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Braid Group Actions on Rational Maps

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Joint with Sarah Koch.

PIctures of Julia sets due to Sarah Koch and Curt McMullen

Outline

- I. Combinatorics of Rational Maps
- II. Deformation Space
- III. Teichmüller Parameter Spaces
- IV. Braid Group Actions

Rational maps

A *degree d rational map* is a map

$$F : \mathbb{P}^1 \to \mathbb{P}^1$$

that can be written as a ratio $F(z) = G(z)/H(z)$ where G and H are polynomials in $\mathbb{C}[z]$ of maximum degree d.

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Question: To what extent is F determined by its action on $\mathcal{P}$?

Some examples

Template: $d = 2, \quad a \xrightarrow{2} a, \quad b \xrightarrow{2} c \longrightarrow b$

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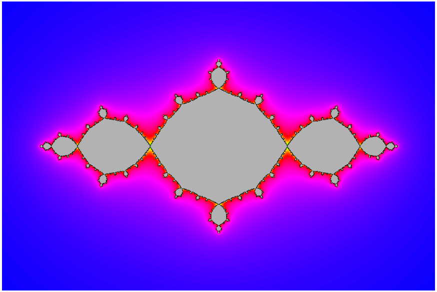

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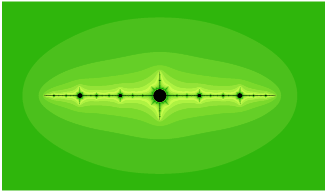
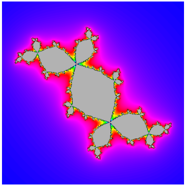
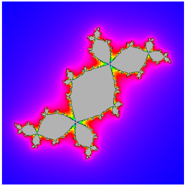

Question: How to distinguish these?

Branched coverings

A rational map $F : \mathbb{P}^1 \to \mathbb{P}^1$ defines a branched covering

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where P is a finite set containing the branch locus of f.
Two branched coverings of pairs $f_i : (S^2, P_i) \to (S^2, P_i)$, $i = 1, 2$, are *topologically equivalent* if there is a homeomorphism $\phi : (S^2, P_1) \to (S^2, P_2)$ so that the diagram commutes:

$$\begin{array}{ccc}
(S^2, P_1) & \xrightarrow{\phi} & (S^2, P_2) \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\
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\end{array}$$

Thurston equivalence for branched coverings

Two branched coverings of pairs $f_i : (S^2, P_i) \to (S^2, P_i)$, are *Thurston equivalent* if the diagram commutes:

$$\begin{array}{ccc} (S^2, P_1) & \xrightarrow{\psi} & (S^2, P_2) \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ (S^2, P_1) & \xrightarrow{\phi} & (S^2, P_2) \end{array}$$

where $\psi = \eta_2 \circ \phi \circ \eta_1$,

$$\eta_i : (S^2, P_i) \to (S^2, P_i)$$

are homeomorphisms isotopic to the identity map rel P_i, for $i = 1, 2$.

Thurston rigidity theorem for rational maps

Theorem (Thurston)

Let $F : \mathbb{P}^1 \to \mathbb{P}^1$ *be a post-critically finite map that is not Lattès. Then* F *is uniquely determined by the Thurston equivalence class of its associated branched covering*

$$f : (S^2, P) \to (S^2, P).$$

Partially post-critically finite rational maps.

Consider rational maps $F : (\mathbb{P}^1, A) \to (\mathbb{P}^1, B)$, where $A \subset B$ are finite sets and $\mathrm{Crit}_F \subset B$.

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Two branched coverings of pairs $f_i : (S^2, A_i) \to (S^2, B_i)$ are *combinatorially equivalent* if

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Deformation Space

Fix a branched covering of pairs $f : (S^2, A) \to (S^2, B)$.

The *deformation space of f* is defined by

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i.e., $F \in D_f$ if and only if there are homeomorphisms $\phi, \psi : S^2 \to \mathbb{P}^1$ such that

$$\begin{array}{ccc} (S^2, A) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(A)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ (S^2, B) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(B)) \end{array}$$

commutes. This gives an *f-marking* (ϕ, ψ) of F as a rational map of pairs, uniquely defined up to isotopy rel B (resp., rel A).

Teichmüller space

Let $\mathcal{T}_A = \mathrm{Teich}(S^2, A) = \mathsf{Homeo}(S^2, \mathbb{P}^1)/\sim_A$.

Fix $f : (S^2, A) \to (S^2, B)$.

The *Thurston lifting map* is a holomorphic map $\sigma_f : \mathcal{T}_B \to \mathcal{T}_A$ such that

$$\begin{array}{ccc} (S^2, A) & \xrightarrow{\sigma_f(\phi)} & (\mathbb{P}^1, \sigma_f(\phi)(A)) \\ \downarrow f & & \downarrow F \\ (S^2, B) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(B)) \end{array}$$

commutes.

Define $D_f \hookrightarrow \mathcal{T}_B$ that takes each $F \in D_f$ to $[\phi] \in \mathcal{T}_B$, where (ϕ, ψ) is an f-marking for F. Identify D_f with its image: $D_f \subset \mathcal{T}_B$.

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Let $\mathrm{Per}_n^d(0)^*$ be the subspace of $\mathrm{Per}_n^d(0)$ containing rational maps of degree d with one critical orbit A of order n, and which has a critical value outside of A. Fix $F \in \mathrm{Per}_n^d(0)^*$ and let

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So, D_f is connected $\implies \mathrm{Per}_n^d(0)^*$ is connected $\iff \mathrm{Per}_n^d(0)$ is connected.

Counter-example to connectedness of D_f

Theorem (E.H. - S. Koch)

Take $F \in \mathrm{Per}_4^2(0)^*$, *and let* $f : (S^2, A) \to (S^2, B)$ *a topological element in the combinatorial equivalence class. Then* D_f *has infinitely many connected components.*

Main Elements of the Proof

- An intermediate space of rational maps
- Braid group actions on rational maps and branched coverings
- Fundamental groups of complements of plane algebraic curves

Intermediate space of rational maps

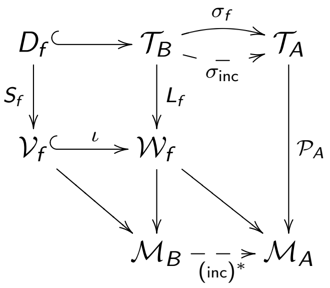
Recall that D_f can be thought of as f-marked rational maps (ϕ, ψ, F). Let

$$\mathcal{W}_f = \{(\phi|_B, \psi|_A, F) \mid \phi \in \mathcal{T}_B, \psi = \sigma_f(\phi)\}$$

and

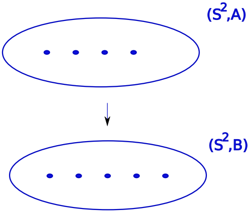
$$\mathcal{V}_f = \{(\phi|_B, \psi|_A, F) \in \mathcal{W}_f \mid \psi \sim_A \phi\}.$$

These play the role of moduli space (rather than Teichmüller space) and fit in the following diagram:



Braid group actions

Let $\mathcal{P}_B = \mathrm{Mod}(S^2, B)$ the *pure braid group*. Fix $f : (S^2, A) \to (S^2, B)$. We say $h \in \mathcal{P}_B$ *lifts* to $h' \in \mathcal{P}_A$ if $h \circ f = f \circ h'$.



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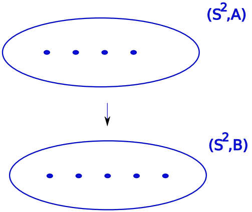

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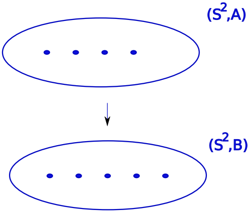

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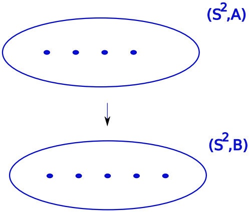

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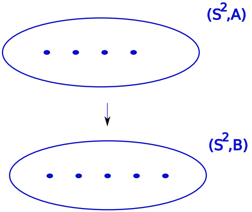

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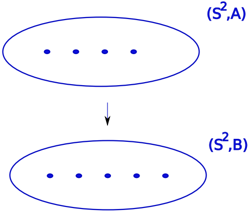
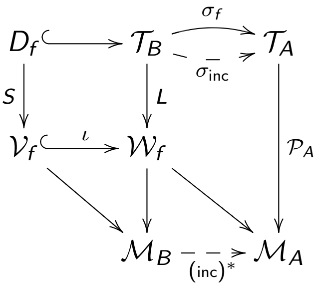
E_f = *equalizer*

$= \{h \in S \ : \ \Phi(h_t) \sim_A h_t \quad \forall t \in [0,1]\}$

Connectivity and coverings

L_f = *liftables*
S_f = *special liftables*
E_f = *equalizer*





$E_f = \iota_*(\pi_1(\mathcal{V}_f))$
Covering Space Lemma: $|D_f| \geq [S_f : E_f]$.

Coordinates for $\mathcal{V}_f$.

Take $f : S^2 \to S^2$ a branched cover of degree 2 with distinct critical points p, q, such that p lies in a degree 4 orbit A and $q, f(q) \notin A$.

In order for a rational map F to have a marking by f, we would have to have

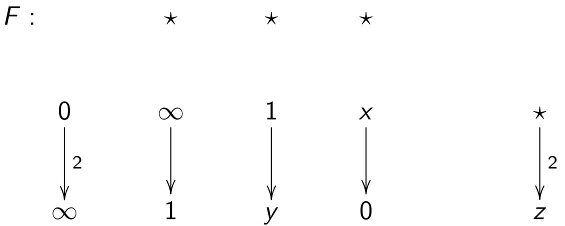
$$F: \; 0 \xrightarrow{2} \infty \longrightarrow 1 \longrightarrow x \longrightarrow 0 \qquad\qquad q \xrightarrow{2} z$$

and hence $F(z) = \frac{(z-x)(z-r)}{z^2}$

- $r = \frac{x}{x-1} + 1$, only depends on x
- $q = \frac{2xr}{x+r}$, only depends on x

Write $\mathcal{V}_f = \{(x, F)\} \simeq \mathbb{C} \setminus K$, where K is a finite set of points.

Coordinates for $\mathcal{W}_f$



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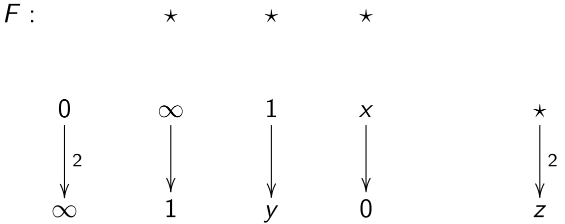


$\Rightarrow F(z) = \frac{(z-x)(z-r)}{z^2}$ where

- r depends on x and y
- z depends on r

Write $\mathcal{W}_f = \{((y, z), x, F)\} \simeq \mathbb{C}^2 \setminus \mathcal{C}$, where $\mathcal{C}$ is an algebraic curve.

Affine embedding of $\mathcal{V}_f$ and $\mathcal{W}_f$

The coordinates give

$$\begin{array}{rcl} \mathcal{V}_f & \hookrightarrow & \mathcal{W}_f \\ (x, F) & \mapsto & ((y, z), x, F). \end{array}$$

and

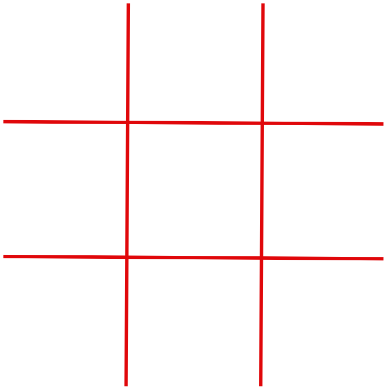
$$\mathcal{V}_f \subset \mathcal{W}_f \hookrightarrow \mathcal{M}_B \times \mathcal{M}_A \to \mathcal{M}_A \times \mathcal{M}_A$$

where

$$\begin{array}{rcl} \mathcal{W}_f & \hookrightarrow & \mathcal{M}_A \times \mathcal{M}_A \\ ((y, z), x, F) & \mapsto & (y, x) \end{array}$$

Useful Property: $\mathcal{W}_f \to \mathcal{M}_A \times \mathcal{M}_A = (\mathbb{C} \setminus \{0, 1\}) \times (\mathbb{C} \setminus \{0, 1\})$ is an injection!

Seeing $\mathcal{W}_f$ and $\mathcal{V}_f$ in $\mathbb{C}^2$.



Picture of $\mathcal{M}_A \times \mathcal{M}_A$.

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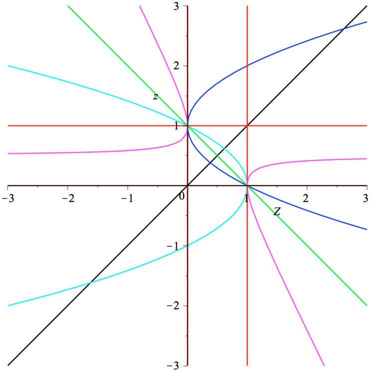

- $\mathcal{W}_f$ = complement of the colored curves
- $\mathcal{V}_f$ = diagonal $\cap\, \mathcal{W}_f$
- $\mathcal{W}_f \to \mathcal{M}_A$ are just projections p_1 and p_2 to vertical and horizontal coordinates.

Finding L_f, S_f and E_f?

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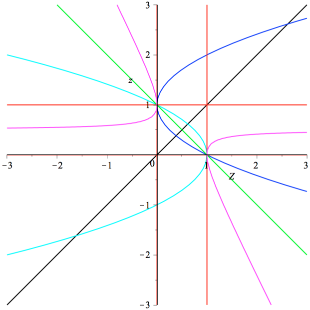

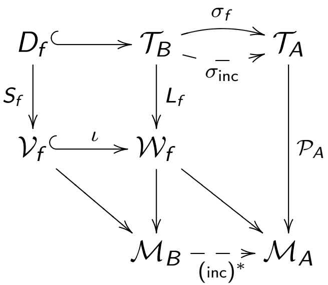

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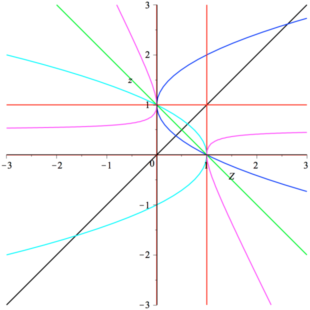

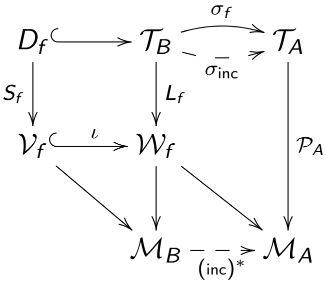

$$L_f = \pi_1(\mathcal{W}_f).$$

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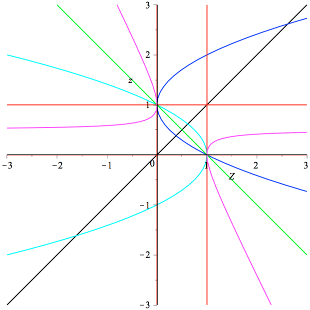
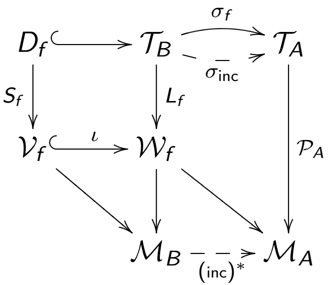


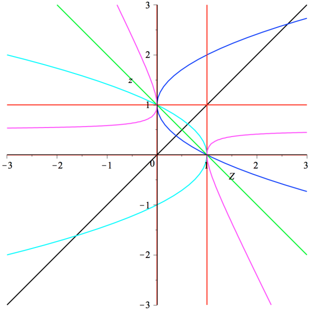


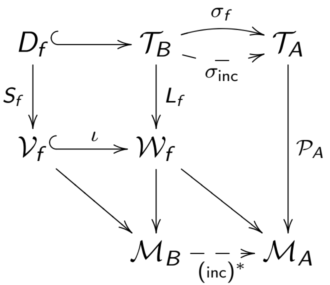
$L_f = \pi_1(\mathcal{W}_f)$.

$S_f = \text{Equalizer}((p_1)_*, (p_2)_*)$

Finding L_f, S_f and E_f?

Goal: Find $[S_f : E_f]$.





$$L_f = \pi_1(\mathcal{W}_f).$$
$$S_f = \text{Equalizer}((p_1)_*, (p_2)_*)$$
$$E_f = \iota_*(\pi_1(\text{Equalizer}(p_1, p_2)))$$

Finding L_f, S_f and E_f?

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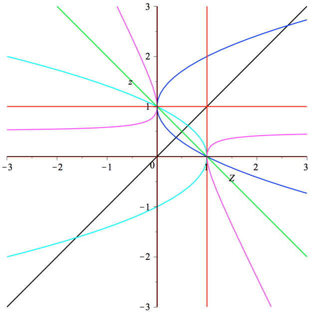

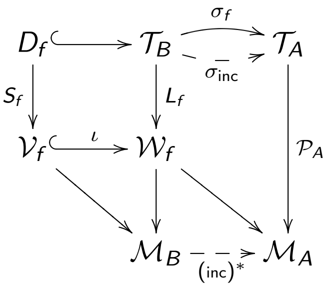

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Question: Is the fundamental group of the equalizer equal to the equalizer of the fundaental groups?

Proof of Theorem

Claim: $[S_f : E_f] = \infty$.

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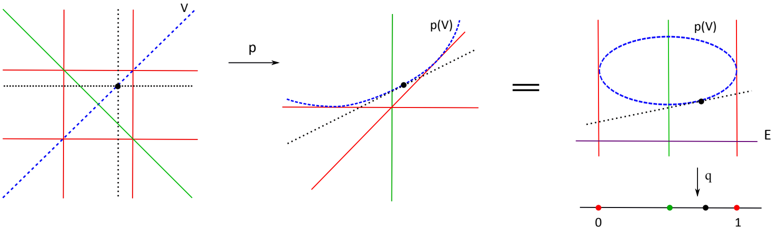
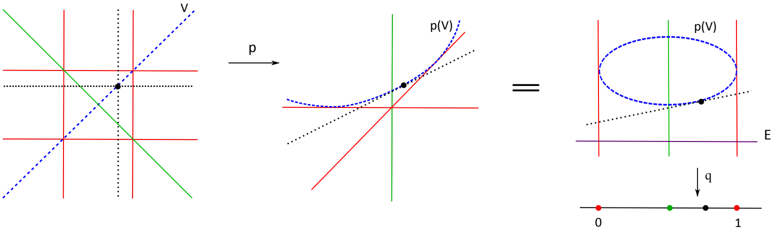
Reduce to an easier topological situation.



- Left picture: fill in some of the deleted curves of $\mathcal{W}_f$ to get $\widehat{\mathcal{W}_f}$
- Right picture: take the quotient by symmetry across diagonal to get $\overline{\mathcal{W}}_f$.

Proof

Claim: $[\pi_1(\mathcal{W}_f) : \iota_*(\pi_1(\mathcal{V}_f))] = \infty$.

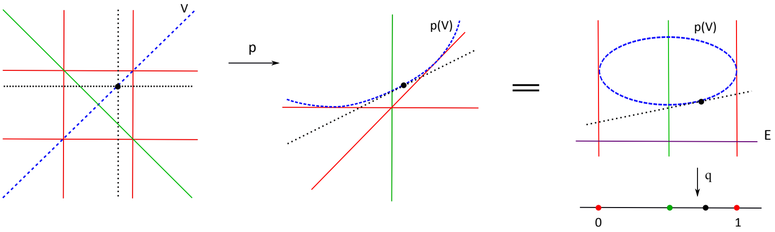


$$\begin{array}{ccccccc} & & & \pi_1(\mathcal{V}_f) & & & \\ & \swarrow^{\iota} & & \downarrow & \searrow & & \searrow^{\text{1-1}} \\ \pi_1(\mathcal{W}_f) & \twoheadrightarrow & \pi_1(\widehat{\mathcal{W}_f}) & \twoheadrightarrow & \pi_1(\overline{\mathcal{W}_f})^q & \twoheadrightarrow & \mathcal{F}_2 \end{array}$$

- $(q \circ \iota)_* : \pi_1(\mathcal{V}_f) \hookrightarrow \pi_1(\mathbb{P}^1 \setminus \{3 \text{ points}\}) = \mathcal{F}_2$
- $1 \to \mathbb{Z} \to \pi_1(\overline{\mathcal{W}_f}) \to \mathcal{F}_2 \to 1$, central extension.

Proof

Claim: $[\pi_1(\mathcal{W}_f) : \iota_*(\pi_1(\mathcal{V}_f))] = \infty$.



- $\exists \gamma \in \pi_1(\mathcal{W}_f)$ of infinite order, that maps to a central element of $\pi_1(\overline{\mathcal{W}}_f)$.
- $\gamma\delta(\gamma) \in S_f$, and maps to the square of a central element in $\pi_1(\overline{\mathcal{W}}_f)$.
- It follows that $[S_f : E_f] = \infty$.

A curious lemma

Lemma

Any element of S_f not in E_f is of the form $\gamma_1\gamma_2$ where γ_1 acts trivially on the base of the covering rel A but non-trivially on the covering (rel A) and γ_2 acts trivially on the covering (rel A) but is non-trivial on the base.

Open Problems

- (Milnor) Is $\mathrm{Per}_n^d(0)$ connected for all n and d?
- Study properties of liftables, special liftables and the equalizer subgroups of the braid group.
- Is there something special about degree two?
- Is the deformation space connected in augmented Teichmüller space?

Thank you for listening