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On Cardinality Bounds Involving the Weak Lindelöf Degree and H-Closed Spaces

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(d) *If X is extremally disconnected then $|X| \leq 2^{wL(X)\pi_\chi(X)\psi(X)}$.*

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(a) $|X| \leq 2^{L(X)\chi(X)}$ (Arhangel'skiĭ, 1969), and
(b) $|X| \leq 2^{\chi(X)}$ if X is H-closed (Dow, Porter 1982).

The weak Lindelöf degree

Definition

For a space X we define the weak Lindelöf degree of X, $wL(X)$, is the least infinite cardinal κ such that for every open cover $\mathcal{U}$ of X there exists $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $X \subseteq cl \bigcup \mathcal{V}$.

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Proposition

For any space X, $wL(X)$ is hereditary on regular closed sets.

- It is clear that $wL(X) \leq L(X)$.

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- Thus $wL(X)$ regarded as a fairly “small” cardinal invariant.
- In fact, there are even normal spaces for which $wL(X) < L(X)$, given in Bell, Ginsburg, Woods, 1978.
- However, if X is paracompact then $wL(X) = L(X)$.
- Also, $wL(X)$ is “small” enough so that $wL(X) \leq c(X)$, where $c(X)$ is the cellularity of X.

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Theorem (Bell, Ginsburg, Woods, 1978)

If X is normal then $|X| \leq 2^{wL(X)\chi(X)}$.

Theorem (BGW)

For every uncountable cardinal κ there exists a weakly Lindelöf, first-countable, Hausdorff space X such that $|X| = \kappa$.

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- *If X is Tychonoff, is $|X| \leq 2^{wL(X)\chi(X)}$?*
- *If X is zero-dimensional, is $|X| \leq 2^{wL(X)\chi(X)}$?*

Main Theorem (Bella, C., 2017)

Let X be a space and κ a cardinal such that $wL(X)t(X) \leq \kappa$. Suppose X has an open π-base $\mathcal{B}$ such that $|B| \leq 2^\kappa$ for all $B \in \mathcal{B}$. Let $\mathcal{C}$ be a cover of X consisting of compact subsets C of X such that $\chi(C, X) \leq \kappa$. Then there exists a subcollection $\mathcal{C}' \subseteq \mathcal{C}$ such that $X = cl_\theta\left(\bigcup \mathcal{C}'\right)$ and $|\mathcal{C}'| \leq 2^\kappa$.

Recall:

$$cl_\theta(A) = \{x \in X : clU \cap A \neq \varnothing \text{ for all } x \in U \in \tau(X)\}$$

The Main Theorem generalizes the following:

Theorem (C., 2013)

Let X be a space and κ a cardinal such that $wL(X)t(X) \leq \kappa$. Suppose X has a dense set of isolated points. Let $\mathcal{C}$ be a cover of X consisting of compact subsets C of X such that $\chi(C,X) \leq \kappa$. Then there exists a subcollection $\mathcal{C}' \subseteq \mathcal{C}$ such that $X = cl_\theta\left(\bigcup \mathcal{C}'\right)$ and $|\mathcal{C}'| \leq 2^\kappa$.

Observations about the proof of the Main Theorem:

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- An increasing chain $\{A_\alpha : \alpha < \kappa^+\}$ of *open* sets is inductively constructed by adding on members of the π-base $\mathcal{B}$.
- Each A_α can be covered by a union of a “small” subfamily of $\mathcal{C}$.

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- The proof is a “closing-off argument”
- An increasing chain $\{A_\alpha : \alpha < \kappa^+\}$ of *open* sets is inductively constructed by adding on members of the π-base $\mathcal{B}$.
- Each A_α can be covered by a union of a “small” subfamily of $\mathcal{C}$.
- In the case where X has a dense set of isolated points D, at each stage of the induction we can find $d \in D$ such that $d \in X \backslash cl \bigcup \mathcal{V}$ for certain families $\mathcal{V}$ of neighborhood base elements.

- In the general case where X has an open π-base $\mathcal{B}$ such that $|B| \leq 2^\kappa$ for all $B \in \mathcal{B}$, we can find $B \in \mathcal{B}$ such that $B \subseteq X \backslash cl \bigcup \mathcal{V}$ for every such family $\mathcal{V}$.

- In the general case where X has an open π-base $\mathcal{B}$ such that $|B| \leq 2^\kappa$ for all $B \in \mathcal{B}$, we can find $B \in \mathcal{B}$ such that $B \subseteq X \backslash cl \bigcup \mathcal{V}$ for every such family $\mathcal{V}$.
- The set $A = \bigcup_{\alpha<\kappa^+} cl(A_\alpha)$ turns out to be *regular-closed*, and thus we can use that $wL(X)$ is hereditary on A.

Corollary

Let X be a regular space. Suppose there exists a dense subset $D \subseteq X$ such that each $d \in D$ has a closed neighborhood that is H-closed, normal, Lindelöf, or c.c.c. Then $|X| \leq 2^{wL(X)\chi(X)}$.

Corollary

Let X be a regular space. Suppose there exists a dense subset $D \subseteq X$ such that each $d \in D$ has a closed neighborhood that is H-closed, normal, Lindelöf, or c.c.c. Then $|X| \leq 2^{wL(X)\chi(X)}$.

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Corollary (Dow, Porter)

If X has a dense set of isolated points then $|X| \leq 2^{wL(X)\chi(X)}$.

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Proof.

- We have $\psi(X) = \chi(X)$ and let $\kappa = wL(X)\psi(X)$.
- There exist an open cover $\mathcal{U}$ of X such that clU is compact for all $U \in \mathcal{U}$. Thus for all $U \in \mathcal{U}$,

$$|U| \leq |clU| \leq 2^{\psi(X)} \leq 2^{\kappa}.$$

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- As $wL(X) \leq \kappa$, there is a subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $|\mathcal{V}| \leq \kappa$ and $\bigcup \mathcal{V}$ is dense in X.

$\square$

Proof.

- As $wL(X) \leq \kappa$, there is a subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $|\mathcal{V}| \leq \kappa$ and $\bigcup \mathcal{V}$ is dense in X.
- As $\bigcup \mathcal{V}$ has cardinality $\leq 2^\kappa$, we have

$$|X| \leq d(X)^{\chi(X)} \leq \left|\bigcup \mathcal{V}\right|^\kappa \leq (2^\kappa)^\kappa \leq 2^\kappa.$$

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Corollary

If X is locally compact then $|X| \leq 2^{wL(X)\psi(X)}$.

Recall that a space is extremally disconnected if clU is open for every open set U.

Proposition

Let X be extremally disconnected. Then

$$c(X) \leq \min\{wL(X)t(X), wL(X)\pi_\chi(X)\}.$$

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Theorem (Shu-Hao, 1988)

If X is Hausdorff then $|X| \leq \pi_\chi(X)^{c(X)\psi_c(X)}$.

Theorem (Bella, C. 2017)

If X is extremally disconnected then $|X| \leq 2^{wL(X)\pi_\chi(X)\psi(X)}$.

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Theorem (De la Vega, 2006 homogeneous, Arhangel'skiĭ, van Mill, Ridderbos, 2007)

If X is compact and power homogeneous then $|X| \leq 2^{t(X)}$.

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- Observe that this generalizes De la Vega's Theorem for compact spaces into the locally compact setting.

Theorem (C., Ridderbos, 2008)

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Theorem (Bella, C., 2017)

If X is extremally disconnected and power homogeneous then $|X| \leq 2^{wL(X)\pi_\chi(X)}$.

H-closed spaces

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A space X is H-closed if for every open cover $\mathcal{V}$ of X there exists $\mathcal{W} \in [\mathcal{V}]^{<\omega}$ such that $X = \bigcup_{W \in \mathcal{W}} clW$.

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A space is H-closed if and only if it is closed in any Hausdorff space in which it is embedded.

In 1982, Dow and Porter proved the following:

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If X is H-closed then $|X| \leq 2^{\chi(X)}$ (in fact, $|X| \leq 2^{\psi_c(X)}$).

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- Porter gave a simplified approach to this theorem at the top in 1993 by embedding X as the remainder in an H-closed extension of a discrete space.
- The approach depended heavily on finiteness and is not known to extend to a general Hausdorff setting.

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- Again, this approach seems not to generalize to a general Hausdorff cardinality bound.

Question (Bella)

Does there exist a cardinality bound for a Hausdorff spaces that generalizes Arhangel'skiĭ's Theorem and the Dow-Porter result?

We can reframe this question:

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Question

Does there exists a property $\mathcal{P}$ of a Hausdorff space that generalizes both the Lindelöf and H-closed properties such that $|X| \leq 2^{\chi(X)}$ for a space X with property $\mathcal{P}$?

The set $\widehat{U}$ and the invariant $\widehat{L}(X)$

- For a space X, fix an open ultrafilter assignment $f : X \to EX$, where

$$EX = \{\mathcal{U} : \mathcal{U} \text{ is a convergent open ultrafilter on } X\}.$$

The set $\widehat{U}$ and the invariant $\widehat{L}(X)$

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Definition

For a non-empty open set $U \subseteq X$, define

$$\widehat{U} = \{x \in X : U \in \mathcal{U}_x\}.$$

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(b) $\widehat{U \cap V} = \widehat{U} \cap \widehat{V}$ and $\widehat{U \cup V} = \widehat{U} \cup \widehat{V}$,

(c) $X \backslash \widehat{U} = \widehat{X \backslash clU}$.

Theorem

A space X is H-closed if and only if for every open cover $\mathcal{V}$ of X there exists $\mathcal{W} \in [\mathcal{V}]^{<\omega}$ such that $X = \bigcup_{W \in \mathcal{W}} \widehat{W}$.

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- This is a formally stronger characterization of H-closed than the standard definition.
- The proof relies on the interaction between finiteness in the definition of H-closed and the f.i.p. property of a filter.

Definition

For a space X, define the cardinal invariant $\widehat{L}(X)$ as the least infinite cardinal κ such that for every open cover $\mathcal{V}$ of X there exists $\mathcal{W} \in [\mathcal{V}]^{\leq\kappa}$ such that $X = \bigcup_{W \in \mathcal{W}} \widehat{W}$.

By the previous Theorem, we see that the property "$\widehat{L}(X) = \aleph_0$" generalizes both H-closed and Lindelöf.

The operator $\widehat{c}$

Definition

For a space X and $A \subseteq X$, define

$$\widehat{c}(A) = \{x \in X : \widehat{U} \cap A \neq \varnothing \text{ for all } x \in U \in \tau(X)\}.$$

A is $\widehat{c}$-closed if $A = c(A)$.

Compare with:

$$cl(A) = \{x \in X : U \cap A \neq \varnothing \text{ for all } x \in U \in \tau(X)\}$$

$$cl_\theta(A) = \{x \in X : clU \cap A \neq \varnothing \text{ for all } x \in U \in \tau(X)\},$$

and recall $U \subseteq \widehat{U} \subseteq clU$.

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(d) *if* U *is open, then* $clU = c(U) \subseteq \widehat{c}(\widehat{U})$.

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(d) if U is open, then $clU = c(U) \subseteq \widehat{c}(\widehat{U})$.

(e) if X is regular then $clA = \widehat{c}(A) = cl_\theta(A)$.

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(h) *If* X *is H-closed then* $c(A)$ *is an H-set.*

Proposition

If X is a space and C is a $\widehat{c}$-closed subset of X, then $\widehat{L}(C, X) \leq \widehat{L}(X)$.

I.e., the invariant $\widehat{L}(X)$ is hereditary on $\widehat{c}$-closed subsets of X.

Proposition

For any Hausdorff space X and for all $x \neq y \in X$ there exist open sets U and V such that $x \in U$, $y \in V$, and $\widehat{U} \cap \widehat{V} = \varnothing$.

The above is formally stronger than the usual definition of Hausdorff.

Proposition

If X is Hausdorff and $A \subseteq X$, then

$$|\widehat{c}(A)| \leq |A|^{\chi(X)}.$$

As $clA \subseteq \widehat{c}(A)$, this improves the well-known result

$$|clA| \leq |A|^{\chi(X)}.$$

Using the previous results, the operator $\widehat{c}$, and a standard closing off-argument, we obtain:

Main Theorem (C., Porter, 2016)

If X is Hausdorff then $|X| \leq 2^{\widehat{L}(X)\chi(X)}$.

As $\widehat{L}(X) = \aleph_0$ for an H-closed space X, it follows that:

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Corollary (Dow, Porter 1982)

If X is H-closed then $|X| \leq 2^{\chi(X)}$ (in fact, $|X| \leq 2^{\psi_c(X)}$).

We can now identify a property $\mathcal{P}$ of a Hausdorff space X that generalizes both the H-closed and Lindelöf properties such that $|X| \leq 2^{\chi(X)}$ for spaces with property $\mathcal{P}$:

$\mathcal{P} =$ for every open cover $\mathcal{V}$ of X there is $\mathcal{W} \in [\mathcal{V}]^{\leq\omega}$ such that

$$X = \bigcup_{W \in \mathcal{W}} \widehat{W}$$

Question

Is $\widehat{L}(X)$ independent of the choice of open ultrafilter assignment?

Thank you!

A. Bella, N. A. Carlson, *On cardinality bounds involving the weak Lindelöf degree*, to appear in Quaestiones Mathematicae.

A. Bella, N. A. Carlson, *On cardinality bounds involving the weak Lindelöf degree*, to appear in Quaestiones Mathematicae.

N. A. Carlson, J.R. Porter *On the cardinality of Hausdorff spaces and H-closed spaces*, Topology Appl. (2017), DOI.