6-2017

Which Topological Groups Arise as Automorphism Groups of Locally Finite Graphs?

Xiao Chang
University of Pittsburgh, xic58@pitt.edu

Paul Gartside
University of Pittsburgh

Follow this and additional works at: http://ecommons.udayton.edu/topology_conf
Part of the Geometry and Topology Commons, and the Special Functions Commons

This Topology + Algebra and Analysis is brought to you for free and open access by the Department of Mathematics at eCommons. It has been accepted for inclusion in Summer Conference on Topology and Its Applications by an authorized administrator of eCommons. For more information, please contact frice1@udayton.edu, mschlangen1@udayton.edu.
Which topological groups arise as automorphism groups of locally finite graphs?

Xiao Chang  
Paul Gartside  

University of Pittsburgh  

June 2017
A **topological group** $G$ is a group with Hausdorff topology such that multiplication and inversion are both continuous.

Let $\Gamma$ be a locally finite graph and $\tau_p$ be the pointwise topology. Then $(\text{Aut}(\Gamma), \tau_p)$ is a topological group.

$G$ is a **profinite group** if $G \cong \lim\leftarrow F_\lambda$ where $F_\lambda$ are finite groups with canonical group homomorphisms $\phi_{\lambda,\mu} : F_\lambda \to F_\mu$ if $F_\mu \leq F_\lambda$.

Profinite group $=$ compact, and 0-dimensional topological group.
**Frucht Theorem**

**Theorem (Frucht, 1939)**

*For every finite group $F$, there exists a finite graph $\Gamma$ such that $\varphi : \text{Aut}(\Gamma) \to F$ is an isomorphism.*

A **rigid graph** is a locally finite graph $R$ such that $\text{Aut}(R) = \{1\}$.

**Examples**

\[
\begin{array}{c}
\mathbb{Z}^2_2 \\
\begin{array}{c}
(0,1) \xleftrightarrow{} (1,1) \\
\end{array} \\
\begin{array}{c}
(0,0) \xleftrightarrow{} (1,0) \\
\end{array} \\
\end{array}
\]
Problem

What is Aut(Γ)? What kind of topological group is Aut(Γ)?

Know:
- If Γ is a locally finite graph and \( v \in \Gamma \), then Aut(Γ)\(_v\) is profinite.
- Further if Γ is a locally finite and countable graph, then Aut(Γ)\(_v\) is an open subgroup of countable index.

Looking for converse...

Let \( G \) be a separable metrizable topological group and let \( U \leq G \) be an open profinite subgroup.

Does there exist a countable locally finite graph Γ such that

\[
\text{Aut}(\Gamma) \cong G?
\]
Answer: YES!

Theorem

Let \( G \) be a separable metrizable topological group with an open profinite subgroup. Then \( \exists \) a countable, connected, locally finite graph \( \Gamma_G \) such that

\[
\text{Aut}(\Gamma) \cong G.
\]

Corollary

For every countable group \( G \), \( \exists \) a countable, connected, locally finite graph \( \Gamma \) s.t.

\[
\text{Aut}(\Gamma) \cong G.
\]

Theorem

A group \( G \) is topologically isomorphic to a separable metrizble topological group with an open profinite subgroup iff \( G \cong \text{Aut}(\Gamma) \) for some connected, locally finite graph.
1. Construct a colored and directed, countably, locally finite graph $C$.

2. Verify $\text{Aut}(C) \cong G$.

3. Replace the colored directed edges by rigid graphs to obtain $\Gamma$.

4. Verify that $\text{Aut}(\Gamma) \cong \text{Aut}(C)$. 
Part 1 - Construct $\mathcal{C}$ (Ingredients)

- $U \leq G$ open, profinite and metrizable.
  
  $U_0 = U \supseteq U_1 \supseteq \cdots \supseteq U_n \supseteq \cdots$ nested sequence of open subgroups of $U$ and $\bigcap_n U_n = \{1\}$.

- Countable dense $T = \{t_0, \ldots, t_n, \ldots\}$.
  Finite subset $T_n = \{t_0, \ldots, t_n\}$ with coloring function $c_n : T_n \to 2\mathbb{N}$ such that the colors are distinct between levels.

- Countably many levels of $C_n = G/U_n$ and $C_{\leq n} = \bigcup_{m \leq n} C_m$.

- $\mathcal{C} = \bigcup_n C_n$.

- ‘Horizontal’ edge from $hU_n$ to $htU_n$ with color $c_n(t)$ for $t \in T_n$.
  ‘Vertical’ edge from $h'U_{n+1}$ to $hU_n$ s.t. $hU_n \supseteq h'U_{n+1}$ with color $2n + 1$. 
Part 1 - Construct $\mathcal{C}$ (Key Properties)

- $\mathcal{C}$ is countable and locally finite and connected.

- For each $n$, $\mathcal{C}_n$ and $\mathcal{C}_{\leq n}$ are invariant under $\text{Aut}(\mathcal{C})$.

- If $\alpha : \mathcal{C} \to \mathcal{C}$ such that $\alpha|_{\mathcal{C}_{\leq n}}$ is an automorphism for all $n$, then $\alpha \in \text{Aut}(\mathcal{C})$. 
Part 2 - $\text{Aut}(C) \cong G$ (Definition)

Define $\text{pt}(\{g\}) = g$.

Define the topological isomorphisms

\[
\Psi : G \to \text{Aut}(C) \quad \text{by} \quad \Psi(g)(htU_n) = ghtU_n
\]

\[
\Phi : \text{Aut}(C) \to G \quad \text{by} \quad \Phi(\alpha) = \text{pt} \left( \bigcap_n \alpha(U_n) \right)
\]
$G, \text{Aut}(\mathcal{C})$ both have topology $\tau_p$. 
Enumerate finite, connected, rigid graphs \( \{ R_n : n \in \mathbb{N} \} \) such that \( |R_n| > |R_m| > 2 \) iff \( n > m \).

Replace colored-edges by isomorphism types of \( R_n \).

Call the resulting graph \( \Gamma \).

It has the same key properties as \( C \).
Hence \( \text{Aut}(\Gamma) \cong \text{Aut}(C)(\cong G) \).
What’s Next?

Construct a countable, connected, non-locally finite graph $\Gamma$ such that $\text{Aut}(\Gamma)$ is topologically isomorphic to a given Polish non-archimedean group.
Thank you