Aperiodic Colorings and Dynamics

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Aperiodic colorings and dynamics

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Let $G$ be a simple graph. A coloring $\phi$ on $G$ is said to be *aperiodic* or *distinguishing* if there are no non-trivial automorphisms of $G$ preserving $\phi$. The *distinguishing number* $D(G)$ is the minimum number of colors needed to produce a distinguishing coloring of $G$. If the degree or valence of $G$, $\deg(G) = \max\{\deg(v) | v \in G\}$, is finite, then $D(G) \leq \deg(G) + 1$, where equality is attained only for complete graphs $K_n$, complete bipartite graphs $K_{n,n}$, and the cyclic graph with 5 vertices $C_5$. 
Finite case

Let $G$ be a simple graph. A coloring $\phi$ on $G$ is said to be \textit{aperiodic} or \textit{distinguishing} if there are no non-trivial automorphisms of $G$ preserving $\phi$. The \textit{distinguishing number} $D(G)$ is the minimum number of colors needed to produce a distinguishing coloring of $G$. 
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is finite, then $D(G) \leq \text{deg}(G) + 1$, where equality is attained only for complete graphs $K_n$, complete bipartite graphs $K_{n,n}$ and the cyclic graph with 5 vertices $C_5$. 
A pointed colored graph \((G', z, \phi')\) is a *limit* of \((G, \phi)\) if there is a sequence of balls \(B_G(x_i, R_i)\), with \(x_i \in G\) and \(R_i \to \infty\) such that \((B_G(x_i, R_i), x_i, \phi)\) and \((B_G(z, R_i), z, \phi')\) are isomorphic as pointed colored graphs.
Limits of graphs

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A coloring \(\phi\) is limit aperiodic or limit distinguishing if every limit graph \((G', \phi')\) has no non-trivial automorphism as a colored graph.

The distinguishing number \(D_l(G)\) is the minimum number of colors needed to produce a distinguishing coloring of \(G\).

We want to calculate \(D_l(G)\) for connected graphs of bounded degree.

Theorem: If an infinite connected graph \(G\) has bounded degree \(\deg G < \infty\), then it admits a limit distinguishing coloring by \(\deg G\) colors.
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*If an infinite connected graph \(G\) has bounded degree \(\deg G < \infty\), then it admits a limit distinguishing coloring by \(\deg G\) colors.*
Idea of the proof

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If the graph is large enough, we can do it with one color less, attaining the optimal bound $D(G) \leq \deg G$.
We have a reserve color that we can use to create many different distinguishing colorings.
We want to prove for a coloring $\chi$ an estimate of the following type: there are $R, S > 0$ such that if $B_G(x, R) \rightarrow B_G(y, R)$ is an isomorphism of graphs preserving $\chi$, then either $x = y$ or $d(x, y) > S$. 
Idea of the proof 2

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In fact for $\chi$ to be limit distinguishing is equivalent to countably many such estimates for pairs $R_n, S_n$ with $R_n, S_n \rightarrow \infty$. 
Idea of the proof 3

Divide carefully the graph $G$ into connected clusters $G = \bigsqcup C_n$. These clusters determine a graph $G_1$, where each cluster is a vertex.
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A word on repetitiveness

A graph is *repetitive* if every ball $B(x, r) \subset G$ appears uniformly on $G$, that is, there is some $K_{x,r}$ such that for every $y \in G$ there is some $z$ with $d(y, z) \leq K_{x,r}$ and $B(x, r) \simeq B(z, r)$. 

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**Theorem**

*If the graph $G$ is of bounded degree and repetitive, then there is a limit distinguishing repetitive coloring by $\text{deg } G$ colors.*
Foliated spaces

A foliated space is a topological space $X$ with an equivalence class of atlas consisting of charts of the form $\phi_i : U_i \subset X \to \mathbb{R}^n \times Z_i$, and such that the change of coordinates $\phi_i \circ \phi_j^{-1}$ sends the plaques $\mathbb{R}^n \times \{z\}$ to plaques smoothly.
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In this context, we have a differentiable structure along the leaves, but only topological structure in the transverse direction.

This structure gives rise to a special type of dynamical system, a pseudogroup. If this dynamical system is in some sense trivial, we say that the foliation is without holonomy.
We will consider triples \((M, x, f)\), where \(M\) is a Riemannian \(n\)-manifold, \(x \in M\) a distinguished point, and \(f : M \to \mathbb{H}\) a smooth map into a separable Hilbert space.
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Two such triples \((M, x, f)\) and \((N, y, g)\) are equivalent if there is an isometry \(\phi : M \to N\) sending \(x\) to \(y\) and such that \(g\phi = f\).
Universal space

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Let \(\tilde{M}_\infty(n)\) be the space of equivalence classes of triples, then choosing a manifold \(M\) and a map \(f : M \to \mathbb{H}\) determines an inclusion \(M \hookrightarrow \tilde{M}_\infty(n)\) sending \(x \in M\) to \([M, x, f]\).
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Let \(\hat{M}_\infty^* (n)\) be the space of equivalence classes of triples, then choosing a manifold \(M\) and a map \(f : M \to \mathbb{H}\) determines an inclusion \(M \hookrightarrow \hat{M}_\infty^* (n)\) sending \(x \in M\) to \([M, x, f]\).

If \(f\) is an isometric embedding with some minor additional conditions, we get an embedding of \(M\) into a compact lamination \(X \subset \hat{M}_\infty^* (n)\).
A sort of translator

A nice thing about this approach, is that to construct foliated spaces containing $M$ and satisfying additional properties can be reduced to the existence of maps $f$ defined on $M$ and satisfying certain conditions.
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Theorem

Every manifold of bounded geometry can be realized as a leaf in a compact foliated space without holonomy. Moreover, if the manifold is repetitive, then the space can be taken to be minimal.
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**Theorem**

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Theorem
Given a tiling as before, we can color it by finitely many colors so that the associated colored tiling is limit aperiodic. Moreover, if the tiling was repetitive, then the colored tiling can be chosen to be repetitive.
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Thank you!