2022

Fixed Points of Functions below the Line y = x

Grace Fryling
Baldwin Wallace University

Harrison Rouse
Baldwin Wallace University

ELEC. PROC. UNDERGRAD. MATH. DAY
7 (2022), 20–26.

FIXED POINTS OF FUNCTIONS BELOW THE LINE $y = x$

GRACE A. FRYLING AND HARRISON M. ROUSE

Communicated by Paul Eloe

ABSTRACT. This paper concerns fixed points of functions whose graphs lie on or below the line $y = x$. Using the Monotone Convergence Theorem we show that positive fixed points of such functions are "attracting on the right" so long as we include a couple of further assumptions about these functions near their fixed points. As an illustrative example, we confirm that this is the case for the function $y = x \sin x$; the positive fixed points of this function "attract on the right" and "repel on the left." Further, we generalize by showing that differentiability is in fact not needed to conclude that a fixed point is attracting on the right. Continuing in this direction, we identify a class of discontinuous functions whose fixed points are attracting on the right.

KEYWORDS: *Fixed points of real valued functions, attracting fixed point, repelling fixed point*

MSC (2010): Primary 26A18, Secondary 37C25

1. INTRODUCTION

First, it is important to define exactly what a fixed point is, and what it means for a fixed point to be attracting or repelling. Let f be a function defined on the real numbers.

Definition 1.1. A real number p is called a *fixed point* of f if $f(p) = p$.

Definition 1.2. Let p be a fixed point of f. We say p is an *attracting fixed point* if there exists $\epsilon > 0$ such that every real number x with $|x - p| < \epsilon$ satisfies $\lim_{n\to\infty} f^n(x) = p$.

Definition 1.3. Let p be a fixed point of f. We say p is a *repelling fixed point* if there exists $\epsilon > 0$ such that for every real number x with $|x - p| < \epsilon$ and $x \neq p$, there exists $k \in \mathbb{N}$ such that $|f^k(x) - p| \geq \epsilon$.

Definition 1.4. Let p be an attracting fixed point of f. The set of all real numbers x with the property that the sequence $\{f^n(x)\}$ converges to p is called the *basin of attraction* of p.

For many fixed points of differentiable functions, we can determine whether they are attracting or repelling using the value of the derivative [1, p.12].

Theorem 1.1. *Let f be a continuously differentiable function with a fixed point $x = p$. If $|f'(p)| < 1$, then the fixed point $x = p$ is attracting.*

Proof. Assume $|f'(p)| < 1$, so there is a real number k such that

$$|f'(p)| < k < 1.$$

By the continuity of f' there is an $\epsilon > 0$ such that

$$|f'(x)| < k$$

for all $x \in (p - \epsilon, p + \epsilon)$. Choose $x \in (p - \epsilon, p + \epsilon)$, $x \neq p$. From the Mean Value Theorem,

$$\frac{f(x) - f(p)}{x - p} = f'(c)$$

where c lies between x and p. Therefore,

$$\frac{|f(x) - f(p)|}{|x - p|} = |f'(c)| < k,$$

or

$$|f(x) - f(p)| < k|x - p|.$$

Because p is a fixed point, this gives

$$|f(x) - p| < k|x - p|.$$

Therefore, the distance from $f(x)$ to p is smaller than the distance from x to p because $0 < k < 1$. This implies that $f(p)$ lies in the interval $(p - \epsilon, p + \epsilon)$ since x was chosen in that interval. Now we can apply the above process to $f(x)$ and $f(p)$, giving

$$|f^2(x) - p| = |f^2(x) - f^2(p)| < k|f(x) - f(p)| < k^2|x - p|.$$

Continuing in this way gives

$$|f^n(x) - p| < k^n|x - p|.$$

Note that $k^n \to 0$ as $n \to \infty$ because $0 < k < 1$. Hence $f^n(x) \to p$ as $n \to \infty$. Therefore p is an attracting fixed point. ∎

Theorem 1.2. *Let f be a continuously differentiable function with a fixed point $x = p$. If $|f'(p)| > 1$, then the fixed point $x = p$ is repelling.*

Proof. Assume that $|f'(p)| > 1$. There exists a real number k such that

$$|f'(p)| > k > 1.$$

Let ϵ be the distance between k and $|f'(p)|$, so $\epsilon > 0$. By the continuity of f' and of the absolute value function, we can find $\delta > 0$ such that for all $x \in (p - \delta, p + \delta)$,

$$|\ |f'(p)| - |f'(x)|\ | < \epsilon$$

and thus $|f'(x)| > k$. Choose $x \in (p - \delta, p + \delta)$, $x \neq p$. By the Mean Value Theorem there exists c between p and x such that

$$\frac{|f(x) - f(p)|}{|x - p|} = |f'(c)|.$$

Because $c \in (p - \delta, p + \delta)$

$$|f'(c)| > k > 1.$$

Because p is a fixed point, we can put these inequalities together:

$$\frac{|f(x) - p|}{|x - p|} = \frac{|f(x) - f(p)|}{|x - p|} = |f'(c)| > k > 1.$$

Therefore

$$|f(x) - p| > k|x - p|.$$

Repeating this argument, we can show that

$$|f^2(x) - p| > k|f(x) - p|.$$

With the preceding inequality, we conclude that

$$|f^2(x) - p| > k^2|x - p|.$$

This argument can be repeated as long as $f^{n-1}(x)$ is in $(p - \delta, p + \delta)$ to give the inequality

$$|f^n(x) - p| > k^n|x - p|.$$

Since $k > 1$, there is an n such that $k^n|x - p| > \delta$ and $f^n(x)$ is not in the interval $(p - \delta, p + \delta)$. Thus, p is a repelling fixed point. ∎

If, however, $|f'(p)| = 1$, then the fixed point is not necessarily attracting or repelling. The following section will explore one such case of this behavior.

2. A Revealing Example

The following two theorems show that the positive fixed points of the function $f(x) = x \sin x$ defined on $(-\infty, \infty)$ are, so to speak, attracting on the right and repelling on the left.

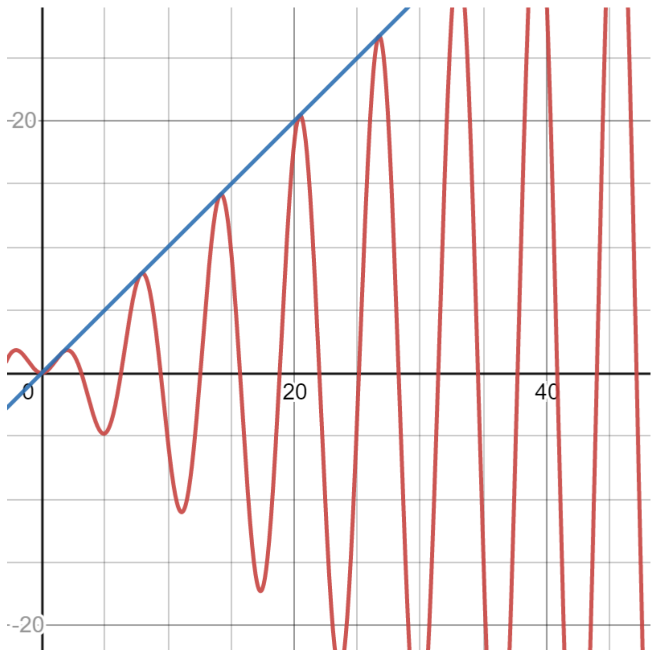

FIGURE 1. The graph of $f(x) = x \sin x$.

Theorem 2.1. *For each positive fixed point p of $f(x) = x \sin x$, there exists $\epsilon > 0$ such that*

$$\lim_{n\to\infty} f^n(x) = p$$

whenever $p \leq x < p + \epsilon$.

Proof. The fixed points of f are the solutions of the equation $x \sin x = x$, or $x(\sin x - 1) = 0$. Thus, the fixed points consist of $x = 0$ together with $x = \frac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$. The positive fixed points, then, are given by

$$\frac{\pi}{2} + 2\pi k, \quad k = 0, 1, 2, \ldots$$

Now we choose a fixed value of $k = 0, 1, 2, \ldots$ and let $p = \frac{\pi}{2} + 2\pi k$ be the corresponding positive fixed point. Our immediate goal is to show that there is an interval to the right of p on which $0 < f'(x) < 1$.

First, the calculations

$$f'(x) = x \cos x + \sin x$$

and

$$f''(x) = -x \sin x + 2 \cos x$$

yield

$$f'(p) = 1 \quad \text{and} \quad f''(p) = -\left(\frac{\pi}{2} + 2\pi k\right) < 0,$$

respectively. Because f'' is continuous, there exists $\epsilon_1 > 0$, ϵ_1 sufficiently small, such that $f''(x) < 0$ if $x \in (p, p + \epsilon_1)$. Therefore f' is a strictly decreasing function on this interval, and so

$$f'(x) < f'(p) = 1, \quad x \in (p, p + \epsilon_1). \tag{2.1}$$

In the same way, because f' is continuous and $f'(p) = 1$ we can find $\epsilon_2 > 0$, ϵ_2 sufficiently small, such that

$$f'(x) > 0, \quad p \in (p, p + \epsilon_2). \tag{2.2}$$

If we define

$$\epsilon = \min\{\epsilon_1, \epsilon_2, 2\pi\} > 0, \tag{2.3}$$

then (2.1) and (2.2) together imply that

$$0 < f'(x) < 1, \quad x \in I, \tag{2.4}$$

where I denotes the interval $(p, p + \epsilon)$. (The 2π appearing in (2.3) is added as a convenience to ensure that p is the only fixed point of f in the closure of I.)

Now choose $x_0 \in I$. It will suffice to show that

$$\lim_{n\to\infty} f^n(x_0) = p.$$

By the Mean Value Theorem,

$$f(x_0) - f(p) = f'(c_0)(x_0 - p)$$

for some c_0 strictly between p and x_0. Inequalities (2.4) imply that $f'(c_0)$ lies strictly between 0 and 1, and so

$$0 < f(x_0) - f(p) < x_0 - p.$$

Because p is a fixed point of f, we can add p throughout to conclude that

$$p < f(x_0) < x_0. \tag{2.5}$$

Thus, if we define $x_1 = f(x_0)$, then (2.5) shows that $x_1 \in I$, and so the above argument applied to x_1 implies that

$$p < f(x_1) < x_1 < x_0,$$

and then likewise $x_2 = f(x_1)$ satisfies

$$p < f(x_2) < x_2 < x_1 < x_0.$$

Continuing in this way, we find that the sequence $\{f^n(x_0)\}$ is strictly decreasing and bounded below by p, and thus converges to a limit $x^\star \in [p, p + \epsilon)$ by the Monotone Convergence Theorem [2, p.180]. It remains only to show that $x^\star$ is in fact p. However, by the continuity of f and the definition of $x^\star$,

$$f(x^\star) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x^\star.$$

Thus $x^\star$ is a fixed point of f. Because p is the only fixed point of f in the closure of I, $x^\star$ must equal p. ∎

Theorem 2.2. *For each positive fixed point p of $f(x) = x \sin x$, there exists $\epsilon > 0$ such that if $x \in (p - \epsilon, p)$, then there exists $k \in \mathbb{N}$ with $f^k(x) \notin (p - \epsilon, p)$.*

Proof. We have already noted that $f'(p) = 1$ and that f'' is continuous with $f''(p) < 0$. So once again by the properties of continuous functions there exists $\epsilon > 0$ such that $f''(x) < 0$ if $x \in I$, where I denotes the interval $(p - \epsilon, p)$, meaning that f' is strictly decreasing on this interval. We further assume that $\epsilon < \frac{\pi}{2}$, which implies that p is the only fixed point of f in the closure of I. Thus $f'(x) > f'(p) = 1$ for all $x \in I$. Now choose $x_0 \in I$. By the Mean Value Theorem,

$$f(p) - f(x_0) = f'(c_0)(p - x_0) \tag{2.6}$$

for some c_0 strictly between x_0 and p. Because $f'(c_0) > 1$, and because p is a fixed point of f, (2.6) implies that

$$p - f(x_0) > p - x_0,$$

and so $f(x_0) < x_0$. Define $x_1 = f(x_0)$. If $x_1 \notin I$, then we are done. Else we can repeat the preceding argument to find that $x_2 = f(x_1) < x_1$, etc. If all of the x_n, $n = 0, 1, 2, \ldots$ generated in this way are in I, then the sequence $\{x_n\}$ is decreasing and bounded below (by $p - \epsilon$), and thus approaches a limit $x^\star$, where $p - \epsilon \leq x^\star < p$, by the Monotone Convergence Theorem. Once again the continuity of f and the definition of $x^\star$ give

$$f(x^\star) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x^\star,$$

so that $x^\star$ is a fixed point of f. However, because p is the only fixed point of f in the closure of I, $x^\star$ must in fact equal p itself, a contradiction. We conclude that not all of the x_n are in I, proving the theorem. ∎

3. A Generalization

Here we will notice that actually, differentiability is not needed to arrive at the conclusion of Theorem 2.1. As long as f is continuous and satisfies $f(x) \leq x$ for all x, with some further assumptions we can show that a fixed point p is "attracting on the right."

Theorem 3.1. *Let f be a continuous function on $[0, \infty)$ such that $f(x) < x$ except at $x = p$, where $f(p) = p$. Suppose there is some interval (p, q), where $q > p$, such that $f(x) \geq p$ for $p \leq x < q$. Then $\lim_{n\to\infty} f^{(n)}(x) = p$ for each $x \in (p, q)$.*

Proof. Let x_0 be any number in the interval (p, q). Our assumption implies that

$$p \leq f(x_0) < x_0.$$

So either $f(x_0) = p$, proving the theorem, or $f(x_0)$ is in the interval (p, q). So, we will assume $f(x_0)$ is in (p, q). Let $x_1 = f(x_0)$. We see that

$$p \leq f(x_1) < x_1.$$

Once again, we see that either $f(x_1) = p$, proving the theorem, or $f(x_1) \in (p, q)$. We will assume the latter, and let $x_2 = f(x_1) < x_1$. This process generates a sequence, $\{x_n\}$, where $x_n = f^n(x_0)$. Because $\{x_n\}$ is a decreasing sequence bounded below by p, the Monotone Convergence Theorem implies that this sequence has a limit $x^* \geq p$. Because $f(x)$ is continuous,

$$f(x^*) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x^*$$

as before. So, x^* is a fixed point of f. Because $x = p$ is the only fixed point of f, $x^* = p$. So, for any $x_0 \in (p, q)$, $\lim_{n\to\infty} f^n(x_0) = p$. Therefore, (p, q) lies in the basin of attraction of $x = p$. ∎

4. Is Continuity Required?

Here we ask to what extent the above results depend on continuity. The following example suggests that the assumption of continuity cannot be wholly abandoned. For $x \geq 0$ define

$$f(x) = \begin{cases} 1 + \dfrac{1}{n+1} & \text{for} \quad x = 1 + \dfrac{1}{n}, \quad n = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Thus f has a unique fixed point at the origin and satisfies $0 \leq f(x) < x$ for $x > 0$. However, the sequence $f(2) = \frac{3}{2}$, $f\left(\frac{3}{2}\right) = \frac{4}{3}$, $\ldots$ converges to 1, which means that these points are not in the basin of attraction of the fixed point.

However, the condition of continuity across the interval does not seem to be entirely necessary, so long as we define it to be continuous from the right.

Definition 4.1. A function f defined on an interval is said to be *continuous from the right* if

$$\lim_{x \to a^+} f(x) = f(a)$$

for all real numbers a in the interval.

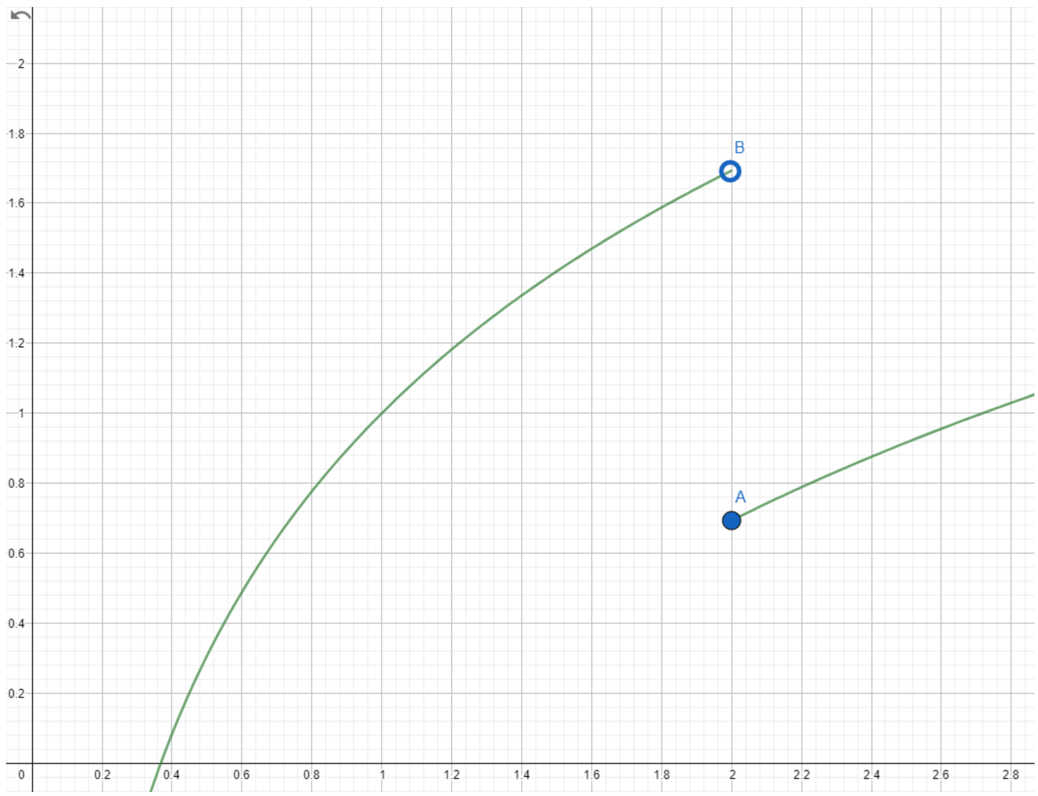

Figure 2. Continuity from the right

We will use this assumption to extend Theorem 3.1.

Theorem 4.1. *Let f be a function such that $f(x) < x$ except for a point $x = p$ at which $f(p) = p$. Suppose there is some interval (p, q), where $q > p$, on which $f(x) \geq p$. Further assume that on this interval, f is continuous from the right. Then (p, q) lies in the basin of attraction for the fixed point $x = p$.*

Proof. Let x_0 be any number in the interval (p, q). Because it is in this interval,

$$p \leq f(x_0) < x_0.$$

So either $f(x_0) = p$, proving the theorem, or $f(x_0)$ is in the interval (p, q). So, we will assume $f(x_0)$ is in (p, q). Let $x_1 = f(x_0)$. We see that

$$p \leq f(x_1) < x_1.$$

Once again, we see that either $f(x_1) = p$, proving the theorem, or $f(x_1) \in (p, q)$. We will assume the latter, and let $x_2 = f(x_1)$. This process generates a sequence $\{x_n\}$, where $x_n = f^n(x_0)$. Because $\{x_n\}$ is a decreasing sequence bounded below by p, the Monotone Convergence Theorem implies that this sequence has a limit $x^* \geq p$. Now $f(x)$ is continuous from the right in this interval and our sequence $\{x_n\}$ is always decreasing; therefore,

$$f(x^*) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x^*.$$

So, $x = x^*$ is a fixed point of f. Because $x = p$ is the only fixed point of f, $x^* = p$. So, for any $x_0 \in (p, q)$, $\lim_{n\to\infty} f^n(x_0) = p$. Therefore, (p, q) lies in the basin of attraction of $x = p$. ∎

5. Further Questions

While it is possible to remove the requirement for full continuity within the basin of attraction by replacing it with the requirement of continuity on the right, there may be other conditions which can further generalize this theorem. This paper also focuses on functions bounded above by the line $y = x$, so there would need to be further theorems describing behavior of functions bounded below by $y = x$, or bounded by other lines or functions entirely. Finally, this paper only proves the existence of basins of attractions for such functions, but further writing could be done on the sizes and behaviors of these basins, especially in functions like $f(x) = x \sin x$ or other similar functions with infinitely many fixed points.

REFERENCES

[1] A.S.A. Johnson, K.M. Madden and A.A. Şahin. *Discovering Discrete Dynamical Systems*, Mathematical Association of America, Washington, DC 2017.

[2] S.R. Lay. *Analysis*, 5th ed., Pearson 2014.

Department of Mathematics, Baldwin Wallace University, Berea, OH 44017, USA
E-mail address: gfryling18@bw.edu

Department of Mathematics, Baldwin Wallace University, Berea, OH 44017, USA
E-mail address: hrouse18@bw.edu

Received December 20, 2021; revised February 3, 2022.