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The Isbell-hull of an asymmetrically normed space

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1. Introduction

2. Injective asymmetrically normed spaces

3. The injective hull of an asymmetrically normed space
Asymmetrically normed spaces

A function \( p : X \rightarrow [0, \infty) \) on a real vector space \( X \) will be called an
asymmetric seminorm on \( X \) if for all \( x, y \in X \) and \( \lambda \in [0, \infty) \),
(a) \( p(\lambda x) = \lambda p(x) \);
(b) \( p(x + y) \leq p(x) + p(y) \).
If in addition we also have
(c) \( p(x) = p(-x) = 0 \) if and only if \( x = 0 \),
\( p \) will be called an asymmetric norm, and the pair \((X, p)\) an asymmetrically
normed space.
If (a) is replaced by by:
\( (a') \) \( p(\lambda x) = |\lambda| p(x) \) for all \( \lambda \in \mathbb{R} \), then \( p \) is called a semi-norm.
If \( p \) is an asymmetric norm on \( X \), the function \( p^t : X \rightarrow [0, \infty) \) defined by
\[
p^t(x) = p(-x), \quad x \in X
\]
is also an asymmetric norm, the asymmetric norm conjugate to \( p \).
The symmetrisation of the asymmetric norm \( p \) is the function \( p^s : X \rightarrow [0, \infty) \)
given by
\[
p^s(x) = \max\{p(x), p(-x)\}, \quad x \in X
\]
and this is easily seen to be a norm on \( X \).
An asymmetric norm $p$ induces a $T_0$-quasi-metric $d_p$ on $X$ defined by

$$d_p(x, y) = p(y - x) \text{ for all } x, y \in X.$$ 

For $x \in X$, $r > 0$ we define the balls

$$B^p_r(x) = \{ y \in X : d_p(x, y) < r \} = \{ y \in X : p(y - x) < r \}$$ 

and

$$B^p_r[x] = \{ y \in X : d_p(x, y) \leq r \} = \{ y \in X : p(y - x) \leq r \}.$$ 

The family $\{B^p_r(x) : r > 0 \}$ forms a fundamental system of neighbourhoods for $x$ for a $T_0$ topology $\tau_p$ on $X$, which we shall refer to as the topology induced by $p$.

The Hausdorff topology $\tau_{p^s}$ induced by the norm $p^s$ is clearly finer than the topologies $\tau_p$ and $\tau_{p^t}$. 
As a simple but important example we mention the asymmetric norm $p_1$ on $\mathbb{R}$ (regarded as a real vector space) defined for all $x \in \mathbb{R}$ by

$$p_1(x) = x^+,\$$

where $x^+ = x \vee 0 = \max\{x, 0\}$ is the positive part of $x$. In this case

$$p_1^t(x) = \max\{-x, 0\} = x^-\$$

$$p_1^s(x) = \max\{x^+, x^-\} = |x|.$$
An asymmetrically normed space \((X, p)\) will be called (a) \textit{Isbell-convex} if for every family \((x_i)_{i \in I}\) of elements of \(X\) and families of non-negative real numbers \((r_i)_{i \in I}\) and \((s_i)_{i \in I}\) it follows that if
\[
p(x_j - x_i) \leq r_i + s_j
\]
whenever \(i, j \in I\), then
\[
\bigcap_{i \in I} B_{r_i}^p[x_i] \cap B_{s_i}^t[x_i] \neq \emptyset.
\]
(b) \textit{metrically convex} if for every two elements \(x, y \in X\) and non-negative numbers \(r\) and \(s\) such that \(p(y - x) \leq r + s\), there exists a \(z \in X\) such that 
\[
p(z - x) \leq r \quad \text{and} \quad p(y - z) \leq s.
\]
(c) \textit{Isbell-complete} if for each family \((x_i)_{i \in I}\) of elements in \(X\) and families of non-negative real numbers \((r_i)_{i \in I}\) and \((s_i)_{i \in I}\) such that if \(B_{r_i}^p[x_i] \cap B_{s_j}^t[x_j] \neq \emptyset\) whenever \(i, j \in I\), then
\[
\bigcap_{i \in I} B_{r_i}^p[x_i] \cap B_{s_i}^t[x_i] \neq \emptyset.
\]
Examples

1. $X = \mathbb{R}^2$, \[\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}\] has the binary intersection property.

2. $X = \mathbb{R}^2$, \[\|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2}\] does not have the binary intersection property.

3. Hence $\mathbb{C}$, with its usual norm, does not have the binary intersection property.
Note that an asymmetrically normed space \((X, p)\) is Isbell-convex (metrically convex, Isbell-complete) if and only if the \(T_0\)-quasi-metric space \((X, d_p)\) has the same property.
Note that an asymmetrically normed space \((X, p)\) is Isbell-convex (metrically convex, Isbell-complete) if and only if the \(T_0\)-quasi-metric space \((X, d_p)\) has the same property.

**Lemma**

*Every asymmetrically normed space \((X, p)\) is metrically convex.*
Note that an asymmetrically normed space \((X, p)\) is Isbell-convex (metrically convex, Isbell-complete) if and only if the \(T_0\)-quasi-metric space \((X, d_p)\) has the same property.

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**Lemma**

*An asymmetrically normed space \((X, p)\) is Isbell-convex if and only if it is Isbell-complete.*
Definition

An asymmetrically normed space \((Y, q)\) is called **1-injective** if for every asymmetrically normed space \((X, p)\) and every linear subspace \(X_0\) of \(X\), every continuous linear map \(T_0 : (X_0, p) \to (Y, q)\) has a continuous extension \(T : X \to Y\) such that \(\|T\|_{p,q} \leq \|T_0\|_{p,q}\).

Lemma

*If the asymmetrically normed space \((X, p)\) is Isbell-convex, then so is \((X, p^\dagger)\), and the normed space \((X, p^s)\) is a hyperconvex Banach space, and therefore 1-injective (as a Banach space).*
Definition

An asymmetrically normed space \((Y, q)\) is called \textbf{1-injective} if for every asymmetrically normed space \((X, p)\) and every linear subspace \(X_0\) of \(X\), every continuous linear map \(T_0: (X_0, p) \rightarrow (Y, q)\) has a continuous extension \(T: X \rightarrow Y\) such that \(\|T\|_{p,q} \leq \|T_0\|_{p,q}\).

Lemma

If the asymmetrically normed space \((X, p)\) is Isbell-convex, then so is \((X, p^t)\), and the normed space \((X, p^s)\) is a hyperconvex Banach space, and therefore 1-injective (as a Banach space).

Theorem

An Isbell-convex (equivalently, Isbell-complete) asymmetrically normed space \((Y, q)\) is 1-injective.
Let \((X, p)\) be an asymmetrically normed space. Recall that a function pair \(f = (f_1, f_2)\), where \(f_i : X \to [0, \infty)\) for \(i = 1, 2\), is called ample if

\[
p(y - x) \leq f_2(x) + f_1(y),
\]

and that \(f\) is minimal whenever \(g = (g_1, g_2)\) is an ample pair such that if \(g_1(x) \leq f_1(x), g_2(x) \leq f_2(x)\) for all \(x \in X\), then \(g_1 = f_1, g_2 = f_2\). The set of all minimal function pairs on \(X\) will be denoted by \(\mathcal{E}(X, p)\). The following characterisation of the elements of \(\mathcal{E}(X, p)\) is useful; recall that for \(a \in \mathbb{R}\), we write \(a^+ = a \vee 0 = \max\{a, 0\} \).
Let \((X, p)\) be an asymmetrically normed space. Recall that a function pair \(f = (f_1, f_2)\), where \(f_i : X \to [0, \infty)\) for \(i = 1, 2\), is called **ample** if

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p(y - x) \leq f_2(x) + f_1(y),
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\[
g_1(x) \leq f_1(x),
\]

\[
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for all \(x \in X\), then \(g_1 = f_1, g_2 = f_2\). The set of all minimal function pairs on \(X\) will be denoted by \(E(X, p)\). The following characterisation of the elements of \(E(X, p)\) is useful; recall that for \(a \in \mathbb{R}\), we write \(a^+ = a \vee 0 = \max\{a, 0\}\).

A function pair \(f = (f_1, f_2)\) belongs to \(E(X, p)\) if and only if for every \(x \in X\),

\[
f_2(x) = \sup_{s \in X} (p(s - x) - f_1(s))^+
\]

and

\[
f_1(x) = \sup_{s \in X} (p(x - s) - f_2(s))^+.
\]
For every $z \in X$, we define the minimal function pair $f_z = (f_{z,1}, f_{z,2})$ by

\[
f_{z,1}(x) = p(x - z), \quad f_{z,2}(x) = p(z - x).
\]

The mapping $z \mapsto f_z$ is an injection of $X$ into $\mathcal{E}(X, p)$. 
For every $z \in X$, we define the minimal function pair $f_z = (f_{z,1}, f_{z,2})$ by

$$f_{z,1}(x) = p(x - z), \quad f_{z,2}(x) = p(z - x).$$

The mapping $z \mapsto f_z$ is an injection of $X$ into $\mathcal{E}(X, p)$. We now define scalar multiplication on $\mathcal{E}(X, p)$. For $\lambda \in \mathbb{R}$ and $f \in \mathcal{E}(X, p)$, we define the function pair $f^\lambda = (f_1^\lambda, f_2^\lambda)$ by

$$f_1^\lambda(x) = \begin{cases} \lambda f_1(\lambda^{-1}x) & \text{if } \lambda > 0, \\ p(x) & \text{if } \lambda = 0, \text{ and} \\ |\lambda| f_2(\lambda^{-1}x) & \text{if } \lambda < 0 \end{cases}$$

$$f_2^\lambda(x) = \begin{cases} \lambda f_2(\lambda^{-1}x) & \text{if } \lambda > 0, \\ p(-x) & \text{if } \lambda = 0, \\ |\lambda| f_1(\lambda^{-1}x) & \text{if } \lambda < 0. \end{cases}$$
Lemma

If $f = (f_1, f_2) \in \mathcal{E}(X, p)$ and $\lambda \in \mathbb{R}$, then $f^\lambda \in \mathcal{E}(X, p)$

It now follows that we can define scalar multiplication in $\mathcal{E}(X, p)$ by putting

$$\lambda f = f^\lambda.$$  

We now turn to defining addition on $\mathcal{E}(X, p)$. If $f = (f_1, f_2), g = (g_1, g_2) \in \mathcal{E}(X, p), x \in X$ we put $f \oplus g = ((f \oplus g)_1, (f \oplus g)_2)$, where

$$(f \oplus g)_1(x) = \sup\{(f_1(x - s) - g_2(s))^+ : s \in X\},$$

$$(f \oplus g)_2(x) = \sup\{(f_2(x - s) - g_1(s))^+ : s \in X\}.$$  

Lemma (Olela Otafudu, Topology Appl. 166 (2014))

If $f = (f_1, f_2) \in \mathcal{E}(X, p)$, then for $x \in X$,

$$\sup\{(f_1(x + s) - f_1(s))^+ : s \in X\} = p(x),$$

$$\sup\{(f_2(x + s) - f_2(s))^+ : s \in X\} = p(-x).$$
If $f, g \in \mathcal{E}(X, p)$, then $f \oplus g$ is ample.
If $f, g \in \mathcal{E}(X, p)$, then $f \oplus g$ is ample.

Suppose $x, y, z \in X$. Then

\[(f_y \oplus f_z)_1(x) = f_{(y+z),1}(x)\]

and

\[(f_y \oplus f_z)_2(x) = f_{(y+z),2}(x).\]

**Lemma**

If $f, g \in \mathcal{E}(X, p)$ and $x \in X$, then

\[\sup_{s \in X}(f_1(x - s) - g_2(s))^+ = \sup_{s \in X}(g_1(s) - f_2(x - s))^+\]

and

\[\sup_{s \in X}(f_2(x - s) - g_1(s))^+ = \sup_{s \in X}(g_2(s) - f_1(x - s))^+\].
Let \( f, g, h \in \mathcal{E}(X, p) \).

Then

\[
f \oplus g = g \oplus f
\]

and

suppose \( f \oplus g, g \oplus h \in \mathcal{E}(X, p) \). Then

\[
(f \oplus g) \oplus h = f \oplus (g \oplus h).
\]
Let $f, g, h \in \mathcal{E}(X, p)$. Then
\[ f \oplus g = g \oplus f \]
and suppose $f \oplus g, g \oplus h \in \mathcal{E}(X, p)$. Then
\[ (f \oplus g) \oplus h = f \oplus (g \oplus h). \]

In the light of the definition of scalar multiplication, the only candidate for the additive identity is $f^0 = (f_1^0, f_2^0)$, with $f_1^0(x) = p(x)$, $f_2^0(x) = p(-x))$. We check this: Let $f = (f_1, f_2) \in \mathcal{E}(X, p)$ and $x \in X$. Then,
\[ (f^0 \oplus f)_1(x) = \sup \{(p(x - s) - f_2(s))^+ : s \in X\} = f_1(x), \]
and
\[ (f^0 \oplus f)_2(x) = \sup \{(p(s - x) - f_1(x))^+ : s \in X\} = f_2(x). \]
The only candidate for the additive inverse of \( f = (f_1, f_2) \) is 
\((-1)f = (f_1^{-1}, f_2^{-1})\). We check this:

\[
(f \oplus f^{-1})_1(x) = \sup \{(f_1(x - s) - f_2^{-1}(s))^+ : s \in X\}
\]
\[
= \sup \{(f_1(x + s) - f_1(s))^+ : s \in X\}
\]
\[
= p(x) = f_1^0(x).
\]

A similar calculation shows that
\[
(f \oplus f^{-1})_2(x) = p^{-1}(x) = f_2^0(x).
\]

We denote the additive inverse of \( f = (f_1, f_2) \in \mathcal{E}(X, p) \) by \(-f\); thus
\(-f = ((-f)_1, (-f)_2), \) where

\[
(-f)_1(x) = f_2(-x),
\]
\[
(-f)_2(x) = f_1(-x)
\]

whenever \( x \in X \). If \( f, g \in \mathcal{E}(X, p) \), then \( f \oplus g \in \mathcal{E}(X, p) \).
Theorem

If scalar multiplication on $\mathcal{E}(X, p)$ is defined by $\lambda f = f^\lambda$ and addition $\oplus$ by

$$(f \oplus g)_1(x) = \sup\{(f_1(x - s) - g_2(s))^+ : s \in X\}$$

and

$$(f \oplus g)_2(x) = \sup\{(f_2(x - s) - g_1(s))^+ : s \in X\}$$

whenever $f, g \in \mathcal{E}(X, p)$ and $\lambda \in \mathbb{R}$, then $\mathcal{E}(X, p)$ is a vector space and the map $x \mapsto f_x$ is a linear isomorphism of $X$ into $\mathcal{E}(X, p)$. 
To define an asymmetric norm on \( E(X, p) \) we take our cue from the \( T_0 \)-quasi-metric \( D \) defined on the injective hull of a \( T_0 \)-quasi-metric space \((X, d)\) in [Kemajou, Künzi, Otafudu, Topology Appl. 159 (2012)] by
\[
D(f, g) = \max\{\sup_{s \in X} (f_1(s) - g_1(s))^+, \sup_{s \in X} (g_2(s) - f_2(s))^+\}
\]
for \( f, g \in E(X, p) \), it is shown that
\[
D(f, g) = \sup_{s \in X} (f_1(s) - g_1(s))^+ = \sup_{s \in X} (g_2(s) - f_2(s))^+.
\]
Recall that the additive identity \( f^0 = (f_0^1, f_0^2) \) is defined by
\[
f_1^0(s) = p(s), f_2^0(s) = p(-s).
\]
For \( f \in E(X, p) \) we now put
\[
\tilde{p}(f) = D(f^0, f) = \sup_{s \in X} (f_2(s) - p(-s))^+ = \sup_{s \in X} (p(x) - f_1(s))^+ = f_2(0)
\]
and
\[
\tilde{p}(-f) = D(f, f^0) = f_1(0).
\]
The function \( \tilde{p} : \mathcal{E}(X, p) \to [0, \infty) \) defined above is an asymmetric norm on \( \mathcal{E}(X, p) \) and the map \( x \mapsto f_x \) is an isometry.
The function $\tilde{p} : \mathcal{E}(X, p) \to [0, \infty)$ defined above is an asymmetric norm on $\mathcal{E}(X, p)$ and the map $x \mapsto f_x$ is an isometry.

**Theorem**

*An 1-injective asymmetrically normed space $(X, p)$ is Isbell-convex.*
The function $\tilde{p} : \mathcal{E}(X, p) \to [0, \infty)$ defined above is an asymmetric norm on $\mathcal{E}(X, p)$ and the map $x \mapsto f_x$ is an isometry.

**Theorem**

An 1-injective asymmetrically normed space $(X, p)$ is Isbell-convex.

**Lemma**

$\mathcal{E}(X, p)$ is Isbell-convex.
