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Fiber Strong Shape Theory for Topological Spaces

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Mathematics

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ABSTRACT. **In the paper we construct and develop a fiber strong shape theory for arbitrary spaces over fixed metrizable space $\mathbf{B_0}$. Our approach is based on the method of Mardešic'-Lisica and instead of resolutions, introduced by Mardešic', their fiber preserving analogues are used. The fiber strong shape theory yields the classification of spaces over $\mathbf{B_0}$ which is coarser than the classification of spaces over $\mathbf{B_0}$ induced by fiber homotopy theory, but is finer than the classification of spaces over $\mathbf{B_0}$ given by usual fiber shape theory.**

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1 Resolution and Strong Expansions of Spaces over $\mathbf{B_0}$

An inverse system of the category $\mathbf{Top_{B_0}}$ is a collection $\mathbf{X} = (X_\alpha, p_{\alpha\alpha'}, \mathscr{A})$ of space X_α over B_0 indexed by a directed set $\mathscr{A}$ and f.p. maps $p_{\alpha\alpha'} : X_{\alpha'} \to X_\alpha$, $\alpha \le \alpha'$, such that $p_{\alpha\alpha'}\, p_{\alpha'\alpha''} = p_{\alpha\alpha''}$ and $p_{\alpha\alpha} = 1_{X_\alpha}$, $\alpha \in \mathscr{A}$.

A morphism $(f_\beta, \varphi) : \mathbf{X} \to \mathbf{Y} = (Y_\beta, q_{\beta\beta'}, \mathscr{B})$ of inverse systems of the category $\mathbf{Top_{B_0}}$ consists of a function $\varphi : \mathscr{B} \to \mathscr{A}$ and of f.p. maps $f_\beta : X_{\varphi(\beta)} \to Y_\beta$, $\beta \in \mathscr{B}$, such that whenever $\beta \le \beta'$, then there is an index $\alpha \ge \varphi(\beta), \varphi(\beta')$ for which $f_\beta\; p_{\varphi(\beta)} = q_{\beta\beta'}\; f_{\beta'}\; p_{\varphi(\beta')\alpha}$.

Two morphisms $(f_\beta, \varphi), (g_\beta, \psi) : \mathbf{X} \to \mathbf{Y}$ are said to be equivalent, $f \underset{B_0}{\simeq} g$, provided for each $\beta \in \mathscr{B}$ there is an $\alpha \in \mathscr{A}$, $\alpha \ge \varphi(\beta), \psi(\beta)$, such that $f_\beta\; p_{\varphi(\beta)\alpha} = g_\beta\; p_{\psi(\beta)\alpha}$.

Let $\mathbf{pro\text{-}Top_{B_0}}$ be a category, whose objects are the inverse systems $\mathbf{X}$ of the category $\mathbf{Top_{B_0}}$ and whose morphisms are the equivalence classes $\mathbf{f}$ of morphisms $(f_{\beta,\varphi}) : \mathbf{X} \to \mathbf{Y}$ with respect to relation $\underset{B_0}{\simeq}$.

A morphism $\mathbf{p} = (p_\alpha) : X \to \mathbf{X} = (X_\alpha, p_{\alpha\alpha'}, \mathscr{A})$ from a rudimentary system (X) to an inverse system $\mathbf{X}$ consists of the f.p. maps $p_\alpha : X \to X_\alpha$, $\alpha \in \mathscr{A}$, such that $p_\alpha = p_{\alpha\alpha'}\; p_{\alpha'}$, $\alpha \le \alpha'$.

Definition 1.1 *Let X be a space over B_0 and let $\mathbf{X} = (X_\alpha, p_{\alpha\alpha'}, \mathscr{A})$ be an inverse system of the category $\mathbf{Top_{B_0}}$. We say that $\mathbf{p} : X \to \mathbf{X}$ is a resolution over B_0 or fiber resolution of the space X*

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over B_0 *provided it satisfies the following two conditions:*

$\mathbf{R_{B_0}}$ 1). *Let* $P \in \mathrm{ANR}_{B_0}$, *let* $\mathcal{U}$ *be an open covering of* P *and let* $h: X \to P$ *be a f.p. map. Then there exist an index* $\alpha \in \mathcal{A}$ *and a f.p. map* $f: X_\alpha \to P$ *such that* h *and* $f\,p_\alpha$ *are* $\mathcal{U}$ *-near.*

$\mathbf{R_{B_0}}$ 2). *Let* $P \in \mathrm{ANR}_{B_0}$ *and let* $\mathcal{U}$ *be an open covering of* P. *Then there is an open cover* $\mathcal{U}'$ *of* P *with the following property: if* $\alpha \in \mathcal{A}$ *and* $f, f': X \to P$ *are f.p. maps such that the f.p. maps* $f\,p_\alpha$ *and* $f'\,p_\alpha$ *are* $\mathcal{U}'$ *-near, then there is an index* $\alpha' \geq \alpha$ *such that the f.p. maps* $f\,p_{\alpha\alpha'}$ *and* $f'\,p_{\alpha\alpha'}$ *are* $\mathcal{U}$ *-near.*

If in a fiber resolution $\mathbf{p}: X \to \mathbf{X} = (X_\alpha, p_{\alpha\alpha'}, \mathscr{A})$ *of the space* X *over* B_0 *each* X_α *is an* ANR_{B_0}, *then we say that* $\mathbf{p}$ *is a fiber* ANR_{B_0} *-resolution.*

The next theorem is essential in the construction of the fiber shape category for arbitrary spaces over B_0.

Theorem 1.2 *Every space* X *over a metrizable space* B_0 *admits an* ANR_{B_0} *-resolution over* B_0.

In the proof of Theorem 1.2 we shall need the following lemma.

Lemma 1.3 *Let* $f: X \to Y$ *be a f.p. map from the topological space* X *over* B_0 *to an* ANR_{B_0} *-space* Y. *Then there exists an* ANR_{B_0} *-space* Z *of weight* $w(Z) \leq max\{w(X), w(B_0), \aleph_0)\}$ *and f.p. maps* $g: X \to Z$ *and* $h: Z \to Y$ *such that* $f\,h = g$.

Definition 1.4 *Let* X *be a topological space over* B_0, $\mathbf{X} = (X_\alpha, p_{\alpha\alpha'}, \mathscr{A})$ *an inverse system in* $\mathbf{Top_{B_0}}$ *and* $\mathbf{p} = (p_\alpha): X \to \mathbf{X}$ *a morphism of* $\mathbf{pro\text{-}Top_{B_0}}$. *We call* $\mathbf{p}$ *an expansion over* B_0 *of the space* X *over* B_0 *provided it has the following properties:*

$\mathbf{E_{B_0}}$ 1). *For every* ANR_{B_0} *-space* P *over* B_0 *and f.p. map* $f: X \to P$ *there is an index* $\alpha \in \mathcal{A}$ *and a f. p. map* $h: X_\alpha \to P$ *such that* $h\,p_\alpha \underset{B_0}{\simeq} f$.

$\mathbf{E_{B_0}}$ 2). *If* $f, f': X_\alpha \to P$ *are f. p. maps,* $P \in \mathrm{ANR}_{B_0}$ *and* $f\,p_\alpha \underset{B_0}{\simeq} f'\,p_\alpha$, *then there is an index* $\alpha' \geq \alpha$ *such that* $f\,p_{\alpha\alpha'} \underset{B_0}{\simeq} f'\,p_{\alpha\alpha'}$.

Definition 1.5 *A morphism* $\mathbf{p}: X \to (X_\alpha, p_{\alpha\alpha'}, \mathscr{A})$ *is called a strong expansion over* B_0 *provided it satisfies condition* $E_{B_0}1)$ *and the following condition:*

$\mathbf{SE_{B_0}}$ 2). *Let* P *be an* ANR_{B_0} *-space, let* $f_0, f_1: X_\alpha \to P$, $\alpha \in \mathscr{A}$ *be f.p. maps and let* $F: X \times I \to P$ *be a f.p. homotopy such that*

$$S(x,0) = f_0 p_\alpha(x), \quad x \in X,$$
$$S(x,1) = f_1 p_\alpha(x), \quad x \in X.$$

Then there exists a $\alpha' \geq \alpha$ *and a f.p. homotopy* $H: X_{\alpha'} \times I \to P$, *such that*

$$H(x,0) = f_0 p_{\alpha\alpha'}(z), \quad z \in X_{\alpha'},$$
$$H(x,1) = f_1 p_{\alpha\alpha'}(z), \quad z \in X_{\alpha'},$$
$$H(p_{\alpha'} \times 1_I) \underset{B_0}{\simeq} S(\mathrm{rel}(X \times \partial I)).$$

It is clear that, every strong expansion over B_0 is an expansion over B_0.

If all $X_\alpha \in \mathrm{ANR}_{B_0}$, then $\mathbf{p}$ is called an ANR_{B_0} -expansion and strong ANR_{B_0} -expansion, respectively.

The main result of section 1 is the following theorem.

Theorem 1.6 *Let X be a topological space over B_0. Then every resolution $p: X \to \mathbf{X}$ over B_0 induces a strong ANR_{B_0}-expansion.* □

Corollary 1.7 *Every ANR_{B_0}-resolution over B_0 induces ANR_{B_0}-expansion.* □

Corollary 1.8 *Every space X over B_0 admits a cofinite strong ANR_{B_0}-expansion.* □

In the proof of Theorem 1.6 we need the following lemma.

Lemma 1.9 *Let X be a topological space over metrizamble space B_0, let P, P' be ANR_{B_0}-spaces, let $f: X \to P'$, $h_0, h_1 : P' \to P$ be f.p. maps and let $S: X \times I \to P$ be a f.p. homotopy such that*

$$S(x,0) = h_0 f(x), \quad x \in X,$$
$$S(x,1) = h_1 f(x), \quad x \in X.$$

Then there exists an ANR_{B_0}-space P'', f.p. maps $f': X \to P''$, $h: P'' \to P'$ and a f.p. homotopy $K: P'' \times I \to P$ such that

$$h f' = f,$$
$$K(z,0) = h_0 h(z), \quad z \in P''$$
$$K(z,1) = h_1 h(z), \quad z \in P''$$
$$K(f' \times 1_I) = S.$$

Lemma 1.10 *Let $p: X \to \mathbf{X}$ be a resolution over B_0 and let α, P, f_0, f_1 and F be as in $SE_{B_0}2)$. Then for every open covering $\mathcal{U}$ of P, there exist a $\alpha' \geq \alpha$ and a f.p. homotopy $H: X_{\alpha'} \times I \to P$ such that*

$$H(y,0) = f_0\, p_{\alpha\alpha'}(y), \qquad y \in X_{\alpha'}$$
$$H(y,1) = f_1\, p_{\alpha\alpha'}(y), \qquad y \in X_{\alpha'}$$
$$(S, H(1 \times p_{\alpha'})) \leq \mathcal{U}$$

2 On Fiber Strong Shape Category

The objects of category $\mathbf{SSH_{B_0}}$ are all topological spaces over B_0. The morphisms of category $\mathbf{SSH_{B_0}}$ are defined by the following way.

Let $\mathbf{p}: X \to \mathbf{X}$ and $\mathbf{q}: Y \to \mathbf{Y}$ be an ANR_{B_0}-resolutions of X and Y, respectively. Let $[f]: \mathbf{X} \to \mathbf{Y}$ be a some morphism of category $\mathbf{CPHTop_{B_0}}$. Let $\mathbf{p}': X \to \mathbf{X}'$, $\mathbf{q}': Y \to \mathbf{Y}'$, $[f']: \mathbf{X}' \to \mathbf{Y}'$ be another triple of fiber resolutions of spaces X and Y over B_0 and morphism of category $\mathbf{CPHTop_{B_0}}$.

Now define the following equivalence relation. We say the triples $(\mathbf{p}, \mathbf{q}, [f])$ and $(\mathbf{p}', \mathbf{q}', [f'])$ are equivalent if

$$[f'][i] = [j][f],$$

where $[i]: \mathbf{X} \to \mathbf{X}'$ and $[j]: \mathbf{Y} \to \mathbf{Y}'$ are isomorphisms of category $\mathbf{CPHTop_{B_0}}$.

The fiber strong shape morphisms $\mathrm{F}: X \to Y$ are the equivalence classes of triples $(\mathbf{p}, \mathbf{q}, [f])$ with respect to the above defined relation $\sim$.

Let $F: X \to Y$ and $G: Y \to Z$ be the fiber strong shape morphisms, defined by triples $(\mathbf{p}, \mathbf{q}, [f])$

and $(\mathbf{p}',\mathbf{q}',[g])$, where $\mathbf{p}':Y \to \mathbf{Y}$, $\mathbf{q}:Z \to \mathbf{Z}$ and $[g]:\mathbf{Y} \to \mathbf{Z}$.

As we know there exists an unique morphism $[h]:\mathbf{Y} \to \mathbf{Y}'$ of category $\mathbf{CPHTop_{B_0}}$ such that $[h][q]=[q']$. Note that

$$[j][q]=[q']=[h][q].$$

Hence, $[j]=[h]$. Besides, $[g][j]=[g][h][1_{\mathbf{Z}}]$.

Thus, we can assume that the morphisms F and G are given by triples $(\mathbf{p},\mathbf{q},[f])$ and $(\mathbf{p}',\mathbf{q}',[g])$.

Consequently, we can define the composition $\mathrm{GF}:X \to Z$ as the morphism given by triple $(\mathbf{p},\mathbf{r},[g][f])$.

In the role an identity morphism $\mathscr{I}:X \to X$ we can take the morphism defined by triple $(\mathbf{p},\mathbf{p},[1_X])$.

The obtained category $\mathbf{SSH_{B_0}}$ call the fiber strong shape category.

Let $X \in ob(\mathbf{SSH_{B_0}})$. By symbol $\mathrm{ssh_{B_0}}(X)$ denote the equivalence class of topological space X and call the fiber strong shape of X.

For each f.p. map $\varphi:X \to Y$ choose $\mathrm{ANR_{B_0}}$-resolutions $\mathbf{p}:X \to \mathbf{X}$ and $\mathbf{q}:Y \to \mathbf{Y}$. There exists a unique morphism $[f]:\mathbf{X} \to \mathbf{Y}$ of category $\mathbf{CPHTop_{B_0}}$ such that $[q][\varphi]=[f][p]$.

We can define a functor $\mathrm{SS'_{B_0}}:\mathbf{Top_{B_0}} \to \mathbf{SSH_{B_0}}$. By definition,

$$\mathrm{SS'_{B_0}}(X)=X,\ X \in ob(\mathbf{Top_{B_0}})$$

and

$$\mathrm{SS'_{B_0}}(\varphi)=\Phi,\ \varphi \in \mathrm{Mor}_{\mathbf{Top_{B_0}}}(X,Y).$$

Here Φ is a fiber strong shape morphism defined by triple $(\mathbf{p},\mathbf{q},[f])$.

As in [L-M] we can prove that functor $\mathrm{SS'_{B_0}}$ induces a functor $\mathrm{SS_{B_0}}:\mathbf{HTop_{B_0}} \to \mathbf{SSH_{B_0}}$, which we call the fiber strong shape functor. By definition,

$$\mathrm{SS_{B_0}}(X)=X,\ \ X \in ob(\mathbf{HTop_{B_0}})$$

and

$$\mathrm{SS_{B_0}}([\varphi]_{\mathrm{B_0}})=\mathrm{SS'_{B_0}}(\varphi),\ \ [\varphi]_{\mathrm{B_0}} \in \mathrm{Mor}_{\mathbf{HTop_{B_0}}}(X,Y).$$

Let us define a functor $\mathrm{S}:\mathbf{SSH_{B_0}} \to \mathbf{SH_{B_0}}$. Assume that $\mathrm{S}(X)=X$ for each object $X \in ob(\mathbf{SSH_{B_0}})$. Let $\mathrm{F}:X \to Y$ be a fiber strong shape morphism given by a triple $(\mathbf{p},\mathbf{q},[f])$.

Consider the morphism $\mathrm{E}([f])$ as an image of $[f]$ with respect the functor $\mathrm{E}:\mathbf{CPHTop_{B_0}} \to \mathbf{pro\text{-}HTop_{B_0}}$. The triple $(\mathrm{H}\mathbf{p},\mathrm{H}\mathbf{q},\mathrm{E}[f])$ generates a fiber shape morphism, which we denote by $\mathrm{S(F)}:X \to Y$.

Now we can formulate the following

Theorem 2.5 *There exists the following commutative diagram*

$$\begin{tikzcd}
 & \mathbf{SH_{B_0}} \\
\mathbf{HTop_{B_0}} \arrow[ur, "\mathrm{S_{B_0}}"] \arrow[dr, "\mathrm{SS_{B_0}}"'] & \\
 & \mathbf{SSH_{B_0}}. \arrow[uu, "\mathrm{S}"']
\end{tikzcd}$$

where $\mathrm{S_{B_0}}$ *is V.Baladze fiber shape functor* $[B_4]$. □

Corollary 2.6 *Let* X *and* Y *be topological spaces over* B_0. *If* $\mathrm{ssh_{B_0}}(X)=\mathrm{ssh_{B_0}}(Y)$, *then* $\mathrm{sh_{B_0}}(X)=\mathrm{sh_{B_0}}(Y)$. □

Remark 2.7 *Using the methods developed in this paper and papers ([B_6], [L-M],[M_1], [M_2]) it is possible to construct fiber strong shape theory for category of arbitrary continuous maps.* □

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