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ABSTRACT

A new free-space multistage optical interconnection network which is called the Comega interconnection network is presented. It has the same topological construction for the cascade stages of the Comega interconnection. The concept of the left Comega and the right Comega interconnection networks are given to describe the whole Comega interconnection network. The matrix theory for the Comega interconnection network is presented. The route controlling of the Comega interconnection network is decided based on the matrix analysis. The node switching states in cascade stages of the 8×8 Comega interconnection network for the route selection are given. The data communications between arbitrary input channel with arbitrary output channel can be performed easily.

Keywords: Optical interconnection, Photonic switching, the multistage Comega interconnection network, Matrix theory, Route control

1. INTRODUCTION

Over the years, free-space optical interconnects have received considerable interest for their advantage over the electrical interconnects. Multistage interconnection networks (MIN's) are members of the class of regular interconnection network. They are widely used in data communications, optical computing and information processing. Various MIN's such as the Omega network, the crossover network and the banyan network have been proposed[1-3]. In this paper, we propose a new multistage optical interconnect network which is called the Comega network. It has the same topological construction for each cascade stage, which is similar to the Omega network. The input ports are duplicated and then rearranged so as to one copy of the input ports follows the other one. The Comega network can be divided into two parts: the left Comega network and the right Comega network. A mathematical description is given for the left and the right Comega network functions. The matrix analysis for the Comega network is made in this paper. The route controlling of the Comega interconnection network is decided based on the matrix analysis. The node switching states in cascade stages of the 8×8 Comega interconnection network for the route selection are given. As the configuration of the Comega network is very simple, it can be implemented easily by a 1×2 binary phase grating (BPG) under this arrangement. All of the light beams split by the BPG are used in the Comega network without loss. As the topological construction of each cascade stage for the multistage Comega network is quite the same, the complexities of the hardware are reduced. It can be implement the whole free-space multistage Comega interconnection network by a single stage recirculating setup easily[4].

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2. CONCEPT OF COMEGA INTERCONNECTION NETWORK

The Comega interconnection network is a kind of novel free-space multistage regular optical interconnection network. It has the feature of stage-invariable like the Omega interconnection network[1]. For a $N \times N$ Comega interconnection network there are $\log_2 N$ interconnection stages and $N\log_2 N$ switching nodes. Each cascade interconnection stage has the same configuration. An 8×8 Comega interconnection network is shown in Fig. 1. Each interconnection stage can be described by two permutations: the left Comega interconnection network and the right Comega interconnection network, which are shown in Fig. 2(a) and (b). The mathematical expressions for the left Comega interconnection network and the right Comega interconnection network are given by the equations (1) and (2), respectively.

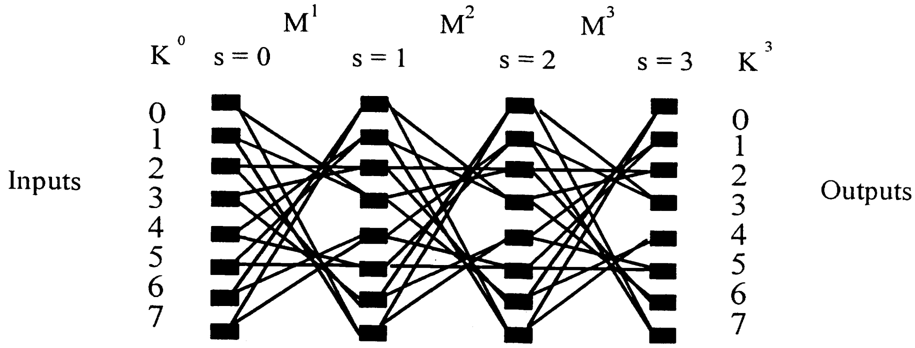
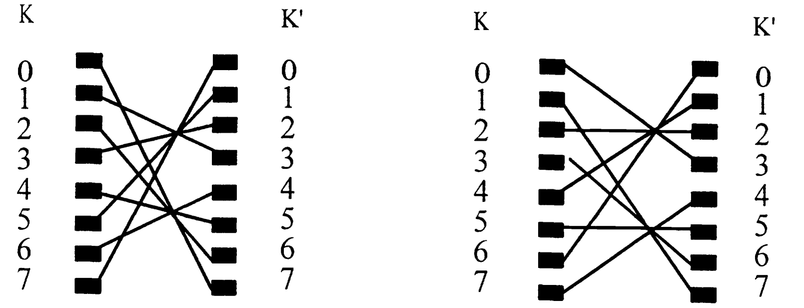


Fig. 1. The 8×8 Comega interconnection network



(a) (b)

Fig. 2. (a) the left Comega interconnection network and (b) the right Comega interconnection network

For the left Comega interconnection network:

$$K' = (N-1-[K/2]+4K)MOD(N) \tag{1}$$

For the right Comega interconnection network:

$$K' = (N-5-[K/2]+4K)MOD(N) \tag{2}$$

Where [x] means maximum integral of less than or equal to x, yMOD(N) means getting remainder of y divided by N.

3. MATRIX DESCRIPTION OF COMEGA INTERCONNECTION NETWORK

In order to gain insight into the Comega interconnection network, a matrix analysis approach can be used to describe the Comega interconnection network[5]. A column vector K^s is used to denote the permutation of the input and output channels in each stage:

$$K^s = (k_0^s, k_1^s, k_2^s, k_3^s, k_4^s, k_5^s, k_6^s, k_7^s)^T \qquad (s= 0, 1, 2, 3) \tag{3}$$

where k_i^s (i = 0, 1, 2, 3, 4, 5, 6, 7) is the order number of the s-th stage. We can describe the operations in each stage of the network with the transform matrix M^s of size 8 × 8.

$$K^s = M^s K^{s-1} \qquad (s = 1, 2, 3) \tag{4}$$

According to the notation of the Comega interconnection network, each stage of the Comega interconnection network consists of left and right Comega interconnection networks which are shown in Fig. 2(a) and (b), so the characteristic matrix $N_{Co}(i, j)$ of the Comega network can be expressed by

$$N_{Co}(i, j) = N_{Co,\,l}(i, j) + N_{Co,\,r}(i, j) \tag{5}$$

Where the matrix $N_{Co,\,l}(i, j)$ corresponds to the left Comega interconnection network, and the matrix $N_{Co,\,r}(i, j)$ corresponds to the right Comega interconnection network. The matrices $N_{Co,\,l}(i, j)$ and $N_{Co,\,r}(i, j)$ can be expressed as follows:

$$N_{Co,\,l}(i, j) = \begin{cases} n_j(l) & (for \quad j = (N-1-[i/2]+4i)MOD(N) \\ 0 & otherwise, \qquad (i, j = 0,1,2,......N-1) \end{cases} \tag{6}$$

$$N_{Co,\,r}(i, j) = \begin{cases} n_j(r) & (for \quad j = (N-5-[i/2]+4i)MOD(N) \\ 0 & otherwise, \qquad (i, j = 0,1,2,......N-1) \end{cases} \tag{7}$$

For the 8×8 Comega interconnection network, three stages of the Comega interconnection network are needed. Each stage of the Comega interconnection network has the same interconnection pattern, the representation of the transform matrix M^s of the s-th stage is as follows:

$$M^s = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & n_6^s(r) & n_7^s(l) \\ 0 & 0 & 0 & 0 & n_4^s(r) & n_5^s(l) & 0 & 0 \\ 0 & 0 & n_2^s(r) & n_3^s(l) & 0 & 0 & 0 & 0 \\ n_0^s(r) & n_1^s(l) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_6^s(l) & n_7^s(r) \\ 0 & 0 & 0 & 0 & n_4^s(l) & n_5^s(r) & 0 & 0 \\ 0 & 0 & n_2^s(l) & n_3^s(r) & 0 & 0 & 0 & 0 \\ n_0^s(l) & n_1^s(r) & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \qquad (s=1, 2, 3) \tag{8}$$

where $n_i^s(l)$ and $n_i^s(r)$ denote the switching state of the i-th switching node in the s-th stage. They must satisfy the following conditions.

$n_i^s(l) = 1$, and $n_i^s(r) = 0$ for the left Comega interconnection state,

$n_i^s(l) = 0,$ and $n_i^s(r) = 1$ for the right Comega interconnection state

As a result, the input-output relationship of the 8×8 Comega interconnection network is given by

$$K^3 = M^3 M^2 M^1 K^0 \tag{9}$$

From the equations shown above, we can see that the matrix theory for the multistage Comega interconnection network provides a convenient way to understand the features of the Comega interconnect network.

4. THE ROUTE CONTROLLING OF COMEGA SWITCHING NETWORK

In order to realize the data signal transmission from arbitrary input channel to arbitrary output channel, the switching states of the switching nodes of each node stage must be selected. In this section, we will use the matrix theory of the Comega interconnection network to describe the routing selection and controlling of the 8×8 Comega switching network. The input and output channels of each stage of the Comega interconnection network can be expressed by the input vector K^0 and output vector K^s (s=1, 2, 3) as follows:

$$K^0 = [A, B, C, D, E, F, G, H]^T \tag{10}$$

$$K^s = [A^s, B^s, C^s, D^s, E^s, F^s, G^s, H^s]^T \tag{11}$$

The output vector K^1 of the first stage of the Comega interconnection network can be expressed by:

$$K^1 = M^1 K^0$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & n_6^1(r) & n_7^1(l) \\ 0 & 0 & 0 & 0 & n_4^1(r) & n_5^1(l) & 0 & 0 \\ 0 & 0 & n_2^1(r) & n_3^1(l) & 0 & 0 & 0 & 0 \\ n_0^1(r) & n_1^1(l) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_6^1(l) & n_7^1(r) \\ 0 & 0 & 0 & 0 & n_4^1(l) & n_5^1(r) & 0 & 0 \\ 0 & 0 & n_2^1(l) & n_3^1(r) & 0 & 0 & 0 & 0 \\ n_0^1(l) & n_1^1(r) & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{bmatrix} \tag{12}$$

That is

$$\begin{bmatrix} A^1 \\ B^1 \\ C^1 \\ D^1 \\ E^1 \\ F^1 \\ G^1 \\ H^1 \end{bmatrix} = \begin{bmatrix} n_6^1(r)G + n_7^1(l)H \\ n_4^1(r)E + n_5^1(l)F \\ n_2^1(r)C + n_3^1(l)D \\ n_0^1(r)A + n_1^1(l)B \\ n_6^1(l)G + n_7^1(r)H \\ n_4^1(l)E + n_5^1(r)F \\ n_2^1(l)C + n_3^1(r)D \\ n_0^1(l)A + n_1^1(r)B \end{bmatrix} \tag{13}$$

The output vector K^1 of the first stage of the Comega interconnection network is also the input vector of the second stage of the Comega interconnection network, so the output vector K^2 of the second stage of the Comega interconnection network is expressed by:

$$K^2 = M^2 K^1$$

$$= M^2 M^1 K^0$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & n_6^2(r) & n_7^2(l) \\ 0 & 0 & 0 & 0 & n_4^2(r) & n_5^2(l) & 0 & 0 \\ 0 & 0 & n_2^2(r) & n_3^2(l) & 0 & 0 & 0 & 0 \\ n_0^2(r) & n_1^2(l) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_6^2(l) & n_7^2(r) \\ 0 & 0 & 0 & 0 & n_4^2(l) & n_5^2(r) & 0 & 0 \\ 0 & 0 & n_2^2(l) & n_3^2(r) & 0 & 0 & 0 & 0 \\ n_0^2(l) & n_1^2(r) & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A^1 \\ B^1 \\ C^1 \\ D^1 \\ E^1 \\ F^1 \\ G^1 \\ H^1 \end{bmatrix} \tag{14}$$

That is

$$\begin{bmatrix} A^2 \\ B^2 \\ C^2 \\ D^2 \\ E^2 \\ F^2 \\ G^2 \\ H^2 \end{bmatrix} = \begin{bmatrix} n_6^2(r)[n_2^1(l)C + n_3^1(r)D] + n_7^2(l)[n_0^1(l)A + n_1^1(r)B] \\ n_4^2(r)[n_6^1(l)G + n_7^1(r)H] + n_5^2(l)[n_4^1(l)E + n_5^1(r)F] \\ n_2^2(r)[n_2^1(r)C + n_3^1(l)D] + n_3^2(l)[n_0^1(r)A + n_1^1(l)B] \\ n_0^2(r)[n_6^1(r)G + n_7^1(l)H] + n_1^2(l)[n_4^1(r)E + n_5^1(l)F] \\ n_6^2(l)[n_2^1(l)C + n_3^1(r)D] + n_7^2(r)[n_0^1(l)A + n_1^1(r)B] \\ n_4^2(l)[n_6^1(l)G + n_7^1(r)H] + n_5^2(r)[n_4^1(l)E + n_5^1(r)F] \\ n_2^2(l)[n_2^1(r)C + n_3^1(l)D] + n_3^2(r)[n_0^1(r)A + n_1^1(l)B] \\ n_0^2(l)[n_6^1(r)G + n_7^1(l)H] + n_1^2(r)[n_4^1(r)E + n_5^1(l)F] \end{bmatrix} \tag{15}$$

Similarly, the output vector K^3 of the third stage of the Comega interconnection network is expressed by:

$$K^3 = M^3 K^2$$

$$= M^3 M^2 M^1 K^0$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & n_6^3(r) & n_7^3(l) \\ 0 & 0 & 0 & 0 & n_4^3(r) & n_5^3(l) & 0 & 0 \\ 0 & 0 & n_2^3(r) & n_3^3(l) & 0 & 0 & 0 & 0 \\ n_0^3(r) & n_1^3(l) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_6^3(l) & n_7^3(r) \\ 0 & 0 & 0 & 0 & n_4^3(l) & n_5^3(r) & 0 & 0 \\ 0 & 0 & n_2^3(l) & n_3^3(r) & 0 & 0 & 0 & 0 \\ n_0^3(l) & n_1^3(r) & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A^2 \\ B^2 \\ C^2 \\ D^2 \\ E^2 \\ F^2 \\ G^2 \\ H^2 \end{bmatrix} \tag{16}$$

That is

$$
\begin{bmatrix} A^3 \\ \\ B^3 \\ \\ C^3 \\ \\ D^3 \\ \\ E^3 \\ \\ F^3 \\ \\ G^3 \\ \\ H^3 \\ \\ \end{bmatrix}
=
\begin{bmatrix}
n_6^3(r)\{n_2^2(l)[n_2^1(r)C + n_3^1(l)D] + n_3^2(r)[n_0^1(r)A + n_1^1(l)B]\} + \\
n_7^3(l)\{n_0^2(l)[n_6^1(r)G + n_7^1(l)H] + n_1^2(r)[n_4^1(r)E + n_5^1(l)F]\} \\
n_4^3(r)\{n_6^2(l)[n_2^1(l)C + n_3^1(r)D] + n_7^2(r)[n_0^1(l)A + n_1^1(r)B]\} + \\
n_5^3(l)\{n_4^2(l)[n_6^1(l)G + n_7^1(r)H] + n_5^2(r)[n_4^1(l)E + n_5^1(r)F]\} \\
n_2^3(r)\{n_2^2(r)[n_2^1(r)C + n_3^1(l)D] + n_3^2(l)[n_0^1(r)A + n_1^1(l)B]\} + \\
n_3^3(l)\{n_0^2(r)[n_6^1(r)G + n_7^1(l)H] + n_1^2(l)[n_4^1(r)E + n_5^1(l)F]\} \\
n_0^3(r)\{n_6^2(r)[n_2^1(l)C + n_3^1(r)D] + n_7^2(l)[n_0^1(l)A + n_1^1(r)B]\} + \\
n_1^3(l)\{n_4^2(r)[n_6^1(l)G + n_7^1(r)H] + n_5^2(l)[n_4^1(l)E + n_5^1(r)F]\} \\
n_6^3(l)\{n_2^2(l)[n_2^1(r)C + n_3^1(l)D] + n_3^2(r)[n_0^1(r)A + n_1^1(l)B]\} + \\
n_7^3(r)\{n_0^2(l)[n_6^1(r)G + n_7^1(l)H] + n_1^2(r)[n_4^1(r)E + n_5^1(l)F]\} \\
n_4^3(l)\{n_6^2(l)[n_2^1(l)C + n_3^1(r)D] + n_7^2(r)[n_0^1(l)A + n_1^1(r)B]\} + \\
n_5^3(r)\{n_4^2(l)[n_6^1(l)G + n_7^1(r)H] + n_5^2(r)[n_4^1(l)E + n_5^1(r)F]\} \\
n_2^3(l)\{n_2^2(r)[n_2^1(r)C + n_3^1(l)D] + n_3^2(l)[n_0^1(r)A + n_1^1(l)B]\} + \\
n_3^3(r)\{n_0^2(r)[n_6^1(r)G + n_7^1(l)H] + n_1^2(l)[n_4^1(r)E + n_5^1(l)F]\} \\
n_0^3(l)\{n_6^2(r)[n_2^1(l)C + n_3^1(r)D] + n_7^2(l)[n_0^1(l)A + n_1^1(r)B]\} + \\
n_1^3(r)\{n_4^2(r)[n_6^1(l)G + n_7^1(r)H] + n_5^2(l)[n_4^1(l)E + n_5^1(r)F]\}
\end{bmatrix}
\tag{17}
$$

According to the equation (17) we can find that all the output vectors associated with all the input vectors. This means that all the output channels can communicate with all the input channels of the Comega interconnection network by selecting the states of the switching nodes in the route control. The route controlling states of the switching nodes of the cascade stages for the interconnection between input and output channels in the 8×8 Comega interconnection network is shown in table 1.

Table 1. The route controlling states for the interconnection between input and output channels in the 8×8 Comega network

	A^3	B^3	C^3	D^3
A	$n_0^1(r)n_3^2(r)n_6^3(r)$	$n_0^1(l)n_7^2(r)n_4^3(r)$	$n_0^1(r)n_3^2(l)n_2^3(r)$	$n_0^1(l)n_7^2(l)n_0^3(r)$
B	$n_1^1(l)n_3^2(r)n_6^3(r)$	$n_1^1(r)n_7^2(r)n_4^3(r)$	$n_1^1(l)n_3^2(l)n_2^3(r)$	$n_1^1(r)n_7^2(l)n_0^3(r)$
C	$n_2^1(r)n_2^2(l)n_6^3(r)$	$n_2^1(l)n_6^2(l)n_4^3(r)$	$n_2^1(r)n_2^2(r)n_2^3(r)$	$n_2^1(l)n_6^2(r)n_0^3(r)$
D	$n_3^1(l)n_2^2(l)n_6^3(r)$	$n_3^1(r)n_6^2(l)n_4^3(r)$	$n_3^1(l)n_2^2(r)n_2^3(r)$	$n_3^1(r)n_6^2(r)n_0^3(r)$
E	$n_4^1(r)n_1^2(r)n_7^3(l)$	$n_4^1(l)n_5^2(r)n_5^3(l)$	$n_4^1(r)n_1^2(l)n_3^3(l)$	$n_4^1(l)n_5^2(l)n_1^3(l)$
F	$n_5^1(l)n_1^2(r)n_7^3(l)$	$n_5^1(r)n_5^2(r)n_5^3(l)$	$n_5^1(l)n_1^2(l)n_3^3(l)$	$n_5^1(r)n_5^2(l)n_1^3(l)$
G	$n_6^1(r)n_0^2(l)n_7^3(l)$	$n_6^1(l)n_4^2(l)n_5^3(l)$	$n_6^1(r)n_0^2(r)n_3^3(l)$	$n_6^1(l)n_4^2(r)n_1^3(l)$
H	$n_7^1(l)n_0^2(l)n_7^3(l)$	$n_7^1(r)n_4^2(l)n_5^3(l)$	$n_7^1(l)n_0^2(r)n_3^3(l)$	$n_7^1(r)n_4^2(r)n_1^3(l)$

	E^3	F^3	G^3	H^3
A	$n_0^1(r)n_3^2(r)n_6^3(l)$	$n_0^1(l)n_7^2(r)n_4^3(l)$	$n_0^1(r)n_3^2(l)n_2^3(l)$	$n_0^1(l)n_7^2(l)n_0^3(l)$
B	$n_1^1(l)n_3^2(r)n_6^3(l)$	$n_1^1(r)n_7^2(r)n_4^3(l)$	$n_1^1(l)n_3^2(l)n_2^3(l)$	$n_1^1(r)n_7^2(l)n_0^3(l)$
C	$n_2^1(r)n_2^2(l)n_6^3(l)$	$n_2^1(l)n_6^2(l)n_4^3(l)$	$n_2^1(r)n_2^2(r)n_2^3(l)$	$n_2^1(l)n_6^2(r)n_0^3(l)$
D	$n_3^1(l)n_2^2(l)n_6^3(l)$	$n_3^1(r)n_6^2(l)n_4^3(l)$	$n_3^1(l)n_2^2(r)n_2^3(l)$	$n_3^1(r)n_6^2(r)n_0^3(l)$
E	$n_4^1(r)n_1^2(r)n_7^3(r)$	$n_4^1(l)n_5^2(r)n_5^3(r)$	$n_4^1(r)n_1^2(l)n_3^3(r)$	$n_4^1(l)n_5^2(l)n_1^3(r)$
F	$n_5^1(l)n_1^2(r)n_7^3(r)$	$n_5^1(r)n_5^2(r)n_5^3(r)$	$n_5^1(l)n_1^2(l)n_3^3(r)$	$n_5^1(r)n_5^2(l)n_1^3(r)$
G	$n_6^1(r)n_0^2(l)n_7^3(r)$	$n_6^1(l)n_4^2(l)n_5^3(r)$	$n_6^1(r)n_0^2(r)n_3^3(r)$	$n_6^1(l)n_4^2(r)n_1^3(r)$
H	$n_7^1(l)n_0^2(l)n_7^3(r)$	$n_7^1(r)n_4^2(l)n_5^3(r)$	$n_7^1(l)n_0^2(r)n_3^3(r)$	$n_7^1(r)n_4^2(r)n_1^3(r)$

From table 1 we can select the controlling states of the switching nodes conveniently to perform the needed interconnection between input and output channels. For example, if we want to transmit the data signal from the input channel G to the output channel C^3, the controlling states of the relative switching nodes of the three cascade stages of the 8×8 Comega interconnection network are set as $n_6^1(r)=n_0^2(r)=n_3^3(l)=1$. That is, the sixth node in the first stage is under the switching state for the right Comega interconnection, the zeroth node in the seond stage is under the switching state for the right Comega interconnection, but the third node in the third stage is under the switching state for the left Comega interconnection. Under this arrangement of the route controlling, the input channel G will communicate with the output channel C^3. Similarly, the route controlling for the communication between other arbitrary input channel with arbitrary output channel can be decided by selecting proper switching node states of the Comega interconnection network which is listed in table 1.

5. CONCLUSIONS

The Comega interconnection network is a new free-space multistage optical interconnection network with the feature of stage-invariant. It is composed of the left Comega and the right Comega interconnection networks. The matrix theory for the multistage Comega interconnection network provides a convenient way to understand the features of the Comega interconnection network. The route controlling of the Comega interconnection network from arbitrary input channel to arbitrary output channel can be decided easily based on the matrix analysis.

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