Classifying Matchbox Manifolds

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Classifying matchbox manifolds

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Plan of the talk

1. What is a matchbox manifold and why we want to study them? What are the examples?

2. Asymptotic discriminant as an invariant, classifying equicontinuous matchbox manifolds (joint work with Steven Hurder, UIC).


3. Matchbox manifolds arising from the actions of profinite Galois groups (joint work with Nicole Looper, Northwestern University).

What is a matchbox manifold?

Matchbox manifolds

A compact connected metrizable space $\mathcal{M}$ is an $n$-dimensional *matchbox manifold* if it admits a *foliated* atlas $\{U_i\}_{1 \leq i \leq \nu}$, where $U_i$ is an open set equipped with a homeomorphism

$$\varphi_i : \overline{U}_i \to [-1, 1]^n \times Z_i,$$

$Z_i$ is totally disconnected.

The term a *matchbox manifold* is due to Aarts and Martens 1988, who studied 1-dimensional matchbox manifolds (for $n = 1$).

A matchbox manifold is a *lamination* in the sense of Lyubich and Minsky.
Example in dimension 1: the Vietoris solenoid

Let \( f^i_{i-1} : S^1 \to S^1 \) be a \( p_i \)-to-1 covering map.

A Vietoris solenoid is the inverse limit space

\[
\Sigma = \{(x_i) = (x_0, x_1, x_2, \ldots) \mid f^i_{i-1}(x_i) = x_{i-1} \} \subset \prod_{i \geq 0} S^1
\]

with subspace topology from the Tychonoff topology on \( \prod_{i \geq 0} S^1 \).

Let \( f_i : \Sigma \to S^1 \), then \( F_0 = f_0^{-1}(x_0) \) is a Cantor set, and \( \Sigma \) is a matchbox manifold.

**Remark:** If \( p_i = 2 \) for \( i \geq 1 \), then the *dyadic* solenoid \( \Sigma \) provides a topological model for the Smale attractor \( \Lambda \).

Thus, a primary motivation to study matchbox manifolds comes from the fact that they arise as attractors of dynamical systems and exceptional minimal sets of foliations.
Motivating questions

There are many examples of matchbox manifolds arising from topological and algebraic constructions (like the Vietoris solenoid).

Far fewer examples arising as subsets of smooth foliated manifolds or as attractors in dynamical systems (like the Smale attractor), especially if \( n \geq 2 \).

Embedding problem

Let \( \mathcal{M} \) be a matchbox manifold with leaf dimension \( n \geq 2 \). Find necessary and sufficient conditions on \( \mathcal{M} \), so that there exists a bi-Lipschitz homeomorphism

\[ \Phi : \mathcal{M} \to M \]

onto a subset of a smooth (real-analytic) manifold \( M \) with a smooth (real-analytic) foliation \( \mathcal{F} \) which maps path-connected components of \( \mathcal{M} \) to the leaves of \( \mathcal{F} \).
Our program

1. Find algebraic, dynamical or topological invariants which classify matchbox manifolds up to a homeomorphism.

2. Construct non-trivial examples.

3. Determine if these invariants can serve as obstructions to the existence of a bi-Lipschitz embedding $\Phi : M \to M$. 
Some references

The inverse limit of coverings of a closed $n$-dimensional manifold

$$X_\infty = \lim_{\leftarrow} \{ f_i^{i-1} : X_i \to X_{i-1} \}$$

is called a weak solenoid.

Weak solenoids were introduced and studied by
- McCord 1965
- Schori 1966,

Fokkink and Oversteegen 2002 used group chains to study the homogeneity of weak solenoids.

In the rest of the talk, we restrict to the class of matchbox manifolds which are homeomorphic to weak solenoids.
Group chains in finitely presented groups

Let $G_0$ be a finitely presented group, and consider a chain of proper subgroups of $G_0$ of finite index

$$\{G_i\}_{i \geq 0} : G_0 \supset G_1 \supset G_2 \supset \cdots$$

Each $G_0/G_i$ is a finite set with the left action of $G_0$. The inverse limit

$$G_\infty = \lim_{\leftarrow} \{G_0/G_i \rightarrow G_0/G_{i-1}\} = \{(g_i G_i) = (G_0, g_1 G_1, \ldots)\} \subset \prod_i G_0/G_i$$

is a Cantor set with minimal component-wise action of $G_0$,

$$h \cdot (G_0, g_1 G_1, g_2 G_2, \ldots) = (G_0, hg_1 G_1, hg_2 G_2, \ldots)$$

**Example:** If $G_0$ is abelian, then $G_i$ is a normal subgroup of $G_0$, $G_0/G_i$ is a finite group and $G_\infty$ is a profinite group.

If $G_0$ is not abelian and $G_i$ is not normal in $G_0$, then $G_0/G_i$ is just a set and $G_\infty$ is a Cantor set.
Suspension of the group action \((G_\infty, G_0)\)

Since \(G_0\) is a finitely presented group, there exists a compact connected 4-manifold \(X_0\) and \(x_0 \in X_0\) such that \(\pi_1(X_0, x_0) \cong G_0\).

Let \(\hat{X}\) be the universal cover of \(X\), then \(\text{Aut}(\hat{X}) = G_0\).

For each \(i \geq 1\), divide by the action of \(G_i\) to obtain a covering map

\[
f_0^i : X_i = \hat{X}/G_i \to X_0,
\]

\(x_i \in X_i\) is so that \(\pi_1(X_i, x_i) \cong G_i\). Since \(G_i \subset G_{i-1}\), then there are covering maps \(f_{i-1}^i : X_i \to X_{i-1}\), and a matchbox manifold

\[
X_\infty = \lim\left\{ f_{i-1}^i : X_i \to X_{i-1} \right\}
\]

The Cantor set \(G_\infty\) embeds as a section \(F_0 = f_0^{-1}(x_0)\) transverse to the path-connected components in \(X_\infty\). The group \(G_0\) acts on \(F_0\) by \(\text{holonomy maps}\).
Choices in the construction

A choice of a smaller section $F_n \subset F_0$ corresponds to truncating the group chain $\{G_i\}_{i \geq 0}$ to $\{G_i\}_{i \geq n}$, $n \geq 0$.

A choice of a different point $y \in F_n$ corresponds to passing to a chain of conjugate subgroups $\{g_i G_i g_i^{-1}\}_{i \geq n}$.

Theorem (Fokkink and Oversteegen 2002)

Let $\{G_i\}_{i \geq 0}$ and $\{H_i\}_{i \geq 0}$ be group chains, $G_0 = H_0$, with matchbox manifolds $M$ and $M'$ respectively. Then $M$ and $M'$ are homeomorphic if and only if $\{H_i\}_{i \geq 0}$ is equivalent to one of the group chains

$$\{g_i G_i g_i^{-1}\}_{i \geq n} \text{ or } \{G_i\}_{i \geq n}, \text{ where } n \geq 0.$$
Invariants independent of choices

Given a matchbox manifold $\mathcal{M}$, we want to find invariants which are independent of the choice of a section $F_n$ and a point $x \in F_n$.

That is, these invariants must only depend on the homeomorphism type of $\mathcal{M}$.

The asymptotic discriminant (*Hurder and Lukina 2017*) is an example of such an invariant.
Profinite group acting on $F_0$

In a group chain $\{G_i\}_{i \geq 0}$, let $C_i = \bigcap_{g \in G_0} gG_i g^{-1}$, then $C_i$ is a finite index normal subgroup of $G_0$,

then $C_\infty = \lim\left\{ G_0/C_i \rightarrow G_0/C_{i-1} \right\}$ is a profinite group.

$C_\infty$ acts transitively on $G_\infty \cong F_0$. The isotropy group at $x$ is

$$D_x = \lim\left\{ G_i/C_i \rightarrow G_{i-1}/C_{i-1} \right\}.$$ 

Since $C_\infty$ acts transitively on $F_0$, for any $x, y \in F_0$

$$D_y = (g_i) D_x (g_i)^{-1}.$$ 

Thus we have that:

- The group $C_\infty$ is independent of $x \in F_0$.
- The group $D_x$ is independent of the choice of $x \in F_0$, up to an isomorphism.
Ellis group of the action on $F_0$

**Theorem (Dyer, Hurder and Lukina 2016)**

The profinite group $C_\infty$ is isomorphic to the *Ellis group* of the action $(F_0, G_0)$, and $D_x$ is isomorphic to the isotropy group of the Ellis group action on $F_0$.

**Remark:** The Ellis group of an *equicontinuous* group action was defined as the closure of $G_0$ in $Homeo(F_0, F_0)$ in the uniform topology, and so it is very difficult to compute. Our result provides an easier method of computing the Ellis group.

We call $D_x$ the *discriminant group* of the action.

Since $D_x$ is a profinite group, it is either finite, or a Cantor group.
Examples

Example (trivial discriminant group):

Consider the Vietoris solenoid

$$\Sigma = \{ f_{i-1}^i : S^1 \to S^1 \}.$$  

Then $G_0 = \pi_1(\mathbb{Z}, 0) = \mathbb{Z}$, and $\{ G_i = \mathbb{Z}/p_1 \cdots p_i \mathbb{Z} \}$, where $p_i$ is the degree of $f_{i-1}^i$.

Since $\mathbb{Z}$ is abelian, $G_i = C_i$, and so $G_i/C_i$ is a trivial group.

Thus $C_\infty \cong F_0$, where $F_0$ is a fibre of $\Sigma$, and the discriminant group $\mathcal{D}_0$ of the Vietoris solenoid is trivial.
Example in $BS(p, 1)$ (Looper and Lukina 2017)

Let $p > 2$ be a prime, and consider the Baumslag-Solitar group

$$BS(p, 1) = \langle \tau, \sigma \mid \sigma \tau \sigma^{-1} = \tau^p \rangle.$$ 

Consider the subgroups $G_i = \langle \tau^{d^i}, \sigma \rangle$, where $d > p$, $d$ is a prime.

Lemma (Dudkin 2010)

Let $\ell$ and $p$ be coprime. Every subgroup $H$ of $BS(p, 1)$ of index $n = \ell m$ is of the form

$$H = \langle \tau^{\ell}, \sigma^m \tau^s \rangle, \quad 0 \leq s \leq \ell - 1,$$

and $H$ is normal if and only if $\ell \mid p^m - 1$ and $\ell \mid s(1 - p)$.

Thus $G_i$ are not normal in $BS(p, 1)$, since $d^i$ do not divide $p - 1$. 


Example in $BS(p, 1)$

Solving the congruence $p^m = 1 \mod d^i$, obtain $m = \phi(d^i)$, where $\phi$ is the Euler $\phi$-function.

Then the maximal normal subgroup of $G_i = \langle \tau^{d^i}, \sigma \rangle$ is

$$C_i = \langle \tau^{d^i}, \sigma^{\phi(d^i)} \rangle.$$ 

We have $\phi(d^i) = d^i - d^{i-1}$, so $\phi(d^i) \to \infty$ as $i \to \infty$.

Since $|G_i : C_i| = \phi(d^i)$, the discriminant group $D_0$ is an infinite profinite group.

Proposition (Looper and Lukina 2017)

For each $d > p$, the group chain $\{G_i\}_{i \geq 0} = \langle \tau^{d^i}, \sigma \rangle$ in $BS(p, 1)$ defines an action of $BS(p, 1)$ on a Cantor set $G_{\infty}$. This action is effective but not free.
Discriminant group under the change of sections

The truncated group chain \( \{G_i\}_{i \geq n}, n \geq 0 \), defines an action on a clopen subset \( F_n \subset F_0 \).

Apply the construction of the discriminant group to the action \((F_n, G_n)\) to obtain the discriminant group \( D_{n,x} \).

**Remark**

The group \( D_{n,x} \) need not be isomorphic to \( D_x \), since the maximal normal subgroup \( E_{n,i} = \bigcap_{g \in G_n} gG_ig^{-1} \) need not be equal to \( C_i \).

Inclusions of clopen sets \( F_m \subset F_n \subset F_0 \) lead to surjective homomorphisms

\[
\Lambda_{\infty}^{n,m} : D_{n,x} \to D_{m,x}.
\]
Definition of stable and wild group actions

Stable group actions (Dyer, Hurder and Lukina 2017)

Let \((F_0, G_0)\) be a group action with group chain \(\{G_i\}_{i \geq 0}\). The action is \textit{stable} if there exists \(N \geq 0\) such that for all \(m \geq n \geq N\) the restriction

\[
\Lambda_{m,n}^\infty : D_{n,x} \rightarrow D_{m,x}
\]

is an isomorphism. Otherwise, the action is said to be \textit{wild}.

\textbf{Example:} The action on the fibre of a Vietoris solenoids is stable, since the discriminant group is trivial.

\textbf{Example:} If \(D_x\) is finite for some choice of \(F_0\) and \(\{G_i\}_{i \geq 0}\), then the action is stable.
Example in $BS(p, 1)$

In $BS(p, 1) = \langle \tau, \sigma \mid \sigma \tau \sigma^{-1} = \tau^p \rangle$, consider the groups $G_i = \langle \tau^{d_i}, \sigma \rangle$.

For $n \geq 0$, consider the truncated chain $\{G_i\}_{i \geq n}$.

**Proposition (Looper and Lukina 2017)**

For $i > n$, the maximal subgroup of $G_i$ normal in $G_n$ is given by

$$E_{n,i} = \langle \tau^{d_i}, \sigma^{\phi(d_i-n)} \rangle.$$

For $\{G_i\}_{i \geq n}$, the discriminant group $D_n$ is an infinite profinite group.

The maps $\Lambda_{m,n}^\infty : D_m \rightarrow D_n$ are isomorphisms for $m > n \geq 0$.

Thus the action in our example is stable.
Realization theorems for stable actions

Every finite group can be realized as a discriminant group of a stable equicontinuous action on a Cantor set.

**Theorem (Dyer, Hurder and Lukina 2017)**

Given a finite group $F$, there exists a group chain $\{G_i\}_{i \geq 0}$ in $SL(n, \mathbb{Z})$ for $n$ large enough, such that for any truncated chain $\{G_i\}_{i \geq n}$ the discriminant group $D_{n,x} \cong F$, and $D_{n,x} \to D_{n+1,x}$ is an isomorphism.

Every separable profinite group can be realized as a discriminant group of a stable equicontinuous action on a Cantor set.

**Theorem (Dyer, Hurder and Lukina 2017)**

Given a separable profinite group $K$, there exists a finitely generated group $G$ and a group chain $\{G_i\}_{i \geq 0} \subset G$, such that for any truncated chain $\{G_i\}_{i \geq n}$ the discriminant group $D_{n,x} \cong K$, and $D_{n,x} \to D_{n+1,x}$ is an isomorphism.
Stable and wild actions

To summarize,

1. The discriminant group of an action $(F_0, G_0)$ depends on the choice of a section $F_0$ and a point $x \in F_0$.

2. The action $(F_0, G_0)$ is stable if the discriminant groups for decreasing sections are eventually become isomorphic.

3. Every finite and every separable profinite group can be realized as the discriminant group of an action on a Cantor set $F_0$.

We now introduce an invariant which characterizes wild actions, that is, the actions where the chain of discriminant groups never stabilizes as we restrict to smaller sections.
Tail equivalence of discriminant groups

Let $\mathcal{A} = \{\phi_i : A_i \to A_{i+1}\}$ and $\mathcal{B} = \{\psi_i : B_i \to B_{i+1}\}$ be sequences of surjective group homomorphisms. Then $\mathcal{A}$ and $\mathcal{B}$ are tail equivalent if there are infinite subsequences $A_{i_n}$ and $B_{j_n}$ such that the sequence

$$A_{i_0} \to B_{j_0} \to A_{i_1} \to B_{j_1} \to \cdots$$

is a sequence of surjective group homomorphisms.

**Asymptotic discriminant (Hurder and Lukina 2017)**

Let $\{G_i\}_{i \geq 0}$ be a group chain with discriminant group $D_0$, and let $D_n$ be the discriminant group of the truncated chain $\{G_i\}_{i \geq n}$. The *asymptotic discriminant* of the group chain $\{G_i\}_{i \geq 0}$ is the tail equivalence class of the sequence of surjective group homomorphisms $\{D_n \to D_{n+1} \mid n \geq 0\}$. 
Wild actions

Theorem (Hurder and Lukina 2017)

Distinct tail equivalence classes of group chains correspond to non-homeomorphic matchbox manifolds. That is, the asymptotic discriminant of a matchbox manifold is an invariant of its homeomorphism class.

Our next theorem shows that the class of group actions on Cantor sets is extremely rich, as the same group can act on a Cantor set in an uncountable number of different ways.

Theorem (Hurder and Lukina 2017)

For $n \geq 3$, let $G \subset \text{SL}(n, \mathbb{Z})$ be a torsion-free subgroup of finite index. Then there exists uncountably many distinct homeomorphism types of weak solenoids which are wild, all with the same base manifold $M_0$ with fundamental group $G$. 
Strongly quasi-analytic actions

We next give a dynamical meaning to the *wild* condition.

*Quasi-analytic* actions of pseudogroups on locally connected spaces appear in the work of *Haefliger*.

Álvarez López and Candel 2009 introduced a more general concept of *strongly quasi-analytic* actions of pseudogroup which is applicable to the actions on totally disconnected spaces.

We adopt the notion of a *strongly quasi-analytic* action to the case of matchbox manifolds, where the holonomy action on the transverse section is given by a group action.
LCSQA actions for matchbox manifolds

Definition (SQA actions)

Let \((F_0, G_0)\) be a group action, and let \(U \subset F_0\) be a clopen subset. Then the restricted action \((U, G_0|U)\) is **strongly quasi-analytic (SQA)** if for every \(g \in G_0\) and every clopen subset \(V \subset U\), if \(g|V\) is the identity, then \(g|U\) is the identity.

Then we have the following concepts.

Definition (LCSQA actions)

A minimal action of a group \(G_0\) on a Cantor set \(F_0\) is **locally SQA** if there is a clopen set \(U \subset F_0\) such that the restricted action \(G_0|U\) is SQA. The action \((F_0, G_0)\) is **locally completely SQA (LCSQA)** if the action of the Ellis group of the restricted action \(G_0|U\) is SQA.
**Stability and the LCSQA condition**

**Theorem (Hurder and Lukina 2017)**

Let $\mathcal{M}$ be a matchbox manifold with a group chain $\{G_i\}_{i \geq 0}$ and the holonomy action $(F_0, G_0)$. Then the action $(F_0, G_0)$ is LCSQA if and only if the action is stable, that is, there exists $N > 0$ such that for every $m > n > N$ the map on the discriminant groups

$$\Lambda_{m,n} : \mathcal{D}_n \to \mathcal{D}_m$$

is an isomorphism.

**Consequence (Hurder and Lukina 2017)**

If a matchbox manifold $\mathcal{M}$ is wild, then it cannot be embedded by a bi-Lipschitz map as a subset of a foliated manifold $\mathcal{M}$ with a real-analytic foliation $\mathcal{F}$. 
Galois theory

Question

The study of profinite groups has been motivated by the study of Galois theory. What can our theory tell us about Galois groups?

Let \( f(x) \) be an irreducible polynomial of degree \( d \) over a field \( K \), and \( \alpha \) be a root of \( f(x) \).

Let \( K(\alpha) \rightarrow K \) be a Galois extension. It then contains \( d \) roots of \( f(x) \) which are all distinct.

A *Galois group* \( \text{Gal}(K(\alpha)/K) \) is a subgroup of automorphisms of the field \( K(\alpha) \) which fixes \( K \).

The Galois group \( \text{Gal}(K(\alpha)/K) \) permutes the roots of \( f(x) \).

Thus \( \text{Gal}(K(\alpha)/K) \subset S_d \), where \( S_d \) is the symmetric group on \( d \) elements.
Arboreal representations of Galois groups

For every root $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ consider an equation

$$f(x) = \alpha_i, \text{ so } f \circ f(x) = f(\alpha_i) = 0.$$  

The roots of $f^2(x)$ are all preimages of 0 under the 2nd iteration of $f$. It is convenient to picture the roots as vertices of a $d$-ary tree.

Suppose there are $d^2$ distinct roots, and $\beta$ is one of them.

One can show that the action of $\text{Gal}(K(\beta)/K)$ permutes paths in the tree, so there is a surjective homomorphism of finite groups

$$\text{Gal}(K(\beta)/K) \rightarrow \text{Gal}(K(\alpha)/K).$$

Thus $\text{Gal}(K(\beta)/K)$ acts on the $d$-ary finite tree $T_2$ by tree automorphisms.

$\text{Gal}(K(\beta)/K)$ is a subgroup of the wreath product $[S_d]_2$. 
Arboreal representations of Galois groups

Continue by induction, assuming that for each $i > 0$ the polynomial $f^i(x)$ has $d^i$ distinct roots.

In the limit, we get a $d$-ary infinite tree $T$ of preimages of 0 under the iterations of $f(x)$, and the profinite group

$$\text{Gal}_\infty(f) = \lim\left\{ \text{Gal}(K(\beta)/K) \to \text{Gal}(K(\alpha)/K) \right\} \subset \text{Aut}(T).$$

The set of infinite paths $X$ in a $d$-ary tree $T$ is a Cantor set, so we obtain the action of the profinite group $\text{Gal}_\infty(f)$ on a Cantor set $T$.

The study of the arboreal representations of Galois groups was started by Odoni 1985 in order to answer certain questions in number theory.

There has been much development in this area of arithmetic dynamics.
LCSQA property and wreath products

The Galois group of an arboreal representation is a subgroup of the infinite wreath product $[S_d]_{\infty}$.

The group $[S_d]_{\infty}$ contains elements which permute some subtrees of the infinite $d$-ary tree, but fix other subtrees. So the action of $[S_d]_{\infty}$ on the path space of the $d$-ary tree is not LCSQA.

The group $[S_d]_{\infty}$ is topologically infinitely generated, while our theory is for topologically finitely generated profinite groups.

Question

Suppose the Galois group $\text{Gal}_{\infty}(f)$ is topologically finitely generated.

1. Is it possible to apply the machinery of group chains to this action?
2. What is the relation between the structure of Cantor actions and profinite Galois theory?
Asymptotic discriminant for arboreal representations

Theorem (Looper and Lukina 2017)

Let \( f(x) \) be a polynomial of degree \( d > 1 \) over a field \( K \), and suppose all roots of \( f^n(x) \) are distinct for all \( n \geq 0 \). Suppose \( \text{Gal}_\infty(f) \) is topologically finitely generated and acts transitively on the roots at each level. Then there exists a discrete finitely generated group \( G_0 \) and a group chain \( \{ G_i \}_{i \geq 0} \) in \( G_0 \), such that the Ellis group of the action \( (G_\infty, G_0) \) is isomorphic to \( \text{Gal}_\infty(f) \).

Example (Looper and Lukina 2017): Let \( f(x) = x^d - a \), and let \( K \) be the \( p \)-adic numbers \( \mathbb{Q}_p \), \( p \neq 2 \). Suppose \( d > 2 \) is a prime, \( d \neq p \), and \( a \) is chosen so that \( f^i \) is irreducible over \( K \) for all \( i \geq 0 \).

Then \( \text{Gal}_\infty(f) \) is isomorphic to the Ellis group of the action with group chain \( \{ G_i = \langle \tau^{d^i}, \sigma \rangle \}_{i \geq 0} \) in \( BS(p, 1) \). This action is stable.
Work in progress

1. **Theorem (Looper and Lukina 2017)** Let $f(x)$ be a polynomial of a degree $d$ prime to $p$, and suppose that the Galois group $\text{Gal}_\infty(f)$ is contained in the maximal tamely ramified extension of $\mathbb{Q}_p$. Then the action of $\text{Gal}_\infty(f)$ is stable.

2. Find a polynomial $f(x)$ with topologically finitely generated $\text{Gal}_\infty(f)$, whose action of the $d$-ary tree is wild.

3. Find necessary and/or sufficient conditions on a polynomial $f(x)$ so that the action of the Galois group $\text{Gal}_\infty(f)$ is stable.
References


Thank you for your attention!