuous iff f is continuous wrt Scott topologies.

D domain $\Rightarrow\; \mathcal{B}_D = \{\twoheaduparrow x \mid x \in D\}$ basis for $\sigma_D = \{U \mid U$ Scott open$\}$.

Transitivity: $x \leq y \ll y' \leq z \;\Rightarrow x \ll z$; Implies $\uparrow(\twoheaduparrow x) = \twoheaduparrow x$.

Interpolation: $x \ll z \;\Rightarrow\; (\exists y)\, x \ll y \ll z$. Implies $\twoheaduparrow x$ Scott open.

Scott Topology

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Lawson Topology

Basis: $\{\twoheaduparrow x \setminus \uparrow F \mid F \in \mathcal{P}_{<\omega} D\}$

Hausdorff refinement of Scott topology.

D is *coherent* if Lawson topology is compact.

All CCCs of domains consist of coherent domains

Basic Models

Interval domain: $(\mathbb{I}[0,1], \supseteq)$ – restriction of $(\mathbb{IR}, \supseteq)$

Cantor Tree: $\mathbb{CT} = \{0,1\}^* \cup \{0,1\}^\omega$ in prefix order.

Topology

Upper space: X – locally compact Hausdorff space

$\Gamma(X)$ – nonempty compact subsets of X under reverse inclusion:

$A \sqsubseteq B$ iff $B \subseteq A$. $A \ll B$ iff $B \subseteq A^\circ$.

Generalizes to the *upper power domain:*

$\mathcal{P}_U(D) = (\{X \subseteq D \mid \emptyset \neq X = \uparrow X \textit{ Scott compact}\}, \supseteq)$.

EDALAT: Used $(\Gamma(X), \supseteq)$ to model fractals, weakly hyperbolic Iterated Function Systems, neural nets...

Domain Environments

(Lawson) D is a *domain environment* for X if $(X, \tau_X) \simeq \mathrm{Max}\, D$ in relative Scott topology.

Example: $(\Gamma(X), \supseteq)$; $X \simeq \mathrm{Max}\, \Gamma(X)$ by $x \mapsto \{x\}$.

Computational Models:

X – metrizable space;

M – countably-based bounded complete domain.

Lawson; Ciesielski, Flagg & Kopperman:

$$(\exists M)\ (X, \tau_M) \simeq (\mathrm{Max}\, M, \sigma_M|_{\mathrm{Max}\, M}) \quad \text{iff} \quad X \text{ is Polish.}$$

Banach (1933)

X complete metric space

$C_b(X,\mathbb{R}) = \{f\colon X \to \mathbb{R} \mid f \text{ continuous, bounded}\}$ - Banach space;

$C_b(X,\mathbb{R})^* = \{\varphi\colon C_b(X) \to \mathbb{R} \mid \varphi \text{ continuous, linear}\}$ – dual space

$\operatorname{Prob} X$ – unit sphere of $C_b(X,\mathbb{R})^*$ in weak*-topology.

$\operatorname{SProb} X$ – unit ball of $C_b(X,\mathbb{R})^*$ in weak*-topology.

Banach-Alaoglu Theorem: Unit ball is weak*-compact.

So, $\operatorname{SProb} X$ and, since it's a closed subset, $\operatorname{Prob} X$ are weak*-compact.

Kolmogorov (1936)

Developed *abstract theory* of *measure spaces* and *probability*:

Probability space: $(\Omega, \Sigma_\Omega, \mu)$ – Set, σ-algebra, probability measure;

Random variable:

$X\colon (\Omega, \Sigma_\Omega) \to (\mathbb{R}, \Sigma_{\mathcal{B}(\mathbb{R})})$ *measurable map* to $\mathbb{R}$ with Borel σ-algebra.

Approach introduced:

- Probability measures on infinite product spaces; • 0–1 Laws;
- Probability measure as a set function: $\mu\colon \Sigma_\Omega \to [0,1]$ satisfying:

(*i*) $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$;

(*ii*) $\mu(\dot{\bigcup}_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n)$ if $\{A_n\}_{n\in\mathbb{N}} \subseteq \Sigma_\Omega$ pairwise disjoint.

Note: Condition (ii) implies:

- If $A \subseteq B$, then $\mu(A) \le \mu(B)$, and
- If $m \le n \Rightarrow A_m \subseteq A_n$, then $\mu(\bigcup_n A_n) = \sup_n \mu(A_n)$.

Relating Banach and Kolmogorov

Riesz Representation Theorem:

$\mu \mapsto (f \mapsto \int f \, d\mu) : \mathcal{M}(X) \simeq C_b(X, \mathbb{R})^*$ is an isometric isomorphism.

The weak*-topology is the weak topology, so:

$\mu_n \to \mu$ *weakly* iff $\int f \, d\mu_n \to \int f \, d\mu$ for $f : X \to \mathbb{R}$ bounded, continuous.

Portmanteau Theorem

Let $\mu_n, \mu \in \operatorname{Prob} X$ for X complete metric space. TAE:

- $\mu_n \to \mu$ in the weak topology
- $\int f \, d\mu_n \to \int f \, d\mu$ for all $f : X \to \mathbb{R}$ bounded, uniformly continuous
- $\limsup_n \mu_n(F) \le \mu(F)$ for all $F \subseteq X$ closed
- $\liminf_n \mu_n(O) \ge \mu(O)$ for all $O \subseteq X$ open
- $\lim_n \mu_n(A) = \mu(A)$ for all $A \subseteq X$ *μ-continuity sets:* $\mu(\overline{A} \setminus A) = 0$

Simple Measures Weak*-dense

X - separable metric space.

$A \subseteq X$ measurable $\Rightarrow A^\varepsilon = \{x \in X \mid (\exists a \in A)\, d(a,x) < \varepsilon\}$.

Definition: *(Lévy-Prokhorov metric)*

$d(\mu,\nu) = \inf\{\varepsilon > 0 \mid \mu(A) \leq \nu(A^\varepsilon)\ \&\ \nu(A) \leq \mu(A^\varepsilon)\ \forall A \in \mathcal{B}(X)\}$

The Lévy-Prokhorov metric generates the weak*-topology.

Prokhorov's Theorem: If X is a separable metric space, then $\{\sum_{x\in F} r_x\delta_x \mid 0 \leq r_x, \sum_{x \in F} r_x = 1, F \subseteq X \text{ finite}\} \subseteq \operatorname{Prob} X$ is dense in the Lévy-Prokhorov metric, and similarly for $\operatorname{SProb} X$.

Measures From a Domain Perspective

Valuations

Let D be a domain and let σ_D denote its family of Scott-open sets.
A *continuous valuation* is a mapping $\mu\colon \sigma_D \to [0,1]$ satisfying:

Strictness $\mu(\emptyset) = 0$

Modularity $\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$

Monotonicity $U \subseteq V \quad \Longrightarrow \quad \mu(U) \leq \mu(V)$

Continuity $\{U_i\} \subseteq \sigma_D$ directed $\quad \Rightarrow \quad \mu(\bigcup_i U_i) = \sup_i \mu(U_i)$.

$\mathbb{V}D$ – valuations on D, ordered pointwise:
$\mu \sqsubseteq \nu$ iff $\mu(U) \leq \nu(U)$ $(\forall U \in \sigma_D)$.

$\mathbb{V}D \subseteq [D \to [0,1]]$ is a subdcpo; but domain structure is mysterious.

Measures From a Domain Perspective

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Continuity $\{U_i\} \subseteq \sigma_D$ directed $\quad \Rightarrow \quad \mu(\bigcup_i U_i) = \sup_i \mu(U_i)$.

Every Borel subprobability measure μ induces a valuation on σ_D by $\mu(U) = \int \chi_U \, d\mu$;

The converse – *every valuation extends to a Borel subprobability measure* – was shown by LAWSON for countably-based bounded complete domains, and by ALVAREZ-MANILLA, EDALAT AND SAHEB-DJARHOMI for general domains.

The correspondence $\mu \in \mathrm{Prob}\, D \iff \mu \in \mathbb{V}D$ is bijective.

The Domain Order from the Classical Approach

Recall for a compact space X and $\mu, \nu \in \mathrm{Prob}\, X$,

$$\int f\, d\mu \leq \int f\, d\nu\ (\forall f \colon X \to \mathbb{R}_+) \iff \mu = \nu.$$

Theorem: If D is a coherent domain and $\mu, \nu \in \mathbb{V}D \simeq_{\mathsf{Set}} \mathrm{SProb}\, D$, then TAE:

- $\mu \sqsubseteq \nu$, i.e., $\mu(U) = \int \chi_U\, d\mu \leq \int \chi_U\, d\nu = \nu(U)$ $(\forall U \in \sigma(D))$.
- $\int f\, d\mu \leq \int f\, d\nu$ for all $f \colon D \to \mathbb{R}_+$ Scott continuous.
- $\int f\, d\mu \leq \int f\, d\nu$ for all $f \colon D \to \mathbb{R}_+$ monotone Lawson continuous.

When Scott is Weak on the Top (Edalat 1996)

If D is a countably-based domain and $\mu_n, \mu \in \mathbb{V}D$, then TAE:

❶ $\mu_n \to \mu$ in the Scott topology on $\mathbb{V}D$.

❷ $\liminf_n \mu_n(U) \geq \mu(U)$ ($\forall U \in \sigma_D$).

Corollary: If

- X separable metric space, and
- $e\colon (X, \tau_X) \hookrightarrow (\mathrm{Max}\, D, \sigma|_{\mathrm{Max}\, D})$ embedding as a G_δ

Then

- $e_*\colon (\mathrm{Prob}\, X, w^*) \hookrightarrow (\mathrm{Max}\, \mathbb{V}D, \sigma|_{\mathrm{Max}\, \mathbb{V}D})$ is an embedding.

When Scott is Weak on the Top (Edalat 1996)

If D is a countably-based domain and $\mu_n, \mu \in \mathbb{V}D$, then TAE:

1. $\mu_n \to \mu$ in the Scott topology on $\mathbb{V}D$.
2. $\liminf_n \mu_n(U) \geq \mu(U)$ $(\forall U \in \sigma_D)$.

Testing LPMs (van Breugel, M., Ouaknine & Worrell 2003)

Theorem: If D is a countably-based coherent domain, and $\mu_n, \mu \in \mathbb{V}D$, then $\mu_n \to \mu$ in the Lawson topology on $\mathbb{V}D$ iff:

- $\liminf_n \mu_n(U) \geq \mu(U)$ $(\forall U \in \sigma_D)$, and
- $\limsup_n \mu_n(\uparrow F) \leq \mu(\uparrow F)$ $(\forall F \subseteq D$ finite).

Corollary: If D is countably-based coherent, then the Lawson topology on $\mathbb{V}D$ agrees with the weak topology on $\mathrm{SProb}\, D$, so $\mathbb{V}D$ is coherent.

The proof uses the Portmanteau Theorem to establish the weak topology is finer than the Lawson topology.

Applications in Domain Theory

$\mathbb{V}$ extends to a monad on DCPO and on DOM by

$f \in [P \to Q] \;\mapsto\; \mathbb{V}f \in [VP \to \mathbb{V}Q]$ by $\mathbb{V}f\,\nu(U) = \nu(f^{-1}(U))$,

the push forward of ν by f.

The Jung-Tix Problem

Is there a Cartesian closed category of domains A for which $\mathbb{V}$: A $\to$ A?

What's known: A cannot be BCD (Jones, 1989).

A = RB or A = FS are only possibilities.

Recorded Knowledge of Domain Structure of $\mathbb{V}$ (Jung & Tix 1988)

- $\mathbb{V}$: COH $\to$ COH is a monad.
- $\mathbb{V}T \in$ BCD for any finite rooted tree T.
- $\mathbb{V}T^{rev} \in$ RB for any finite reverse tree T.

Expanding the Examples

New examples for which $\mathbb{V}D$ has known domain structure:

Tree Domains

D is a *tree domain* if $K\,D$ is a countable rooted tree and D is algebraic.

Example: $\mathbb{CT} := \{0,1\}^* \cup \{0,1\}^\omega$ – use prefix order.

$s \ll t$ iff $s \leq t$ & $s \in \{0,1\}^*$.

$C := \{0,1\}^\omega$ – Cantor set of infinite words, with inherited Scott topology.

Theorem: (Jung-Tix) $\mathbb{V}D$ is bounded complete if D is a tree domain.

Proof: Any tree domain is a bilimit of finite, rooted trees.

New examples for which $\mathbb{V}D$ has known domain structure:

Tree Domains

Theorem: (Jung-Tix) $\mathbb{V}D$ is bounded complete if D is a tree domain.

Chains

D – complete chain

The *cumulative distribution function* of $\mu \in \mathbb{V}D$ is

$$F_\mu \colon D \to [0,1] \text{ by } F_\mu(x) = \mu(\downarrow x).$$

F_μ preserves all infs, so F_μ has an upper adjoint $G_\mu \colon [0,1] \to D$.

If λ is Lebesgue measure, then $\nu = G_{\mu *}\,\lambda \in \mathbb{V}D$ satisfies:

$F_\nu(\downarrow x) = F_\mu(\downarrow x)\ \forall x \in D$, so $F_\nu = F_\mu$, so $\nu = \mu$.

It follows that $G \mapsto G_{\mu *}\,\lambda \colon [[0,1] \to D] \to \mathbb{V}D$ is an order-isomorphism.

Theorem: $\mathbb{V}D$ is a continuous lattice if D is a complete chain.

The Splitting Lemma and Simple Valuations

Intuition: Moving mass from a lower point to a higher point makes the measure higher in the order, e.g.,

$$r\delta_a + s\delta_b < \frac{1}{3}\delta_x + \frac{2}{3}\delta_y, \frac{1}{2}\delta_x + \frac{1}{2}\delta_y < \delta_z, \quad \text{if } a, b < x \| y < z.$$

Splitting Lemma (Jones 1989)

Let $\mu = \sum_{x \in F} r_x \delta_x, \nu = \sum_{y \in G} s_y \delta_y$ in $\mathbb{V}D$. Then

$\mu \leq \nu$ iff there are *transport numbers* $\{t_{x,y}\}_{(x,y) \in F \times G} \subseteq \mathbb{R}_+$ satisfying:

1. $r_x = \sum_y t_{x,y}$ $(\forall x \in F)$
2. $\sum_x t_{x,y} \leq s_y$ $(\forall y \in G)$
3. $t_{x,y} > 0 \;\Rightarrow\; x \leq y$.

   Moreover, $\mu \ll \nu$ iff

4. $t_{x,y} > 0 \implies \sum_x t_{x,y} < s_y$ and $x \ll y$ $(\forall x, y)$.

The proof is an application of the Max Flow – Min Cut Theorem.

The Splitting Lemma and Simple Valuations

$B_D \subseteq D$ is a *basis* if

- $\twoheaddownarrow x \cap B_D$ is directed, and
- $x = \sup(\twoheaddownarrow x \cap B_D)$

for all $x \in D$.

Simple Valuations are Dense

Let D be a domain with basis B_D, and let $\mathcal{B}$ be a basis for $[0,1]$. Then:

$$B_{\mathbb{V}D} = \{\sum_{x \in F} r_x \delta_x \mid r_x \in \mathcal{B}, \sum_x r_x < 1 \ \& \ F \subseteq B_D \text{ finite}\}$$

is a basis for $\mathbb{V}D$.

As a consequence, $\mu = \sup(\twoheaddownarrow \mu \cap B_{\mathbb{V}D})$ for all $\mu \in \mathbb{V}D$.

Random variable:

$X: (S, \Sigma_S, \mu) \to (T, \Sigma_T)$ measurable map from a probability space to a measure space.

A *stochastic process* is a family $\{X_t \mid t \in T \subseteq \mathbb{R}_+\}$ of random variables $X_t: \Omega \to S$, where $(\Omega, \Sigma_\Omega, \mu)$ is a probability space, and S is a Polish space.

Skorohod's Theorem

Let S be a Polish space, let $\nu \in \mathrm{Prob}\, S$, and let λ denote Lebesgue measure on $[0,1]$. Then there is a random variable $X: [0,1] \to S$ satisfying $X_* \lambda = \nu$.

Moreover, if $\nu_n, \nu \in \mathrm{Prob}\, S$ satisfy $\nu_n \to_w \nu$, then the random variables $X_n, X: [0,1] \to S$ can be chosen so that $X_* \lambda = \nu, X_{n*} \lambda = \nu_n$ and $X_n \to X$ λ-a.e.

Proposition: Let D be a domain and let

$$\mu = \sum_{x \in F} r_x \delta_x \leq \sum_{y \in G} s_y \delta_y = \nu \in \mathrm{Prob}\, D.$$

Assume that r_x, s_y are dyadic rationals for each $x \in F, y \in G$,

and assume there is an $m \in \mathbb{N}$ and $f_m \colon \{0,1\}^m \to D$ with

$$f_{m*}\left(\tfrac{1}{2^m} \sum_{i \leq 2^m} \delta_i\right) = \tfrac{1}{2^m} \sum_{i \leq 2^m} \delta_{f_m(i)} = \mu.$$

Then there is $n > m \in \mathbb{N}$ and $f_n \colon \{0,1\}^n \to D$ satisfying:

- $f_{n*}\left(\frac{1}{2^n} \sum_{j \leq 2^n} \delta_j\right) = \frac{1}{2^n} \sum_{j \leq 2^n} \delta_{f_n(j)} = \nu$, and
- $f_m \circ \pi_m \leq f_n$, where $\pi_m \colon \{0,1\}^n \twoheadrightarrow \{0,1\}^m$ is the canonical projection.

Note: $f_* \,\nu(A) = \nu(f^{-1}(A))$, the push forward of ν via f.

The proof relies on the Splitting Lemma and the fact that if r_x, s_y are dyadic, so are the transport numbers $t_{x,y}$.

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- $f_m \circ \pi_m \leq f_n$, where $\pi_m\colon \{0,1\}^n \twoheadrightarrow \{0,1\}^m$ is the canonical projection.

Corollary: *(Skorohod's Theorem for Domains)*

If D is a countably-based coherent domain, then

$$f \mapsto f_*\,\nu_C\colon [\mathbb{CT} \to D] \twoheadrightarrow \mathrm{Prob}\, D$$

is Scott continuous and surjective, where ν_C is Haar measure on the Cantor set $C \simeq \{0,1\}^\infty = \mathrm{Max}\,\mathbb{CT}$.

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Let S be a Polish space, let $\nu \in \mathrm{Prob}\, S$, and let λ denote Lebesgue measure on $[0,1]$. Then there is a random variable $X \colon [0,1] \to S$ satisfying $X_* \lambda = \nu$.

Moreover, if $\nu_n, \nu \in \mathrm{Prob}\, S$ satisfy $\nu_n \to_w \nu$, then the random variables $X_n, X \colon [0,1] \to S$ can be chosen so that $X_* \lambda = \nu, X_{n*} \lambda = \nu_n$ and $X_n \to X$ λ-a.e.

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Testing LPMs (van Breugel, M., Ouaknine & Worrell 2003)

Theorem: If D is a countably-based coherent domain, and $\mu_n, \mu \in \mathbb{V}D$, then $\mu_n \to \mu$ in the Lawson topology on $\mathbb{V}D$ iff:

- $\liminf_n \mu_n(U) \geq \mu(U)$ $\big(\forall U \in \sigma_D\big)$, and
- $\limsup_n \mu_n(\uparrow F) \leq \mu(\uparrow F)$ ($\forall F \subseteq D$ finite).

Corollary: If D is countably-based coherent, then the Lawson topology on $\mathbb{V}D$ agrees with the weak topology on $\mathrm{SProb}\, D$, so $\mathbb{V}D$ is coherent.

Proof: In light of the Theorem, the Portmanteau Theorem implies the weak topology on $\mathrm{SProb}\, D$ is finer than the Lawson topology on $\mathbb{V}D$, but the weak topology is compact and the Lawson topology is Hausdorff. $\square$

Tree Domains

D is a *tree domain* if $K\,D$ is a countable rooted tree and D is algebraic.

Example: $\mathbb{CT} := \{0,1\}^* \cup \{0,1\}^\omega$ – use prefix order.

$s \ll t$ iff $s \leq t$ & $s \in \{0,1\}^*$.

$C := \{0,1\}^\omega$ – Cantor set of infinite words, with inherited Scott topology.

Fact 1: $\mathbb{V}\colon \mathrm{DCPO} \to \mathrm{DCPO}$ is *locally continuous:*

$\mathbb{V}\colon [D \to E] \to [\mathbb{V}D \to \mathbb{V}E]$ Scott continuous for DCPOs D, E.

Then $D \simeq \textbf{bilim}\, D_i \quad \Longrightarrow \quad \mathbb{V}D \simeq \textbf{bilim}\, \mathbb{V}D_i$.

Fact 2: D tree domain $\quad \Longrightarrow \quad D \simeq \textbf{bilim}\, D_n,\ \ D_n$ finite Scott-closed subtree.

Theorem: (Jung-Tix) $\mathbb{V}D$ is bounded complete if D is a tree domain.

Proof: $\mathbb{V}D \simeq \mathrm{bilim}\, \mathbb{V}D_n$ and $\mathbb{V}D_n \in \mathrm{BCD}$ by Jung-Tix. □

Tree Domains

Theorem: (Jung-Tix) $\mathbb{V}D$ is bounded complete if D is a tree domain.

Chains

The *cumulative distribution function* for $\mu \in \mathbb{V}D$ is

$$F_\mu \colon D \to [0,1] \text{ by } F_\mu(x) = \mu(\downarrow x).$$

If D is a complete chain, then $\bigcap_{x \in \mathcal{F}} \downarrow x = \downarrow \inf \mathcal{F}$, so F_μ preserves filtered infs because $\mu \colon \mathcal{O}(D) \to [0,1]$ is Scott continuous.

Since D is a chain, F_μ preserves finite infs, so F_μ preserves all infima.

Thus F_μ is a lower adjoint. Let $G_\mu \colon [0,1] \to D$ be F_μ's upper adjoint.

Tree Domains

Theorem: (Jung-Tix) $\mathbb{V}D$ is bounded complete if D is a tree domain.

Chains

Then $G_\mu \colon [0,1] \to D$ preserves all suprema – i.e., G_μ is Scott continuous.

If λ is Lebesgue measure, then $\nu = G_{\mu*}\,\lambda \in \mathbb{V}D$ satisfies:

$F_\nu(\downarrow x) = \lambda(G_\mu^{-1}(\downarrow x)) \stackrel{*}{=} \lambda(\downarrow F_\mu(x)) = F_\mu(x)$ using $F_\mu \dashv G_\mu$. So $\nu = \mu$.

It's also straightforward to show $G \mapsto G_{\mu*}\,\lambda \colon [[0,1] \to D] \to \mathbb{V}D$ is an order-isomorphism.

Theorem: $\mathbb{V}D$ is a continuous lattice if D is a complete chain.

Corollary: If D is a countably-based coherent domain, then the map $f \mapsto f_* \nu\colon [\mathbb{CT} \to D] \to \mathbb{V}D$ is Scott continuous and surjective, where ν_C is Haar measure on the Cantor set $C \simeq \{0,1\}^\infty = \mathrm{Max}\,\mathbb{CT}$.

Note: If $f\colon \mathbb{CT} \to D$, then $f_* \mu(A) = \mu(f^{-1}(A))$, the push forward of μ via f.

Proof Outline: If $\mu \in \mathbb{V}D$, let $\mu_n \ll \mu$ be simple measures with dyadic coefficients satisfying $\mu = \sup_n \mu_n$.

Apply the Proposition recursively to define Scott-continuous maps $f_m\colon C_{p_m} \to D$ with $f_m(\nu_{p_m}) = \mu_m$ satisfying $m < n$ implies $f_m \circ \pi_m \leq f_n$.

Then $F_m\colon \mathbb{CT} \to D$ by $F_m|_{C_{p_k}} = f_k$ for $k \leq m$, and $F_m(x) = f_m \circ \pi_{p_m}(x)$ otherwise is Scott-continuous satisfying $F_m(\nu_C) = f_m(\nu_{p_m}) = \mu_m$.

Then $F = \sup_m F_m\colon \mathbb{CT} \to D$ is Scott continuous and

$$F(\nu_C) = \sup_m F_m(\nu_C) = \sup_m f_m \circ \pi_{p_m}(\nu_C)$$

$$= \sup_m f_m(\nu_{p_m}) = \sup_m \mu_m = \mu. \qquad \square$$

Proofs and Open Problems

Corollary: If D is a countably-based coherent domain, then the map $f \mapsto f_* \nu \colon [\mathbb{CT} \to D] \to \mathbb{V}D$ is Scott continuous and surjective, where ν_C is Haar measure on the Cantor set $C \simeq \{0,1\}^\infty = \mathrm{Max}\,\mathbb{CT}$.

Skorohod's Theorem

Let S be a Polish space, let $\nu \in \mathrm{Prob}\,S$, and let λ denote Lebesgue measure on $[0,1]$. Then there is a random variable $X\colon [0,1] \to S$ satisfying $X_* \lambda = \nu$.

Moreover, if $\nu_n, \nu \in \mathrm{Prob}\,S$ satisfy $\nu_n \to_w \nu$, then the random variables $X_n, X\colon [0,1] \to S$ can be chosen so that $X_* \lambda = \nu, X_{n*} \lambda = \nu_n$ and $X_n \to X$ λ-a.e.

Proof Outline:

- $S \hookrightarrow M_S$ – countably-based bounded complete domain environment.
- $\mathrm{Prob}\,S \hookrightarrow \mathrm{Max}\,\mathrm{Prob}\,M_S \subseteq \mathbb{V}M_S$; weak topology is the inherited Scott topology.

Corollary: If D is a countably-based coherent domain, then the map $f \mapsto f_* \nu \colon [\mathbb{CT} \to D] \to \mathbb{V}D$ is Scott continuous and surjective, where ν_C is Haar measure on the Cantor set $C \simeq \{0,1\}^\infty = \mathrm{Max}\,\mathbb{CT}$.

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Proof Outline:

- $B_S \subseteq M_S$ – countable basis
  $\mathcal{B} = \{\sum_{x \in F} r_x \delta_x \mid r_x \text{ dyadic}, \sum_x r_x = 1, F \subseteq B_S\}$ countable basis for $\mathrm{Prob}\,M_S$
- Apply the Corollary, and for $\mu \in \mathrm{Prob}\,S$, restrict F to $C = \mathrm{Max}\,\mathbb{CT}$.

Corollary: If D is a countably-based coherent domain, then the map $f \mapsto f_* \nu \colon [\mathbb{CT} \to D] \to \mathbb{V}D$ is Scott continuous and surjective, where ν_C is Haar measure on the Cantor set $C \simeq \{0,1\}^\infty = \mathrm{Max}\,\mathbb{CT}$.

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Open Problems:

What does $f \mapsto f_* \colon [\mathbb{CT} \to D] \twoheadrightarrow \mathbb{V}D$ tell us about the domain structure of $\mathbb{V}D$?

In particular:

Can $f \mapsto f_* \colon [\mathbb{CT} \to D] \twoheadrightarrow \mathbb{V}D$ be used to show $\mathbb{V}D \in$ RB or FS?