3-22-2016

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Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 19, NUMBER 1 (2016) (Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

SMALLEST EIGENVALUES FOR A RIGHT FOCAL BOUNDARY VALUE PROBLEM

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Abstract

We establish the existence of smallest eigenvalues for the fractional linear boundary value problems $D^{\alpha}_{0+}u + \lambda_1 p(t)u = 0$ and $D^{\alpha}_{0+}u + \lambda_2 q(t)u = 0$, $0 < t < 1$, with each satisfying the right focal boundary conditions $u(0) = u'(1) = 0$. A comparison result is then obtained.

MSC 2010: Primary 26A33; Secondary 34A08

Key Words and Phrases: fractional boundary value problem, u_0-positive operator, smallest eigenvalues

1. **Introduction**

We consider the eigenvalue problems

$$D^{\alpha}_{0+}u + \lambda_1 p(t)u = 0, \quad 0 < t < 1, \tag{1.1}$$

$$D^{\alpha}_{0+}u + \lambda_2 q(t)u = 0, \quad 0 < t < 1, \tag{1.2}$$

satisfying the boundary conditions

$$u(0) = u'(1) = 0, \tag{1.3}$$

where $1 < \alpha \le 2$ is a real number, D^{α}_{0+} is the standard Riemann-Liouville derivative, and $p(t)$ and $q(t)$ are continuous nonnegative functions on $[0, 1]$, where neither $p(t)$ nor $q(t)$ vanishes identically on any nondegenerate compact subinterval of $[0, 1]$. In this paper, we modify an approach developed by the authors in [5] to show the existence of smallest eigenvalues (1.1),(1.3) and (1.2),(1.3). We will then compare these smallest eigenvalues under the assumption that $p(t) \le q(t)$.

pp. 11–18 , DOI: 10.1515/fca-2016-0002 DE GRUYTER

Using Krein-Rutman theory [14] to show the existence of and compare smallest eigenvalues for boundary value problems has been a well-studied area (see [1, 2, 3, 4, 7, 8, 10, 15, 16] for some examples). However, just recently, the existence and comparison of smallest eigenvalues and applications of these results have been studied for fractional boundary value problems in [5, 6, 9]. In [5], the authors studied the second order linear fractional eigenvalue problems with conjugate boundary conditions. The standard approach used in the papers cited above was modified to account for the unbounded slope of the Green's function for $-D^{\alpha}_{0+} = 0$, $u(0) = u(1) = 0$ at 0. The Green's function for $-D^{\alpha}_{0+} = 0$, $u(0) = u'(1) = 0$ also has an unbounded slope at 0. Therefore, in this paper, we use the approach established in [5] to show the existence of and compare smallest eigenvalues for the right focal fractional boundary value problems. The difference in this analysis to the work done on the conjugate problem is that the Banach space for the right focal problem does not involve $C^{(1)}$ functions since showing the interior of the cone used in this paper does not involve the mean value theorem or the sign of the derivative at 1.

2. Preliminary Definitions and Theorems

Definition 2.1. Let $1 < \alpha \le 2$. The α-th Riemann-Liouville fractional derivative of the function $u : [0, 1] \to \mathbb{R}$, denoted $D^{\alpha}_{0+}u$, is defined as

$$D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dt^2}\int_0^t (t-s)^{1-\alpha}u(s)ds,$$

provided the right-hand side exists.

Definition 2.2. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{P}$ of $\mathcal{B}$ is said to be a cone, provided:

(i) $\alpha u + \beta v \in \mathcal{P}$, for all $u, v \in \mathcal{P}$ and all $\alpha, \beta \ge 0$, and
(ii) $u \in \mathcal{P}$ and $-u \in \mathcal{P}$ implies $u = 0$.

Definition 2.3. A cone $\mathcal{P}$ is solid if the interior, $\mathcal{P}^{\circ}$, of $\mathcal{P}$, is nonempty. A cone $\mathcal{P}$ is reproducing if $\mathcal{B} = \mathcal{P} - \mathcal{P}$; i.e., given $w \in \mathcal{B}$, there exist $u, v \in \mathcal{P}$ such that $w = u - v$.

Remark 2.1. Krasnosel'skii [13] showed that every solid cone is reproducing.

Cones give rise to a natural partial ordering on a Banach space.

Definition 2.4. Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. If $u, v \in \mathcal{B}$, $u \le v$ with respect to $\mathcal{P}$ if $v - u \in \mathcal{P}$. If both $M, N : \mathcal{B} \to \mathcal{B}$ are bounded linear operators, $M \le N$ with respect to $\mathcal{P}$ if $Mu \le Nu$ for all $u \in \mathcal{P}$.

Definition 2.5. A bounded linear operator $M : \mathcal{B} \to \mathcal{B}$ is u_0-positive with respect to $\mathcal{P}$ if there exists $u_0 \in \mathcal{P}\backslash\{0\}$ such that for each $u \in \mathcal{P}\backslash\{0\}$, there exist $k_1(u) > 0$ and $k_2(u) > 0$ such that $k_1 u_0 \le Mu \le k_2 u_0$ with respect to $\mathcal{P}$.

The following two results are fundamental to our comparison results and are attributed to Krasnosel'skii [13]. The proof of Theorem 2.1 can be found in [13], and the proof of Theorem 2.2 is provided by Keener and Travis [12] as an extension of Krasnosel'skii's results.

Theorem 2.1. *Let $\mathcal{B}$ be a real Banach space and let $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone. Let $L : \mathcal{B} \to \mathcal{B}$ be a compact, u_0-positive, linear operator. Then L has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

Theorem 2.2. *Let $\mathcal{B}$ be a real Banach space and $\mathcal{P} \subset \mathcal{B}$ be a cone. Let both $M, N : \mathcal{B} \to \mathcal{B}$ be bounded, linear operators and assume that at least one of the operators is u_0-positive. If $M \le N$, $Mu_1 \ge \lambda_1 u_1$ for some $u_1 \in \mathcal{P}$ and some $\lambda_1 > 0$, and $Nu_2 \le \lambda_2 u_2$ for some $u_2 \in \mathcal{P}$ and some $\lambda_2 > 0$, then $\lambda_1 \le \lambda_2$. Futhermore, $\lambda_1 = \lambda_2$ implies u_1 is a scalar multiple of u_2.*

3. Comparison of Smallest Eigenvalues

In [11], Kaufmann and Mboumi showed that the Green's function for $-D_{0+}^{\alpha} u(t) = 0$ satisfying (1.3) is given by

$$G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \le t \le s < 1. \end{cases} \tag{3.1}$$

Therefore, if $u(t) = \lambda_1 \int_0^1 G(t,s)p(s)u(s)ds$, $u(t)$ solves (1.1),(1.3). Similarly, if $u(t) = \lambda_2 \int_0^1 G(t,s)q(s)u(s)ds$, $u(t)$ solves (1.2),(1.3). Notice that $G(t,s) \ge 0$ on $[0,1] \times [0,1)$ and $G(t,s) > 0$ on $(0,1] \times (0,1)$.

Define the Banach space

$$\mathcal{B} = \{u : u = t^{\alpha-1}v, v \in C[0,1]\},$$

with the norm

$$\|u\| = |v|_0,$$

where $|v|_0 = \sup_{t\in[0,1]} |v(t)|$ denotes the usual supremum norm. Notice that for $u \in \mathcal{B}$,

$$|u|_0 = |t^{\alpha-1}v|_0 \le t^{\alpha-1}\|u\|,$$

implying

$$|u|_0 \le \|u\|.$$

Define the linear operators

$$Mu(t) = \int_0^1 G(t,s)p(s)u(s)ds, \tag{3.2}$$

and

$$Nu(t) = \int_0^1 G(t,s)q(s)u(s)ds. \tag{3.3}$$

Now,

$$Mu(t) = \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)u(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds$$

$$= t^{\alpha-1}\left(\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds\right).$$

Notice that since $\alpha > 1$,

$$\left|\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)u(s)ds\right| \le \frac{|p|_0|v|_0}{\Gamma(\alpha)}\left|\int_0^1 s^{\alpha-1}(1-s)^{\alpha-2}ds\right|$$
$$= \frac{|p|_0|v|_0\Gamma(\alpha-1)}{\Gamma(2\alpha-1)} < \infty.$$

Therefore, the first term inside the parentheses is well-defined.

Set

$$g(t) = \begin{cases} 0, & t = 0, \\ t^{1-\alpha}\displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds, & 0 < t \le 1. \end{cases}$$

In the proof of Theorem 3.1 in [5], it was shown $g \in C[0,1]$.

Therefore, $M : \mathcal{B} \to \mathcal{B}$. An application of the Arzelà Ascoli theorem shows M is compact. A similar argument can be made for N. Thus, we have the following result.

THEOREM 3.1. *The operators $M, N : \mathcal{B} \to \mathcal{B}$ are compact.*

Next, we define the cone

$$\mathcal{P} = \{u \in \mathcal{B} \mid u(t) \ge 0 \text{ for } t \in [0,1]\}.$$

LEMMA 3.1. *The cone $\mathcal{P}$ is solid in $\mathcal{B}$ and hence reproducing.*

Proof. Define

$$\Omega := \{u = t^{\alpha-1}v \in \mathcal{B} : u(t) > 0 \text{ for } t \in (0,1], v(0) > 0\}. \tag{3.4}$$

We will show $\Omega \subset \mathcal{P}^\circ$. Let $u \in \Omega$. Since $v(0) > 0$, there exists an $\epsilon_1 > 0$ such that $v(0) - \epsilon_1 > 0$. Since $v \in C[0,1]$, there exists an $a \in (0,1)$ such that $v(t) > \epsilon_1$ for all $t \in (0,a)$. So $u(t) = t^{\alpha-1}v(t) > \epsilon_1 t^{\alpha-1}$ for all $t \in (0,a)$. Also, since $u(t) > 0$ on $[a,1]$, there exists an $\epsilon_2 > 0$ with $u(t) - \epsilon_2 > 0$ for all $t \in [a,1]$.

Let $\epsilon = \min\{\frac{\epsilon_1}{2}, \frac{\epsilon_2}{2}\}$. Define $B_\epsilon(u) = \{\hat{u} \in \mathcal{B} : ||u - \hat{u}|| < \epsilon\}$. Let $\hat{u} \in B_\epsilon(u)$. So $\hat{u} = t^{\alpha-1}\hat{v}$, where $\hat{v} \in C[0,1]$. Now $|\hat{u}(t) - u(t)| \leq t^{\alpha-1}\|\hat{u} - u\| < \epsilon t^{\alpha-1}$. So for $t \in (0,a)$, $\hat{u}(t) > u(t) - t^{\alpha-1}\epsilon > t^{\alpha-1}\epsilon_1 - t^{\alpha-1}\epsilon_1/2 = t^{\alpha-1}\epsilon_1/2$. So $\hat{u}(t) > 0$ for $t \in (0,a)$. Also, $|\hat{u}(t) - u(t)| \leq \|\hat{u} - u\| < \epsilon$. So for $t \in [a,1]$, $\hat{u}(t) > u(t) - \epsilon > \epsilon_2 - \epsilon_2/2 > 0$. So $\hat{u}(t) > 0$ for all $t \in [a,1]$. So $\hat{u} \in \mathcal{P}$ and thus $B_\epsilon(u) \subset \mathcal{P}$. So $\Omega \subset \mathcal{P}^\circ$. □

LEMMA 3.2. *The bounded linear operators M and N are u_0-positive with respect to $\mathcal{P}$.*

P r o o f. First, we show $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$. Let $u \in \mathcal{P}$. So $u(t) \geq 0$. Then since $G(t,s) \geq 0$ on $[0,1] \times [0,1)$ and $p(t) \geq 0$ on $[0,1]$,

$$Mu(t) = \int_0^1 G(t,s)p(s)u(s)ds \geq 0,$$

for $0 \leq t \leq 1$. So $M : \mathcal{P} \to \mathcal{P}$.

Now let $u \in \mathcal{P}\backslash\{0\}$. So there exists a compact interval $[\alpha,\beta] \subset [0,1]$ such that $u(t) > 0$ and $p(t) > 0$ for all $t \in [\alpha,\beta]$. Then, since $G(t,s) > 0$ on $(0,1] \times (0,1)$,

$$Mu(t) = \int_0^1 G(t,s)p(s)u(s)ds \geq \int_\alpha^\beta G(t,s)p(s)u(s)ds > 0,$$

for $0 < t \leq 1$.

Now,

$$Mu(t) = t^{\alpha-1}\left(\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds\right).$$

Let

$$v(t) = \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds.$$

So, $v(0) = \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)u(s)ds > 0$, thus $M : \mathcal{P}\backslash\{0\} \to \Omega \subset \mathcal{P}^\circ$.

Now choose $u_0 \in \mathcal{P}\backslash\{0\}$, and let $u \in \mathcal{P}\backslash\{0\}$. So $Mu \in \Omega \subset \mathcal{P}^\circ$. Choose $k_1 > 0$ sufficiently small and k_2 sufficiently large so that $Mu - k_1u_0 \in \mathcal{P}^\circ$ and $u_0 - \frac{1}{k_2}Mu \in \mathcal{P}^\circ$. So $k_1u_0 \leq Mu$ with respect to $\mathcal{P}$ and $Mu \leq k_2u_0$ with respect to $\mathcal{P}$. Thus $k_1u_0 \leq Mu \leq k_2u_0$ with respect to $\mathcal{P}$ and M is u_0-positive with respect to P. A similar argument shows N is u_0-positive. □

Lemma 3.3. *The eigenvalues of* (1.1),(1.3) *are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of* (1.2),(1.3) *are reciprocals of eigenvalues of N, and conversely.*

P r o o f. Let Λ be an eigenvalue of M with corresponding eigenvector $u(t)$. Notice that

$$\Lambda u(t) = Mu(t) = \int_0^1 G(t,s)p(s)u(s)ds,$$

if and only if

$$u(t) = \frac{1}{\Lambda}\int_0^1 G(t,s)p(s)u(s)ds,$$

if and only if

$$D_{0+}^{\alpha}u(t) + \frac{1}{\Lambda}p(t)u(t) = 0, \quad 0 < t < 1,$$

with

$$u(0) = u'(1) = 0.$$

So $\dfrac{1}{\Lambda}$ is an eigenvalue of (1.1),(1.3). A similar argument can be made for eigenvalues of N. □

Theorem 3.2. *Let $\mathcal{B}, \mathcal{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathcal{P}^\circ$.*

P r o o f. Since M is a compact linear operator that is u_0-positive with respect to $\mathcal{P}$, by Theorem 2.1, M has an essentially unique eigenvector, say $u \in \mathcal{P}$, and eigenvalue Λ with the above properties. Since $u \neq 0$, $Mu \in \Omega \subset \mathcal{P}^\circ$ and $u = M\left(\frac{1}{\Lambda}u\right) \in \mathcal{P}^\circ$. □

Theorem 3.3. *Let $\mathcal{B}$, $\mathcal{P}$, M, and N be defined as earlier. Let $p(t) \le q(t)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 3.2 associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathcal{P}^\circ$. Then $\Lambda_1 \le \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(t) = q(t)$ on $[0,1]$.*

P r o o f. Let $p(t) \le q(t)$ on $[0,1]$. So for any $u \in \mathcal{P}$ and $t \in [0,1]$,

$$(Nu - Mu)(t) = \int_0^1 G(t,s)(q(s) - p(s))u(s)ds \ge 0.$$

So $Nu - Mu \in \mathcal{P}$ for all $u \in \mathcal{P}$, or $M \leq N$ with respect to $\mathcal{P}$. Then by Theorem 2.2, $\Lambda_1 \leq \Lambda_2$.

If $p(t) = q(t)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(t) \neq q(t)$. So $p(t) < q(t)$ on some subinterval $[\alpha, \beta] \subset [0, 1]$. Then, using an argument similar to the proof that N was u_0-positive, $(N-M)u_1 \in \Omega \subset \mathcal{P}^\circ$ and so there exists $\epsilon > 0$ such that $(N-M)u_1 - \epsilon u_1 \in \mathcal{P}$. So $\Lambda_1 u_1 + \epsilon u_1 = Mu_1 + \epsilon u_1 \leq Nu_1$, implying $Nu_1 \geq (\Lambda_1 + \epsilon)u_1$. Since $N \leq N$ and $Nu_2 = \Lambda_2 u_2$, by Theorem 2.2, $\Lambda_1 + \epsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda_2$. □

Since the eigenvalues of (1.1),(1.3) are reciprocals of eigenvalues of M and conversely, and the eigenvalues of (1.2),(1.3) are reciprocals of eigenvalues of N and conversely, the following theorem is an immediate consequence of Theorems 3.2 and 3.3.

THEOREM 3.4. *Assume the hypotheses of Theorem 3.3. Then there exists smallest positive eigenvalues λ_1 and λ_2 of (1.1),(1.3) and (1.2),(1.3), respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $p(t) = q(t)$ for all $t \in [0, 1]$.*

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Received: March 5, 2015

Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **19**, No 1 (2016), pp. 11–18,
DOI: 10.1515/fca-2016-0002