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Mathematics

Fiber Strong Shape Theory for Topological Spaces

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ABSTRACT. In the paper we construct and develop a fiber strong shape theory for arbitrary spaces over fixed metrizable space $B_0$. Our approach is based on the method of Mardešić'-Lisica and instead of resolutions, introduced by Mardešić', their fiber preserving analogues are used. The fiber strong shape theory yields the classification of spaces over $B_0$ which is coarser than the classification of spaces over $B_0$ induced by fiber homotopy theory, but is finer than the classification of spaces over $B_0$ given by usual fiber shape theory.


Keywords and Phrases: Fiber shape, Fiber homotopy, Fiber resolution, Fiber shape expansion, Fiber strong expansion, $A(N)R_{B_0}$-space, $A(N)E_{B_0}$-space.

1 Resolution and Strong Expansions of Spaces over $B_0$

An inverse system of the category $\text{Top}_{B_0}$ is a collection $X = (X_\alpha, p_{\alpha\beta}, \mathcal{A})$ of space $X_\alpha$ over $B_0$ indexed by a directed set $\mathcal{A}$ and f.p. maps $p_{\alpha\beta} : X_\alpha \to X_\beta$, $\beta \leq \alpha$, such that $p_{\alpha\beta} \circ p_{\beta\gamma} = p_{\alpha\gamma}$ and $p_{\alpha\alpha} = 1_{X_\alpha}$, $\alpha \in \mathcal{A}$.

A morphism $(f_\beta, \phi) : X \to Y = (Y_\beta, q_{\beta\gamma}, \mathcal{B})$ of inverse systems of the category $\text{Top}_{B_0}$ consists of a function $\phi : \mathcal{B} \to \mathcal{A}$ and of f.p. maps $f_\beta : X_{\phi(\beta)} \to Y_\beta$, $\beta \in \mathcal{B}$, such that whenever $\beta \leq \beta'$, then there is an index $\alpha \geq \phi(\beta), \phi(\beta')$ for which $f_\beta p_{\alpha(\beta)} = q_{\beta\beta'} f_{\beta'} p_{\alpha(\beta')}$.

Two morphisms $(f_\beta, \phi), (g_\beta, \psi) : X \to Y$ are said to be equivalent, $f \sim g$, provided for each $\beta \in \mathcal{B}$ there is an $\alpha \in \mathcal{A}$, $\alpha \geq \phi(\beta), \psi(\beta)$, such that $f_\beta p_{\alpha(\beta)} = g_\beta p_{\alpha(\beta)}$.

Let $\text{pro-}\text{Top}_{B_0}$ be a category, whose objects are the inverse systems $X$ of the category $\text{Top}_{B_0}$ and whose morphisms are the equivalence classes $\mathfrak{f}$ of morphisms $(f_\beta, \phi) : X \to Y$ with respect to relation $\sim$.

A morphism $p = (p_\alpha) : X \to X = (X_\alpha, p_{\alpha\beta}, \mathcal{A})$ from a rudimentary system $(X)$ to an inverse system $X$ consists of the f.p. maps $p_\alpha : X \to X_\alpha$, $\alpha \in \mathcal{A}$, such that $p_\alpha = p_{\alpha\alpha} \circ p_\alpha$, $\alpha \leq \alpha'$.

Definition 1.1 Let $X$ be a space over $B_0$ and let $X = (X_\alpha, p_{\alpha\beta}, \mathcal{A})$ be an inverse system of the category $\text{Top}_{B_0}$. We say that $p : X \to X$ is a resolution over $B_0$ or fiber resolution of the space $X$.

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over $B_0$ provided it satisfies the following two conditions:

**R$_{n_0}$ 1.** Let $P \in \text{ANR}_{n_0}$, let $\mathcal{U}$ be an open covering of $P$ and let $h : X \to P$ be a f.p. map. Then there exist an index $\alpha \in \mathcal{A}$ and a f.p. map $f : X_{\alpha} \to P$ such that $h$ and $f_{p_{\alpha}}$ are $\mathcal{U}$-near.

**R$_{n_0}$ 2.** Let $P \in \text{ANR}_{n_0}$ and let $\mathcal{U}$ be an open covering of $P$. Then there is an open cover $\mathcal{U}$ of $P$ with the following property: if $\alpha \in \mathcal{A}$ and $f, f' : X \to P$ are f.p. maps such that the f.p. maps $f_{p_{\alpha}}$ and $f'_{p_{\alpha}}$ are $\mathcal{U}$-near, then there is an index $\alpha' \geq \alpha$ such that the f.p. maps $f_{p_{\alpha'}}$ and $f'_{p_{\alpha'}}$ are $\mathcal{U}$-near.

If in a fiber resolution $p : X \to \mathcal{X} = (X_{\alpha}, p_{\alpha}, \mathcal{A})$ of the space $X$ over $B_0$ each $X_{\alpha}$ is an ANR$_{n_0}$, then we say that $p$ is a fiber ANR$_{n_0}$-resolution.

The next theorem is essential in the construction of the fiber shape category for arbitrary spaces over $B_0$.

**Theorem 1.2** Every space $X$ over a metrizable space $B_0$ admits an ANR$_{n_0}$-resolution over $B_0$.

In the proof of Theorem 1.2 we shall need the following lemma.

**Lemma 1.3** Let $f : X \to Y$ be a f.p. map from the topological space $X$ over $B_0$ to an ANR$_{n_0}$-space $Y$. Then there exists an ANR$_{n_0}$-space $Z$ of weight $w(Z) \leq \max\{w(X), w(B_0), n_0\}$ and f.p. maps $g : X \to Z$ and $h : Z \to Y$ such that $f h = g$.

**Definition 1.4** Let $X$ be a topological space over $B_0$, $\mathcal{X} = (X_{\alpha}, p_{\alpha}, \mathcal{A})$ an inverse system in $\text{Top}_{n_0}$ and $p = (p_{\alpha}) : X \to X$ a morphism of $\text{pro-Top}_{n_0}$. We call $p$ an expansion over $B_0$ of the space $X$ over $B_0$ provided it has the following properties:

**E$_{n_0}$ 1.** For every ANR$_{n_0}$-space $P$ over $B_0$ and f.p. map $f : X \to P$ there is an index $\alpha \in \mathcal{A}$ and a f.p. map $h : X_{\alpha} \to P$ such that $h p_{\alpha} \cong f$.

**E$_{n_0}$ 2.** If $f, f' : X_{\alpha} \to P$ are f.p. maps, $P \in \text{ANR}_{n_0}$ and $f p_{\alpha} \cong f' p_{\alpha}$, then there is an index $\alpha' \geq \alpha$ such that $f p_{\alpha'} \cong f' p_{\alpha'}$.

**Definition 1.5** A morphism $p : X \to (X_{\alpha}, p_{\alpha}, \mathcal{A})$ is called a strong expansion over $B_0$ provided it satisfies condition $E_{n_0}$ 1) and the following condition:

**SE$_{n_0}$ 2.** Let $P$ be an ANR$_{n_0}$-space, let $f_0, f_1 : X_{\alpha} \to P$, $\alpha \in \mathcal{A}$ be f.p. maps and let $F : X \times I \to P$ be a f.p. homotopy such that

$$S(x, 0) = f_0 p_{\alpha}(x), \quad x \in X,$$
$$S(x, 1) = f_1 p_{\alpha}(x), \quad x \in X.$$

Then there exists a $\alpha' \geq \alpha$ and a f.p. homotopy $H : X_{\alpha'} \times I \to P$, such that

$$H(x, 0) = f_0 p_{\alpha'}(z), \quad z \in X_{\alpha'},$$
$$H(x, 1) = f_1 p_{\alpha'}(z), \quad z \in X_{\alpha'},$$
$$H(p_{\alpha'} \times 1) \cong S(\rel(X \times I)).$$

It is clear that, every strong expansion over $B_0$ is an expansion over $B_0$.

If all $X_{\alpha} \in \text{ANR}_{n_0}$, then $p$ is called an ANR$_{n_0}$-expansion and strong ANR$_{n_0}$-expansion,
respectively.

The main result of section 1 is the following theorem.

**Theorem 1.6** Let $X$ be a topological space over $B_0$. Then every resolution $p : X \to X$ over $B_0$ induces a strong ANR$_{B_0}$-expansion.

**Corollary 1.7** Every ANR$_{B_0}$-resolution over $B_0$ induces ANR$_{B_0}$-expansion.

**Corollary 1.8** Every space $X$ over $B_0$ admits a cofinite strong ANR$_{B_0}$-expansion.

In the proof of Theorem 1.6 we need the following lemma.

**Lemma 1.9** Let $X$ be a topological space over metrizable space $B_0$, let $f : X \to Y$, $h_0, h_1 : P \to P$ be f.p. maps and let $S : X \times I \to P$ be a f.p. homotopy such that

\[
S(x, 0) = h_0 f(x), \quad x \in X.
\]

\[
S(x, 1) = h_1 f(x), \quad x \in X.
\]

Then there exists an ANR$_{B_0}$-space $P'$, f.p. maps $f' : X \to P'$, $h : P' \to P'$ and a f.p. homotopy $K : P' \times I \to P$ such that

\[
h f' = f,
\]

\[
K(z, 0) = h_0 h(z), \quad z \in P'
\]

\[
K(z, 1) = h_1 h(z), \quad z \in P'
\]

\[
K(1 \times x) = S.
\]

**Lemma 1.10** Let $p : X \to X$ be a resolution over $B_0$ and let $\alpha, P, f_0, f_1$ and $F$ be as in SE$_{B_0}$ 2. Then for every open covering $\mathcal{U}$ of $P$, there exist a $\alpha' \geq \alpha$ and a f.p. homotopy $H : X_{\alpha'} \times I \to P$ such that

\[
H(y, 0) = f_0 p_{\alpha'}(y), \quad y \in X_{\alpha'}.
\]

\[
H(y, 1) = f_1 p_{\alpha'}(y), \quad y \in X_{\alpha'}.
\]

\[
(S, H(1 \times p_{\alpha'})) \leq \mathcal{U}
\]

2 On Fiber Strong Shape Category

The objects of category $\mathbf{SSH}_{B_0}$ are all topological spaces over $B_0$. The morphisms of category $\mathbf{SSH}_{B_0}$ are defined by the following way.

Let $p : X \to X$ and $q : Y \to Y$ be an ANR$_{B_0}$-resolution of $X$ and $Y$, respectively. Let $[f] : X \to Y$ be a morphism of category $\mathbf{CPHTop}_{B_0}$. Let $p' : X' \to X'$, $q' : Y' \to Y'$, $[f'] : X' \to Y'$ be another triple of fiber resolutions of spaces $X$ and $Y$ over $B_0$ and morphism of category $\mathbf{CPHTop}_{B_0}$.

Now define the following equivalence relation. We say the triples $(p, q, [f])$ and $(p', q', [f'])$ are equivalent if

\[
[f'] [i] = [j] [f],
\]

where $[i] : X \to X'$ and $[j] : Y \to Y'$ are isomorphisms of category $\mathbf{CPHTop}_{B_0}$.

The fiber strong shape morphisms $F : X \to Y$ are the equivalence classes of triples $(p, q, [f])$ with respect to the above defined relation $\sim$.

Let $F : X \to Y$ and $G : Y \to Z$ be the fiber strong shape morphisms, defined by triples $(p, q, [f])$ and $(q, r, [g])$.
and \((p', q', [g])\), where \(p' : Y \to Y\), \(q : Z \to Z\) and \([g] : Y \to Z\).

As we know there exists an unique morphism \([h] : Y \to Y'\) of category \(\text{CPHTop}_{B_0}\) such that 
\[ [j][q] = [q'] = [h][q]. \]

Hence, \([j] = [h]\). Besides, \([g][j] = [g][h][1_y]\).

Thus, we can assume that the morphisms \(F\) and \(G\) are given by triples \((p, q, [f])\) and \((p', q', [g])\).

Consequently, we can define the composition \(GF : X \to Z\) as the morphism given by triple \((p, p, [1_x])\).

Let \(X \in \text{ob}(\text{SSH}_{B_0})\). By symbol \(\text{ssh}_{B_0}(X)\) denote the equivalence class of topological space \(X\) and call the fiber strong shape of \(X\).

For each f.p. map \(\varphi : X \to Y\) choose ANR_{B_0}-resolutions \(p : X \to X\) and \(q : Y \to Y\). There exists a unique morphism \([f] : X \to Y\) of category \(\text{CPHTop}_{B_0}\) such that \([q][\varphi] = [f][p]\).

We can define a functor \(\text{SS}'_{B_0} : \text{Top}_{B_0} \to \text{SSH}_{B_0}\). By definition,
\[ \text{SS}'_{B_0}(X) = X, \quad X \in \text{ob}(\text{Top}_{B_0}) \]
and
\[ \text{SS}'_{B_0}([\varphi]) = \Phi, \quad \varphi \in \text{Mor}_{\text{Top}_{B_0}}(X, Y). \]

Here \(\Phi\) is a fiber strong shape morphism defined by triple \((p, q, [f])\).

As in [L-M] we can prove that functor \(\text{SS}'_{B_0}\) induces a functor \(\text{SS}_{B_0} : \text{HTop}_{B_0} \to \text{SSH}_{B_0}\), which we call the fiber strong shape functor. By definition,
\[ \text{SS}_{B_0}(X) = X, \quad X \in \text{ob}(\text{HTop}_{B_0}) \]
and
\[ \text{SS}_{B_0}([\varphi]) = \text{SS}'_{B_0}([\varphi]), \quad [\varphi]_{B_0} \in \text{Mor}_{\text{HTop}_{B_0}}(X, Y). \]

Let us define a functor \(S : \text{SSH}_{B_0} \to \text{SH}_{B_0}\). Assume that \(S(X) = X\) for each object \(X \in \text{ob}(\text{SSH}_{B_0})\). Let \(F : X \to Y\) be a fiber strong shape morphism given by a triple \((p, q, [f])\).

Consider the morphism \(E([f])\) as an image of \([f]\) with respect the functor \(E : \text{CPHTop}_{B_0} \to \text{pro-HTop}_{B_0}\). The triple \((Hp, Hq, E[f])\) generates a fiber shape morphism, which we denote by \(S(F) : X \to Y\).

Now we can formulate the following

**Theorem 2.5** There exists the following commutative diagram

\[
\begin{array}{ccc}
\text{SSH}_{B_0} & \xrightarrow{S} & \text{SH}_{B_0} \\
\text{HTop}_{B_0} & \xrightarrow{S} & \text{SSH}_{B_0} \\
\end{array}
\]

where \(S_{B_0}\) is V.Baladze fiber shape functor \([B_0]\). □

**Corollary 2.6** Let \(X\) and \(Y\) be topological spaces over \(B_0\). If \(\text{ssh}_{B_0}(X) = \text{ssh}_{B_0}(Y)\), then \(\text{sh}_{B_0}(X) = \text{sh}_{B_0}(Y)\). □

**Remark 2.7** Using the methods developed in this paper and papers ([B_6], [L-M], [M_1], [M_2]) it is possible to construct fiber strong shape theory for category of arbitrary continuous maps. □
REFERENCES


