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Uncountable Discrete Sets and Forcing

Akira Iwasa

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Uncountable discrete sets and forcing

Akira Iwasa
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**Background**

\[ \textbf{V}: \text{ Ground Model} \]

\[ \textbf{V}^\mathbb{P}: \text{ Extension of } \textbf{V} \text{ by forcing } \mathbb{P} \]

Consider a topological space \((X, \tau)\) in \(\textbf{V}\).

We define a topological space \((X, \tau^\mathbb{P})\) in \(\textbf{V}^\mathbb{P}\) such that

\[ \tau^\mathbb{P} = \text{ the topology generated by } \tau \]

**Observation.**

- \(\tau \subsetneq \tau^\mathbb{P}\) New open sets are added by forcing \(\mathbb{P}\).
- \(\tau\) is a base for \(\tau^\mathbb{P}\).

I am interested in comparing \((X, \tau)\) and \((X, \tau^\mathbb{P})\).
Consider a topological space \((X, \tau)\) in \(V\).

We define a topological space \((X, \tau^P)\) in \(V^P\) such that \(\tau^P = \text{the topology generated by } \tau\).

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- $(X, \tau)$ is Hausdorff $\implies (X, \tau^P)$ is Hausdorff
  "Hausdorffness is preserved by any forcing"

- $(X, \tau)$ is regular $\implies (X, \tau^P)$ is regular

- $(X, \tau)$ is completely regular $\implies (X, \tau^P)$ is completely regular

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Question: Suppose that a space $X$ has no uncountable discrete subspace.

Can forcing create an uncountable discrete subspace of $X$?

In other words:
Suppose that a space $(X, \tau)$ has no uncountable discrete subspace. For some forcing $\mathbb{P}$, can $(X, \tau^\mathbb{P})$ have an uncountable discrete subspace?

No ZFC example so far.
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Definition. A space $X$ has **countable chain condition** (CCC) if every pairwise disjoint family of open sets is countable.

**Observation:** A space has no uncountable discrete subspace if and only if $X$ is **hereditarily CCC**.

So our question can be rephrased as:

**Can forcing destroy hereditarily CCC?**
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Can forcing destroy hereditarily CCC?
HC HL HS

Notation:
- Hereditarily CCC (HC) ⇐⇒ No uncountable discrete subspace
- Hereditarily Lindelof (HL)
- Hereditarily separable (HS)

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\begin{align*}
\text{HL} & \quad \Rightarrow \quad \text{HC} \\
\text{HS} & \quad \Rightarrow
\end{align*}
\]

- HC, HL and HS are similar properties. In fact,

**If forcing destroys HS or HL, then it destroys HC.**
That is, if forcing destroys HS or HL, then it would create an uncountable discrete subspace. **So try to destroy HL or HS.**
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L-space and S-space

- Hereditarily CCC (HC)
- Hereditarily Lindelof (HL)
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**Definition.** L-space = HL but not HS

There is an L-space in ZFC. (Moore)
But forcing cannot destroy Moore’s L-space.
(Tsaban, Zdomskyy)

**Definition.** S-space = HS but not HL.

CH implies there is an S-space.
PFA implies there is no S-space. (Todorcevic)
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Example. Souslin Line

- ♦ implies there is a Souslin line.
- Souslin line is an $L$-space (HL but not HS).
- $\exists$ Souslin line $\iff \exists$ Souslin tree
- Forcing with a Souslin tree destroys Lindelofness of the Souslin line, and so it creates an uncountable discrete subspace of the Souslin line.
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- Strong HFD space is an S-space (a subspace of $2^{\omega_1}$).
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Destroying S-space

Example. (Juhasz) (CH) **Strong hereditarily finally dense (HFD) space.**

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- There is a ccc forcing that destroys hereditarily separability of the strong HFD space, and so the space gets an uncountable discrete subspace in the forcing extension.
How about spaces which are both HL and HS?

Example. Filippov Space.  (1969)

• $E \subseteq [0, 1] \times [0, 1]$ is **Luzin** if every nowhere dense subset is countable.
• CH implies there is a Luzin set.
• Filippov space is:

$$X_E = (E \times S) \cup ([0, 1]^2 \setminus E),$$

where $S$ is the unit circle.

An neighborhood of $(x_1, x_2) \in E \times S$ looks like:
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**Definition.** $E \subseteq [0, 1] \times [0, 1]$ is weakly Luzin if for $E' \subseteq E$, whenever
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is not dense in the unit circle $S$, $E'$ is countable. Luzin sets are weakly Luzin.

**Theorem.** (Kunen) The following are equivalent:

1. $E$ is weakly Luzin.
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Example. (CH + ∃Super compact cardinal) There is a HL and HS space $X$ and a proper forcing $\mathbb{P}$ such that in $V^\mathbb{P}$, $X$ has an uncountable discrete subspace.

Proof.

• CH implies there is a Luzin set $E$.

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Spaces where it is impossible to shoot an uncountable discrete subspace

Spaces where it is **impossible** to shoot an uncountable discrete set by forcing.

- Metrizable.
- Developable.
- Stratifiable.

All the above spaces have a **countable network** if they have no uncountable discrete subspace, and forcing preserves countable network.

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**Theorem.** (Borges) \( \langle X, \tau \rangle \) is **monotonically normal** if and only if for a base \( B \) for \( X \), there is an operator \( H(x, B) \), where \( B \in B \) and \( x \in B \) such that

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H(x, B) \cap H(x', B') \neq \emptyset \implies x \in B' \text{ or } x' \in B.
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Forcing preserves a base of a topology, and so the operator \( H \) will have the same property in any forcing extension. Therefore,

**Forcing preserves monotone normality.**
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Here are useful theorems.

**Theorem.** (Williams, Zhou) The following are equivalent:
1. There is no Souslin tree.
2. Every CCC monotonically normal space is separable.

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**Theorem.** (No Souslin tree) Let $X$ be a monotonically normal space with no uncountable discrete subspace. Then it is impossible to create an uncountable discrete subspace of $X$ by forcing.

**Proof.**
- Suppose that $X$ is a monotonically normal space with no uncountable discrete subspace. In particular, $X$ is CCC.
- By Williams and Zhou’s theorem, $X$ is separable.
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Corollary. (No Souslin tree) Suppose that $X$ is a linearly ordered topological space (LOTS) with no uncountable discrete subspace. Then it is impossible to create an uncountable discrete subspace of $X$ by forcing.

Proof. Linearly ordered topological spaces are monotonically normal. □
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**Proof.** Linearly ordered topological spaces are monotonically normal. □
Definition. A space $X$ is scattered if every subspace contains an isolated point in the relative topology.

Lemma. Assume that there is no $S$-space. Then every scattered space with no uncountable discrete subspace is countable.

Theorem. Assume that there is no $S$-space. Let $X$ be a scattered space with no uncountable discrete subspace. Then it is impossible to create an uncountable discrete subspace of $X$ by forcing (because $X$ is countable).

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(CH) **Kunen Line** is a scattered cometrizable $S$-space.

**Theorem.** (Todocevic) $MA(\aleph_1)$ implies that there is no cometrizable $S$-space.
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Proof.

• Let $X$ be the Kunen line, which is a scattered cometrizable $S$-space.

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- Forcing preserves cometrizability so by the Todorcevic’s theorem, $X$ is not an $S$-space in the forcing extension.
- Hence, $X$ gets an uncountable discrete subspace in the forcing extension. □
Example. (CH & $2^{\aleph_1} = \aleph_2$) There are a scattered $S$-space $X$ with no uncountable discrete subspace and a ccc forcing such that in the forcing extension, $X$ contains an uncountable discrete subspace.

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Can forcing shoot an uncountable discrete set?

(1). NO.
  • Metrizable; developable; stratifiable

(2) Consistently, NO.
  • Monotonically normal; LOTS (if there is no Souslin tree)
  • Scattered (if there is no $S$-space)

(3) Consistently, YES.
  • Monotonically normal; LOTS; scattered
  • Compact; quasi-metrizable; non-archimedean
    (Souslin line can have these properties.)
  • Submetrizable (cometrizable)

(4) So far no ZFC example of a space where forcing create an uncountable discrete subspace.
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Thank you for your attention.