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Uniform Stability In Nonlinear Infinite Delay Volterra Integro-differential Equations Using Lyapunov Functionals

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Abstract: In [10] the first author used Lyapunov functionals and studied the exponential stability of the zero solution of finite delay Volterra Integro-differential equation. In this paper, we use modified version of the Lyapunov functional that were used in [10] to obtain criterion for the stability of the zero solution of the infinite delay nonlinear Volterra integro-differential equation

$$x'(t) = Px(t) + \int_{-\infty}^{t} C(t,s)g(x(s))ds.$$

Keywords: Nonlinear; Volterra; Zero solution; Stability; Infinite delay; Lyapunov functional

MSC: 3K20, 45J05

1 Introduction

In [10] the first author analysed the exponential stability of the zero solution of the nonlinear finite delay Volterra integro-differential equation

$$x'(t) = \int_{t-r}^{t} C(t,s)g(x(s))ds, \tag{1}$$

where $r > 0$ is a constant, and C is continuous in both arguments. The function $g(x) : \mathbb{R} \to \mathbb{R}$ and is continuous in x. Equation (1) has its roots in the study of nuclear reactors and the stability of its zero solution was studied by Brownwell and Ergen [1] in 1954. Later on, the same study was revisited by Nohel [9], in 1960 and then by Levin and Nohel [8], in 1964.

Recently, in [2] and [3], Burton used the notion of fixed point theory to alleviate some of the difficulties that arise from the use of Lyapunov functionals and obtained results concerning the stability and asymptotic stability of the zero solution of (1) when it is scalar. As it is mentioned above, in [10] the author obtained results concerning the exponential stability of (1), which generalized the papers of [1], [8] and [9]. In [10], the author proposed the open problem of extending the results of [10] to the infinite delay nonlinear Volterra integro-differential equation

$$x'(t) = \int_{-\infty}^{t} C(t,s)g(x(s))ds. \tag{2}$$

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In the past six years, we have reviewed three papers concerning the open problem, and none did had the correct solution. It turned out that the problem is still unresolved and remains a mystery. The difficulty arise from the presence of infinite delay, which in turns makes it difficult, if not impossible to construct a suitable Lyapunov functional that is in the spirit of [10]. We have been working on this problem for the last three years and in search for a way to obtain exponential stability, which we failed at, we arrived at stability results of the zero solution, instead. Our aim is to share our results with the research community in hope that it would shed a light on how to solve the open problem. For more on the study of stability, we refer the reader to [4], [6], [9], and [11].

2 Main Results

Now we consider the infinite delay nonlinear Volterra integro-differential equation

$$x'(t) = Px(t) + \int_{-\infty}^{t} C(t,s)g(x(s))ds, \ -\infty < s \leq t \tag{3}$$

and construct a Lyapunov functional; $V(t,x) := V(t)$ and show that for some positive α and under suitable conditions, we have that $V'(t) \leq -\alpha|x|^2$ along the solutions of (3). This will position us to obtain stability results regarding the zero solution and in addition, we will show that every square solution is integrable, or L_1.

Normally P of (3) is expected to be a negative function for stability properties of the zero solution. In this paper we do not require this condition. We use the size of the kernel $C(t,s)$ to offset the positive effect of the constant P. At the end of this paper we provide two examples as an application of our main results showing that the zero solution of (3) is uniformly stable for positive constant P, depending on how fast kernel $C(t,s)$ decays.

It should cause no confusion to denote the norm of a continuous function $\varphi : (-\infty, \infty) \to \mathbb{R}$ with

$$||\varphi|| = \sup_{s\in\mathbb{R}} |\varphi(s)|.$$

Let $\psi : E_{t_0} \to \mathbb{R}$ be a given bounded and continuous initial function, where

$$E_{t_0} = (-\infty, t_0].$$

Then, we say $x(t) \equiv x(t, t_0, \psi)$ is a solution of (3) if $x(t)$ satisfies (3) for $t \geq t_0$ and $x(t, t_0, \psi) = \psi(s)$, $s \in E_{t_0}$. Throughout this paper it is to be understood that when a function is written without its argument, then the argument is t. We begin with the following stability definition. For $t \geq t_0$, the set $C(t)$ denotes the set of all continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ where $\|\phi\| = \sup\{|\phi(s)| : s \leq t\}$.

Definition 2.1. The zero solution of (3) is stable if for each $\varepsilon > 0$ and each t_0, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $[\phi \in E_{t_0} \to \mathbb{R}, \phi \in C(t) : \|\phi\| < \delta]$ implies $|x(t, t_0, \phi)| < \varepsilon$ for all $t \geq t_0$. The zero solution is said to be uniformly stable if it is stable and δ is independent of t_0.

Next, we use a special technique to rewrite (3) so that a suitable Lyapunov functional can be displayed. Let us begin by letting

$$A(t,s) := \int_{-\infty}^{t-s} C(u+s, s)\, du, \ t-s \geq 0.$$

Assume the existence of positive constants λ_1, and λ_2 such that the following conditions hold.

$$xg(x) \geq \lambda_2 x^2 \text{if } x \neq 0, \tag{4}$$

$$|g(x)| \leq \lambda_1 |x| \tag{5}$$

and

$$A(t,t) > 0, \quad \text{for all } t \in [0,\infty). \tag{6}$$

It is clear that conditions (4) and (5) imply that $g(0) = 0$. It can be easily proved that (3) is equivalent to

$$x'(t) = Px(t) - A(t,t)g(x(t)) + \frac{d}{dt}\int_{-\infty}^{t} A(t,s)g(x(s))ds. \tag{7}$$

Theorem 2.1. *Let* (4)- *(6) hold, and suppose there are constants* $\gamma > 0$ *and* $\alpha > 0$ *so that*

$$2P - 2\lambda_2 A(t,t) + P^2 + \lambda_1 A^2(t,t) + \lambda_1^2\gamma \int_{t}^{\infty} |A(u,t)|du \leq -\alpha, \tag{8}$$

$$2\int_{-\infty}^{t} |A(t,s)|ds - \gamma \leq 0, \tag{9}$$

and

$$1 - \lambda_1 \int_{-\infty}^{t} |A(t,s)|ds > 0 \tag{10}$$

then, the zero solution of (3) is stable.

Proof. Let $x(t) = x(t, t_0, \psi)$ be a solution of (3) and define the Lyapunov functional $V(t) = V(t,x)$ by

$$\begin{aligned} V(t) &= \Big(x - \int_{-\infty}^{t} A(t,s)g(x(s))ds\Big)^2 \\ &\quad + \gamma \int_{-\infty}^{t}\int_{t}^{\infty} |A(u,z)|g^2(x(z))du\, dz. \end{aligned} \tag{11}$$

Now, differentiating (11) and using (3) we have

$$\begin{aligned} V'(t) &= 2\Big(x - \int_{-\infty}^{t} A(t,s)g(x(s))ds\Big)[Px - A(t,t)g(x)] + \gamma\int_{t}^{\infty} |A(u,t)|g^2(x(t))du - \gamma\int_{-\infty}^{t} |A(t,z)|g^2(x(z))dz \\ &= 2Px^2 - 2A(t,t)xg(x) - 2Px\int_{-\infty}^{t} A(t,s)g(x(s))ds + 2A(t,t)g(x)\int_{-\infty}^{t} A(t,s)g(x(s))ds \\ &+ \gamma\int_{t}^{\infty} |A(u,t)|g^2(x(t))du - \gamma\int_{-\infty}^{t} |A(t,z)|g^2(x(z))dz. \end{aligned} \tag{12}$$

In what to follow using Schwarz inequality, we perform some calculations to simplify (12).

$$\begin{aligned} 2Px\int_{-\infty}^{t} A(t,s)g(x(s))ds &\leq P^2x^2 + \Big(\int_{-\infty}^{t} A(t,s)g(x(s))ds\Big)^2 \\ &= P^2x^2 + \Big(\int_{-\infty}^{t} |A(t,s)|^{\frac{1}{2}}|A(t,s)|^{\frac{1}{2}}|g(x(s))|ds\Big)^2 \\ &\leq P^2x^2 + \int_{-\infty}^{t} |A(t,s)|ds \int_{-\infty}^{t} |A(t,s)|g^2(x(s))ds. \end{aligned}$$

Similarly,

$$2A(t,t)g(x)\int_{-\infty}^{t} A(t,s)g(x(s))ds \le \lambda_1^2 A^2(t,t)x^2 + \int_{-\infty}^{t} |A(t,s)|ds \int_{-\infty}^{t} |A(t,s)|g^2(x(s))ds.$$

By substituting the above two expressions into (12) and then using (8) and (9), yield

$$\begin{aligned} V'(t) &\le \Big[2P - 2\lambda_2 A(t,t) + P^2 + \lambda_1 A^2(t,t) + \lambda_1^2\gamma \int_{t}^{\infty} |A(u,t)|du\Big]|x|^2 \\ &+ \Big[2\int_{-\infty}^{t} |A(t,s)|ds - \gamma\Big] \int_{-\infty}^{t} |A(t,s)|g^2(x(s))ds \\ &\le -\alpha|x|^2. \end{aligned} \tag{13}$$

Let $\varepsilon > 0$ be given, we will find $\delta > 0$ so that $|x(t,t_0,\psi)| < \varepsilon$ as long as $[\psi \in E_{t_0} \to \mathbb{R} : \|\psi\| < \delta]$. Let

$$L^2 = \Big(1 + \lambda_1^2 \int_{-\infty}^{t_0} |A(t_0,s)|ds\Big)^2 + \lambda_1^2\gamma \int_{-\infty}^{t_0}\int_{t_0}^{\infty} |A(u,z)|du\, dz.$$

By (13), we have V is decreasing for $t \ge t_0$. Thus, using (11) for $t \ge t_0$ we arrive at

$$\begin{aligned} V(t,x) &\le V(t_0,\psi) \\ &\le \Big(|\psi(t_0)| + \lambda_1 \int_{-\infty}^{t_0} |A(t_0,s)||\psi(s)|ds\Big)^2 + \lambda_1^2\gamma \int_{-\infty}^{t_0}\int_{t_0}^{\infty} |A(u,z)|\psi^2(z))du\, dz \\ &= \delta^2\Big[\Big(1 + \lambda_1^2 \int_{-\infty}^{t_0} |A(t_0,s)|ds\Big)^2 + \lambda_1^2\gamma \int_{-\infty}^{t_0}\int_{t_0}^{\infty} |A(u,z)|du\, dz\Big] \\ &\le \delta^2 L^2. \end{aligned} \tag{14}$$

By (11), we have

$$\begin{aligned} V(t,x) &\ge \Big(x - \int_{-\infty}^{t} A(t,s)g(x(s))ds\Big)^2 \\ &\ge \Big(|x| - |\int_{-\infty}^{t} A(t,s)g(x(s))ds|\Big)^2. \end{aligned}$$

Combining the above two inequalities leads to

$$|x(t)| \le \delta L + \int_{-\infty}^{t} |A(t,s)||g(x(s))|ds.$$

So as long as $|x(t)| < \varepsilon$, using (5), we have

$$|x(t)| < \delta L + \varepsilon\lambda_1 \int_{-\infty}^{t} |A(t,s)|ds, \text{ for all } t \ge t_0.$$

Thus, we have from the above inequality that

$|x(t)| < \varepsilon$ for $\delta < \frac{\varepsilon}{L}(1 - \lambda_1 \int_{-\infty}^{t} |A(t,s)|ds)$. Note that by (10), we have $1 - \lambda_1 \int_{-\infty}^{t} |A(t,s)|ds > 0$ and hence, the above inequality regarding δ is valid. □

The next theorem provide conditions under which any solution x satisfies $|x(t)|^2 \in L_{[t_0,\infty)},\ t_0 \in E_k$.

Theorem 2.2. *Assume all the conditions of Theorem 2.1 hold. Let $x(t)$ be any solution of (3). Then we have $|x(t)|^2 \in L_{[t_0,\infty)},\ t_0 \in E_k$.*

Proof. We know from Theorem 2.1 that the zero solution is stable. Thus, for the same δ of stability, we take $|x(t, t_0, \psi)| < 1$. Since V is decreasing, we have by integrating (13) from t_0 to t and using (14) that,

$$V(t,x) \le V(t_0,\psi) - \alpha \int_{t_0}^{t} |x(s)|^2 ds \le \delta^2 L^2 - \alpha \int_{t_0}^{t} |x(s)|^2 ds.$$

Since,

$$V(t,x) \ge \left(x - \int_{-\infty}^{t} A(t,s) g(x(s)) ds\right)^2$$

we have that

$$\left(x - \int_{-\infty}^{t} A(t,s) g(x(s)) ds\right)^2 \le \delta^2 L^2 - \alpha \int_{t_0}^{t} |x(s)|^2 ds. \tag{15}$$

Also, using Schwarz inequality and (5) one obtains

$$\begin{aligned}\left(\int_{-\infty}^{t} |A(t,s)||g(x(s))| ds\right)^2 &= \left(\int_{-\infty}^{t} |A(t,s)|^{1/2}|A(t,s)|^{1/2}|g(x(s))| ds\right)^2 \\ &\le \lambda_1^2 \int_{-\infty}^{t} |A(t,s)| ds \int_{-\infty}^{t} |A(t,s)||x(s)|^2 ds.\end{aligned}$$

As $\int_{-\infty}^{t} |A(t,s)| ds$ is bounded by (9) and $|x|^2 < 1$, we have $\int_{-\infty}^{t} |A(t,s)||x(s)|^2 ds$ is bounded and hence $\int_{-\infty}^{t} |A(t,s)||g(x(s))| ds$ is bounded. Therefore, from (15), we arrive at

$$\begin{aligned}\alpha \int_{t_0}^{t} |x(s)|^2 ds &\le \delta^2 L^2 - \left(x - \int_{-\infty}^{t} A(t,s) g(x(s)) ds\right)^2 \\ &\le \delta^2 L^2 + \left(|x| + |\int_{-\infty}^{t} A(t,s) g(x(s)) ds|\right)^2 \\ &\le \delta^2 L^2 + |x|^2 + 2|x||\int_{-\infty}^{t} A(t,s) g(x(s)) ds| + \left(|\int_{-\infty}^{t} A(t,s) g(x(s)) ds|\right)^2 \\ &\le \delta^2 L^2 + |x|^2 + |x|^2 + \left(|\int_{-\infty}^{t} A(t,s) g(x(s)) ds|\right)^2 + \left(|\int_{-\infty}^{t} A(t,s) g(x(s)) ds|\right)^2 \\ &\le \delta^2 L^2 + 2 + 2\left(\int_{-\infty}^{t} |A(t,s)||g(x(s))| ds\right)^2 \\ &\le K, \text{ for some positive constant } K.\end{aligned}$$

This shows that $|x(t)|^2 \in L_{[t_0,\infty)},\ t_0 \in E_k$. □

It is straight forward to extend the results of this paper to (2). This is remarkable since (2) is totally nonlinear. The absence of a linear term makes it extremely difficult if not impossible to invert or find a suitable Lyapunov functional. Our advantage here is that we were able to rewrite our equation so that it is equivalent to (7).

Theorem 2.3. *Let* (4)- *(6) hold. In addition, we assume*

$$-2\lambda_2 A(t,t) + \lambda_1 A^2(t,t) + \lambda_1^2\gamma \int_t^\infty |A(u,t)|du \le -\alpha. \tag{16}$$

$$\int_{-\infty}^t |A(t,s)|ds - \gamma \le 0, \tag{17}$$

and

$$1 - \lambda_1 \int_{-\infty}^t |A(t,s)|ds > 0 \tag{18}$$

then, the zero solution of (2) is stable and $|x(t)|^2 \in L_{[t_0,\infty)},\ t_0 \in E_k$.

Proof. The proof is immediate consequence of Theorem 2.1 and Theorem 2.2 by setting $P = 0$. □

In the next theorem we show that the zero solution of (3) is uniformly stable by requiring uniform boundedness on the double integral in the Lyapunov functional. For simplicity, we let

$$J = \int_{-\infty}^t |A(t,s)|ds. \tag{19}$$

Theorem 2.4. *Assume all the conditions of Theorem 2.1 hold. Suppose for some positive constant R and for all* $t \ge t_0$,

$$\int_{-\infty}^t \int_t^\infty |A(u,z)|du\, dz \le R, \tag{20}$$

then the zero solution of (3) is uniformly stable.

Proof. Let V be given by (11). Then by expanding V and using Schwartz inequality we arrive at,

$$\begin{aligned} V(t) &= x^2(t) + \Big(\int_{-\infty}^t A(t,s)g(x(s))ds\Big)^2 - 2x(t)\int_{-\infty}^t A(t,s)g(x(s))ds \\ &+ \gamma \int_{-\infty}^t \int_t^\infty |A(u,z)|g^2(x(z))du\, dz \\ &\le 2x^2(t) + 2\int_{-\infty}^t |A(t,s)|ds \int_{-\infty}^t |A(t,s)|g^2(x(s))ds + \gamma \int_{-\infty}^t \int_t^\infty |A(u,z)|g^2(x(z))du\, dz \\ &\le 2x^2(t) + 2\lambda_1^2 J \int_{-\infty}^t |A(t,s)|x^2(s)ds + \gamma\lambda_1^2 \int_{-\infty}^t \int_t^\infty |A(u,z)|x^2(z)du\, dz. \end{aligned} \tag{21}$$

Given an $\epsilon > 0$ and a fixed $t_0 \in E_k$, we choose $\delta > 0$ with $0 < \delta < \epsilon$ such that

$$(2 + 2\lambda_1^2 J + +\gamma\lambda_1^2 R)^{1/2}\delta < \epsilon(1 - \lambda_1 J). \tag{22}$$

Let $x(t) = x(t, t_0, \phi)$ be a solution of (3) with $||\phi|| < \delta$. Then for $t \geq t_0$, using (21) and (22) we have

$$\begin{aligned}\Big(x - \int_{-\infty}^{t} A(t,s)g(x(s))ds\Big)^2 &\leq V(t) \leq V(t_0)\\ &\leq (2 + 2\lambda_1^2 J + +\gamma\lambda_1^2 R)\delta^2. \end{aligned} \tag{23}$$

Also, we notice that

$$\begin{aligned}\Big|x - \int_{-\infty}^{t} A(t,s)g(x(s))ds\Big| &\geq |x| - \int_{-\infty}^{t} |A(t,s)||g(x(s))|ds\\ &\geq |x| - \lambda_1 \int_{-\infty}^{t} |A(t,s)||x(s))ds \end{aligned} \tag{24}$$

We claim that $|x(t)| < \epsilon$ for all $t \geq t_0$. Note that $|x(u)| < \delta < \epsilon$ for all $-\infty \leq u \leq t_0$. If the claim is not true, let $t = t_\star$ be the first t such that $|x(t_\star)| = \epsilon$ and $|x(s)| < \epsilon$ for $t_0 \leq s < t_\star$. Then, using (5), (19), (23), and (24) we obtain

$$\begin{aligned}\epsilon(1 - \lambda_1 J) &= \epsilon(1 - \lambda_1 \int_{-\infty}^{t} |A(t,s)|ds)\\ &\leq |x(t_\star)| - \lambda_1 \int_{-\infty}^{t_\star} |A(t_\star,s)||x(s)|ds\\ &\leq |x(t_\star)| - \int_{-\infty}^{t_\star} |A(t_\star,s)||g(x(s))|ds\\ &\leq |x(t_\star) - \int_{-\infty}^{t_\star} A(t_\star,s)g(x(s))ds|\\ &\leq (2 + 2\lambda_1^2 J + \gamma\lambda_1^2 R)^{1/2}\delta,\end{aligned}$$

which contradicts (22) and this completes the proof. □

We state the following corollary regarding the uniform stability of the zero solution of (2). Its proof follows along the lines of Theorem 2.4.

Corollary 1. Assume all the conditions of Theorem 2.3 hold along with (20). Then the zero solution of (2) is uniformly stable.

Next, we display an example.

3 Examples

Example 3.1. Let $g(x) = x(\frac{\sin^2(x) + 1}{4})$, then $g(0) = 0$, $xg(x) > x^2/4$ and $|g(x)| \leq \frac{1}{2}|x|$. If we choose $C(t,s) = \frac{-1}{16(t-s+1)^4}$, then $C(u+s,s) = \frac{-1}{16(u+1)^4}$, and

$$A(t,s) = \int_{-\infty}^{t-s} C(u+s,s)ds = \frac{1}{4(t-s+1)^3}.$$

This implies that $A(t,t) = \frac{1}{4}$, and hence condition (6) is satisfied. It is easy to verify that

$$\int_t^\infty |A(u,t)|du = 3/4, \quad \int_{-\infty}^t |A(t,s)|ds = 3/4,$$

and

$$\begin{aligned}\int_{-\infty}^t \int_t^\infty |A(u,z)|du\, dz &= \int_{-\infty}^t \int_t^\infty \frac{1}{4(u-z+1)^3} du\, dz \\ &= \int_{-\infty}^t \frac{3}{4(t-z+1)^2}\, dz \\ &= 3/2.\end{aligned}$$

Let $\gamma = 3/2$ and $P = -1$. Then it is easy to verify that (8), (9) and (10) are satisfied since $\lambda_1 = 1/2$, and $\lambda_2 = 1/4$. Thus, we have shown that the zero solution of the nonlinear Volterra integro-differential equation

$$x'(t) = -x(t) - \int_{-\infty}^t \frac{1}{16(t-s+1)^4} x(s)(\frac{\sin^2(x(s))}{4} + 1)ds$$

is uniformly stable.

In the next example we display a kernel C that allows us to have $P > 0$.

Example 3.2. Let $g(x)$ be defined as in the previous example. Let $C(t,s) = -e^{-k(t-s)}$, for some positive constant k to be chosen later. Then, $A(t,s) = \frac{1}{k}e^{-k(t-s)}$ and as a consequence we have $A(t,t) = \frac{1}{k}$. It is easy to verify that

$$\int_t^\infty |A(u,t)|du = \int_{-\infty}^t |A(t,s)|ds = \frac{1}{k^2}.$$

Moreover, $\int_{-\infty}^t \int_t^\infty |A(u,z)|du\, dz = \frac{1}{k^3}$.

Let $k = 4$, and $\gamma = 1/32$. Then conditions (9) and (10) are satisfied since $\lambda_1 = 1/2$, and $\lambda_2 = 1/4$. Left to show that (8) is satisfied for the appropriate choice of P. A substitution of all the parameters in (8) leads to the quadratic equation in P,

$$P^2 + 2P - \eta^2 < 0, \text{ where } \eta^2 = 0.09326171875. \tag{25}$$

Inequality (25) is satisfied for

$$0 \le P < -1 + \sqrt{2 + 2\eta^2}.$$

Thus, we have shown that the zero solution of the nonlinear Volterra integro-differential equation with positive coefficient P

$$x'(t) = Px(t) - \int_{-\infty}^t e^{-k(t-s)} x(s)(\frac{\sin^2(x(s))}{4} + 1)ds$$

is uniformly stable.

4 Discussion and Future Direction

It is always believed that when using Lyapunov theory in the qualitative analysis of functional differential equations, the obtained results are directly related to the characteristic of the Lyapunov function or functional. In this paper, we transformed our nonlinear integro-differential equation to created a linear term, which lead us to the "right" Lyapunov functional. By rewriting (3) in the form (7) gave us a hint on how to approach the creation of the Lyapunov functional given in (11). This can be easily seen by noticing that (7) can be put in the form

$$\frac{d}{dt}\Big(x(t) - \int_{-\infty}^{t} A(t,s)g(x(s))ds\Big) = Px(t) - A(t,t)g(x(t)),$$

which is the first term in the Lyapuniov functional given by (11), squared. In addition, such technique allowed us to deduce stability results on the totally nonlinear integro-differential equation given by (2).
In the paper [7], the authors considered the pairs of the scalar linear Volterra integro-differential equation

$$x'(t) = h(t)x(t) + \int_{0}^{t} C(at-s)x(s)ds \tag{26}$$

and its perturbed form

$$x'(t) = h(t)x(t) + \int_{0}^{t} C(at-s)x(s)ds + g(t,x(t)) \tag{27}$$

where a is a constant, $a > 1$. The function $g(t,x(t))$ is continuous in t and x, and satisfies $|g(t,x(t))| \le \lambda(t)|x(t)|$, where $\lambda(t)$ is continuous. Moreover, $h(t)$ is continuous for all $t \ge 0$ and $C : \mathbb{R} \to \mathbb{R}$ is continuous. We point out that if $C \in L^1[0,\infty)$, then the equations (1.1) and (1.2) become fading memory problems. When $a > 1$, the memory term $\int_0^t C(at-s)ds = \int_{(a-1)t}^{at} C(u)du$ tends to zero as $t \to \infty$, that is the memory fades away completely. On the other hand, if $0 < a < 1$, the memory term never fades away completely; it tends to a constant as $t \to \infty$. For $a = 1$, equation (26) and (27) are the well-known convolution equations. In the same spirit of [7], we conjecture the following:

Conjecture 1. Assume all the conditions of Theorem 2.1 hold. If we require that

$$\int_{t}^{\infty} |A(u,z)|du \in L^1(-\infty,\infty), \tag{28}$$

then the zero solution of (3) is uniformly asymptotically stable.

we end this paper by stating an open problem.

Open Problem 1. In light of [10], what can be said about the exponential stability of the zero solution and the instability of the nonlinear Volterra integro-differential equation with infinite delay

$$x'(t) = Px(t) + \int_{-\infty}^{t} C(t,s)g(x(s))ds, \; -\infty < s \le t?$$

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