Spring 4-2016

Domain Representability and Topological Completeness

Matthew D. DeVilbiss
University of Dayton

Domain Representability and Topological Completeness

Honors Thesis

Matthew D. DeVilbiss

Department: Mathematics

Advisor: Lynne Yengulalp, Ph.D.

April 2016

Domain Representability and Topological Completeness

Honors Thesis

Matthew D. DeVilbiss

Department: Mathematics

Advisor: Lynne Yengulalp, Ph.D.

April 2016

Abstract

Topological completeness properties seek to generalize the definition of complete metric space to the context of topologies. Chapter 1 gives an overview of some of these properties. Chapter 2 introduces domain theory, a field originally intended for use in theoretical computer science. Finally, Chapter 3 examines how this computer-scientific notion can be employed in the study of topological completeness in the form of domain representability. The connections between domain representability and other topological completeness properties are subsequently examined.

Dedication

For Dr. Y, without whom, none of this would be possible.

UNIVERSITY *of* DAYTON

Table of Contents

Chapter 1

Topological Completeness

1.1 Metric Spaces: Motivating Topology, Motivating Completeness

Closeness

For the millennia that mathematics has been studied, the notion of distance has played a particularly important role. From the geometry and trigonometry of the ancient Greeks and ancient Egyptians to the early-modern founders of analysis to the present day, distance has been crucial to many of the most prominent areas of mathematics. For most of that time, distance in one dimension was defined via the absolute value of the difference, that is the distance between x and y in $\mathbb{R}$ is $|x - y|$. More generally in n-dimensions, $\mathbb{R}^n$, the distance between $\langle p_1, p_2, \ldots, p_n \rangle$ and $\langle q_1, q_2, \ldots, q_n \rangle$ is given by the expression

$$\sqrt{\sum_{j=1}^{n} (p_j - q_j)^2}.$$

This single formula was sufficient to support thousands of years of rich mathematical progress in the study of Euclidean space.

However, the late nineteenth century and early twentieth century saw an unprecedented surge in the abstractness and generality of the mathematics being studied. This is due in no small part to the advent of set theory and mathematical logic, providing a rigorous foundation from which more abstract and less intuitive directions could be explored. This generalization and formalization left no area of mathematics unaffected, including the study of distance. For sometime already calculus had existed in the intuitive and less-rigorous language of infinitesimals famously pioneered by Newton and Leibniz. But as the advantages of rigorous definitions and methods of proof became more apparent, the field of analysis came to apply these methods to calculus. The analytic program was rather successful as limits, derivatives, and integrals all came to rely on the more robust foundation of predicate logic and ε's and δ's. But with these developments came the ability to consider increasingly abstract mathematical objects and spaces. Soon arose the need to consider spaces so foreign that the ancient wisdom of distance in Euclidean space no longer applied.

In 1906 [20], Maurice Fréchet developed the definition of metric, taking a critical step towards generality in the context of distance. Pulling from the generalization of sets and functions by the likes of Cantor, Fréchet advanced the following highly influential definition.

Definition 1.1.1. A **metric space** is a pair $\langle X, d \rangle$ with X a set and d, called a **metric** on X, a function $d : X \times X \to [0, \infty)$ satisfying the following three properties for all $x, y, z \in X$:

i) $d(x,y)=0 \leftrightarrow x=y$;

ii) $d(x,y)=d(y,x)$; and

iii) $d(x,z) \leq d(x,y)+d(y,z)$.

This definition should match our intuitive picture of what 'distance' means. If X is thought as the set of points in space and d measures the distance between any two points, the first property means that two points are identical exactly when they are distance 0 apart. Surely it would violate our intuitions to have a point being positive distance from itself or two distinct points being in the exact same location. Property (ii) indicates that the distance going from x to y is the same as the distance in going from y to x, which matches our experience of distance not depending on direction of measurement. Finally property (iii) is called the triangle inequality and formalizes the knowledge of ancient Greek geometers that "the shortest distance between any two points is a straight line". This definition doesn't seem to contain any obvious counter-intuitive notions and apparently captures most of the important properties of Euclidean distance.

It is not difficult to see that distance in Euclidean space in any finite dimension meets this definition and thus constitutes a metric. However, the power of this definition lies in its ability to be applied to *any* set X and *any* function $d: X \times X \rightarrow [0,\infty)$ for which (i)-(iii) holds. Namely, we can apply this concept to spaces and metrics entirely foreign to Euclidean geometry. The examples are (literally) endless, but we will mention just a few here. One example in graph theory can be found by letting the point set be equal to $V(G)$,the vertex set of a graph, and letting the distance function $d(u,v)$ describe the length of the shortest path from u to v along edges in G. On the set of binary strings of length n, denoted 2^n, the function $d(x,y)=|\{m \leq n : x(m) \neq y(m)\}|$ is a metric function called the Hamming distance. For any set X, one may define the discrete metric:

$$d(x,y)=\begin{cases} 0 \text{ if } x=y \\ 1 \text{ otherwise} \end{cases}$$

The discrete metric, while often being highly pathological, meets the definition and therefore is related in some way to our usual notion of distance, provided this axiomatization is sufficiently accurate.

The reason this definition of metric was chosen rather than some other potentially equally valid definition generalizing distance is because, while Fréchet's definition is versatile enough to apply to a myriad of spaces and situations, the definition actually retains everything necessary to study analysis and thus calculus. To demonstrate this, we will translate the notion of convergence of an infinite sequence to a limit in an arbitrary metric space.

Definition 1.1.2. Let $\langle X,d\rangle$ be a metric space and let $x:\mathbb{N}\rightarrow X$ be an infinite sequence in X. We say x **converges** to $\ell \in X$, in symbols $\lim_{n\rightarrow\infty} x(n)=\ell$, if

$$\forall\varepsilon>0\,\exists N\in\mathbb{N}\,\forall n\geq N,\ d(x(n),\ell)<\varepsilon.$$

Of course, Cauchy convergence is just as easy to generalize:

Definition 1.1.3. Let $\langle X,d\rangle$ be a metric space and let $x:\mathbb{N}\rightarrow X$ be an infinite sequence in X. We say x is **Cauchy convergent** if

$$\forall\varepsilon>0\,\exists N\in\mathbb{N}\,\forall n,m\geq N,\ d(x(n),x(m))<\varepsilon.$$

Anyone familiar with basic analysis can identify that these definitions are derived merely from replacing any occurrence of $|x-y|$ with $d(x,y)$. This ease of translation indicates that limit convergence, and analysis in general, can be done in any metric space without making explicit reference to the Euclidean distance function.

With the idea of limits easily established in general metric spaces, we can formalize open and closed sets. Originally generalizations of open and closed intervals, these sets become even more varied as they are applied to general metric spaces. Let $\langle X,d\rangle$ be a metric space:

Definition 1.1.4. A subset $U \subseteq X$ is called an **open set** in X if

$$\forall x \in U \, \exists \varepsilon > 0 \, (\{y \in X : d(x,y) < \varepsilon\} \subseteq U) \, .$$

A subset $C \subseteq X$ is called a **closed set** in X if C contains all of its limit points. Symbolically,

$$\forall x \in X \, \left(\exists a \in C^{\mathbb{N}} \, \left(\lim_{n \to \infty} a(n) = x \right) \to x \in C \right) .$$

Later in the early twentieth century, several important ideas grew out of the study of analysis on these general metric spaces, contributing to Felix Hausdorff's seminal 1914 publication founding the field of topology [24]. Hausdorff became the father of topology [36], a field which would extend Fréchet's generalization beyond imagination. Hausdorff defined topological spaces as follows:

Definition 1.1.5. Let X be a set. A collection of subsets $\tau \subseteq \mathcal{P}(X)$ is called a **topology** if τ satisfies the following properties:

i) $X, \varnothing \in \tau$;

ii) For any $U, V \in \tau$, the intersection $U \cap V \in \tau$;

iii) For any subset $\mathcal{U} \subseteq \tau$, the union $\bigcup \mathcal{U} \in \tau$;

iv) For all distinct $x, y \in X$, there are $U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.[1]

If τ is a topology, we call the pair $\langle X, \tau \rangle$ a **topological space**.

This definition is intended to mimic the behavior of the collection of open sets in a metric space[2]. In fact, it is a standard exercise in an introductory topology course to prove the following statement.

Proposition 1.1.6. *Let $\langle X, d \rangle$ be a metric space. Then $\langle X, \tau \rangle$ is a topological space with $\tau = \{U \subseteq X : U$ is open with w.r.t. $d\}$. Moreover, the closed sets in d exactly constitute the set $\{X \setminus U : U \in \tau\}$.*

Hausdorff himself generalized this notion slightly in 1935 [25] by replacing (iv) with the property that for all distinct $x, y \in X$, there are $U, V \in \tau$ such that $x \in U \setminus V$ and $y \in V \setminus U$[3]. However, in 1922, Kazimierz Kuratowski[4] of the Scottish Café [31] advanced the currently used definition of topological space which does away with (iv) entirely and only requires that τ meet (i)-(iii) to constitute a topology [28].

Perhaps one of the most fortunate coincidences in mathematics is the level of generality that topology achieves while maintaining much of the richness of metric spaces. While analysis and, to a lesser extent, metric spaces make use of arithmetic as a foundation[5], topology makes a complete departure in using only open sets with no mention of $\mathbb{R}$. So essentially, we've abandoned the notion of distance in lieu of a more abstract conception of closeness conveyed via containment in open neighborhoods.

We present a few definitions of interest to general topology but will, for the most part, assume a basic knowledge of topology in the discussion to come[6].

---

[1]Property (iv) in definition 1.1.5 was in the definition of topology as Hausdorff originally defined it. However, that definition is no longer in use and the current definition which was advanced by Kuratowski in 1922 [28] requires only that τ satisfy properties (i)-(iii). Since this is the accepted definition of the past century, Kuratowski's is the one we employ in this thesis. Any topological space which also satisfies property (iv) is called a **Hausdorff space** or a $\mathbf{T_2}$ **space**.

[2]This point is supported by the fact that sets in a topology are ubiquitously referred to as "open".

[3]A topological space in the sense of Kuratowski that satisfies this property is called a **Fréchet space** or a $\mathbf{T_1}$ **space**.

[4]A heartwarming history of this prominent early topologist including references to his early work can be found in [16].

[5]These fields use arithmetic as a foundation in that the notion of distance makes essential use of the real numbers and their arithmetic properties.

[6]Munkres's *Topology* [32] provides an excellent introduction to general topology.

Definition 1.1.7. Let $\langle X, \tau \rangle$ be a topological space. Then $\mathcal{B} \subseteq \tau$ is a **base** for τ if

i) $\bigcup \mathcal{B} = X$, and

ii) For any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \cap B_2$, there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Definition 1.1.8. For topological spaces $\langle X, \tau \rangle$ and $\langle Y, \tau' \rangle$ a function $f : X \to Y$ is **continuous** if $U \in \tau'$ implies $f^{-1}(U) \in \tau$. The function f is called a **homeomorphism** if f is bijective and both f and f^{-1} are continuous.

Completeness

We proceed to the concept that we intend to study for the remainder of this thesis.

Definition 1.1.9. A metric space X[7] is **complete** if every Cauchy sequence converges[8]. Since every convergent sequence is Cauchy in any metric space, completeness entails that the two are equivalent.

One example of a metric space which fails to be complete is the rational numbers $\mathbb{Q}$ equipped with the usual Euclidean distance function $\langle x, y \rangle \mapsto |x - y|$. For instance, consider the sequence $a : \mathbb{N} \to \mathbb{Q}$ defined by

$$a(n) = \left(1 + \frac{1}{n}\right)^n .$$

We know that the limit of this sequence converges to e in $\mathbb{R}$ and thus is Cauchy in $\mathbb{R}$. The space of rationals we are considering makes use of the same metric so the sequence is Cauchy in $\mathbb{Q}$ as well. However, there is no $q \in \mathbb{Q}$ to which $a(n)$ converges since the unique limit is irrational. Thus $\mathbb{Q}$ fails to be complete.

On the other hand, $\mathbb{R}$ is complete and is often even defined as the metric completion or order completion of $\mathbb{Q}$. Intuitively, $\mathbb{R}$ is complete and $\mathbb{Q}$ is not because $\mathbb{R}$ doesn't have any 'holes' detectable by sequences of real numbers whereas $\mathbb{Q}$ does, namely every irrational number constitutes such a 'hole'. Thus in lackadaisical, mostly misleading terms, a space being complete means it doesn't have any holes[9].

Let's consider another subspace of $\mathbb{R}$ as an example: the open unit interval $(0, 1)$. In the Euclidean metric, $(0, 1)$ is also not complete since the sequence $a(n) = \frac{1}{n}$ is Cauchy but not convergent since the limit $0 \notin (0, 1)$. However, this presents a problem since $(0, 1)$ and $\mathbb{R}$ are homeomorphic[10], and thus are topologically equivalent. Then completeness as defined here is *not* a topological property! We side-step this issue by defining a topological analog of metric completeness.

Definition 1.1.10. A topological space $\langle X, \tau \rangle$ is **completely metrizable** if there is a metric $d : X \times X \to [0, \infty)$ such that d generates[11] τ and $\langle X, d \rangle$ is a complete metric space.

Unlike the property of being a complete metric space, the property of complete metrizability is preserved by homeomorphisms, making it a topological property. Thus $(0, 1)$ equipped with the Euclidean topology is completely metrizable: if $h : \mathbb{R} \to (0, 1)$ is a homeomorphism, then $(0, 1)$ along with the metric $d(x, y) = |h(x) - h(y)|$ is a complete metric space. Interestingly, the set of

[7] With both metric spaces and topological spaces, we'll omit the distance function or topology symbol whenever we feel we can get away with it.

[8] Recall the two kinds of convergence defined in 1.1.2 and 1.1.3.

[9] To see examples of how misleading this inspiration might be, the reader need look no further then the immediately following paragraphs.

[10] For instance the function

$$g(x) = \frac{1}{1 + 2^{-x}}$$

constitutes such a homeomorphism.

[11] By 'generates', we mean that the collection of sets which are open with respect to the metric d are exactly those included in the topology τ.

irrational numbers considered as a subspace of $\mathbb{R}$ is not a complete metric space but *is* completely metrizable, a fact that follows from $\mathbb{R} \setminus \mathbb{Q}$ being homeomorphic to ω^ω. On the other hand, we've already shown that $\mathbb{Q}$ is not a complete metric space, but $\mathbb{Q}$ fails to be completely metrizable as well. We'll prove this fact in section 1.3.

However, there is something unsatisfying about this definition to a topologist. This property is defined entirely in terms of metric spaces, making it difficult to verify in more abstract topological spaces. This constrains the generality of topological spaces when considering completeness since topological spaces need to refer back to metric spaces anyways. In fact, it often isn't clear whether a topological space is homeomorphic to one that is generated by a metric space at all. Thus one of the most important lines of inquiry in general topology was the **metrizability problem**. Study of this question took place over decades and developed several rich topological theorems[12].

In the completeness literature, several topological properties were defined which were found to imply completeness in the context of metric spaces but are weaker than complete metrizability in topological spaces. Such properties produced a gradation of completeness strength against which we can compare spaces. Such properties are called topological completeness properties and constitute the object of study for the remainder of this chapter.

1.2 Compactness

Definition 1.2.1. A topological space $\langle X, \tau \rangle$ is **compact** if for any open cover $\mathcal{U}$ of X[13], there is a finite subcover $\{U_i : i \leq n\} \subseteq \mathcal{U}$ with $\bigcup_{i \leq n} U_i \supseteq X$.

Compactness was originally conceived as a generalization of being a bounded closed subset of Euclidean space. While not yet defined, the essence of compactness can be found in the work of Bolzano as far back as 1817 in his proof of his intermediate value theorem [9]. The following theorem occurs as a lemma in that work and was discovered and proved again by Karl Weierstrass about half a century later.

Theorem 1.2.2 (Bolzano-Weierstrass Theorem)**.** *Any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

It was not until 1906 that Fréchet distilled the concepts of Bolzano-Weierstrass Theorem into the definition of what is now called sequential compactness as presented in 1.2.4 [20].

Such studies began turning towards the concept of open covers in the late eighteenth century by mathematicians like Heine and Borel, as exhibited their famous theorem.

Theorem 1.2.3 (Heine-Borel Theorem)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Finally, the open cover definition of compactness in 1.2.1 was defined in 1929 by influential Russian mathematicians Urysohn and Alexandrov [3].

Before we discuss compactness, there are several properties which are closely related to compactness which will be of use. The first has already been alluded to in our historical discussion of Fréchet and the Bolzano-Weierstrass theorem.

Definition 1.2.4. A topological space X is **sequentially compact** if every sequence $x \in X^{\mathbb{N}}$ has a convergent subsequence.

With this definition the Bolzano-Weierstrass theorem 1.2.2 can be restated as follows:

[12] Hodel gives an overview of this area in the survey [26].

[13] The collection $\mathcal{U}$ is an open cover of X if $\mathcal{U} \subseteq \tau$ and $\bigcup \mathcal{U} \supseteq X$.

Theorem 1.2.5 (Alternate Formulation of the Bolzano-Weierstrass Theorem)**.** *A subset of $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded.*

It follows that sequential compactness and compactness are equivalent in $\mathbb{R}^n$. In fact, this is so for all metric spaces, but not for general topological spaces. Outside of metric spaces, it is not difficult to see the compactness is a stronger property than sequential compactness.For instance, the ordinal space ω_1 is sequentially compact since every countable sequence is bounded but w_1 fails to be compact[14].

Another important property related to compactness is local compactness.

Definition 1.2.6. A space X is **locally compact** if every point has a compact neighborhood. In symbols, for all $x \in X$, there is a compact set K and an open set U such that $x \in U \subseteq K$.

Of particular interest is the class of spaces that are Hausdorff and locally compact. These will play an important role in our study beginning with theorem 1.3.4.

By theorem 1.2.3, $\mathbb{R}$ is not compact. This is easy enough to see by considering the open cover $\mathcal{U} = \{(n-1, n+1) : n \in \mathbb{Z}\}$. Similarly, by theorem 1.2.2, $\mathbb{R}$ fails to be sequentially compact. However, $\mathbb{R}$ is locally compact since given any $x \in \mathbb{R}$, we have $[x-1, x+1]$ is compact and $x \in (x-1, x+1) \subset [x-1, x+1]$.

Finally, we present two more generalizations of the open-cover definition of compactness.

Definition 1.2.7. A space X is **Lindelöf** if every open cover has a countable subcover. A space is **countably compact** if every countable open cover has a finite subcover.

Clearly X is compact if and only if X is Lindelöf and countably compact. The real line $\mathbb{R}$ is Lindelöf since $\mathbb{R}$ has a countable basis[15]. Consequently, $\mathbb{R}$ is not countably compact.

We can consider compactness as a completeness property. In fact, compactness is one of the strongest completeness properties we will consider.

Proposition 1.2.8. *Let $\langle X, d \rangle$ be a compact metric space. Then X is a complete metric space.*

Proof. Let $\{x_n : n \in \mathbb{N}\} \subseteq X$ be a Cauchy sequence. Since X is a compact metric space, X is sequentially compact. Then there is a convergent subsequence $\{x_{n_k} : k \in \mathbb{N}\} \subseteq \{x_n : n \in \mathbb{N}\}$. We call the limit of the subsequence x, so $x_{n_k} \to x$. We want to show that $x_n \to x$. We accomplish this by a straightforward epsilon argument.

Let $\varepsilon > 0$. Then since $x_{n_k} \to x$, there is a $K \in \mathbb{N}$ such that for all $k \geq K$, $d(x_{n_k}, x) < \frac{\varepsilon}{2}$. Similarly, since x_n is Cauchy, there is an $N \in \mathbb{N}$ such that for all $n, m \geq N$, $d(x_n, x_m) < \frac{\varepsilon}{2}$. Set $M = \max\{n_K, N\}$. Then by the triangle inequality, for all $m, n_k \geq M$

$$d(x_m, x) \leq d(x_m, x_{n_k}) + d(x_{n_k}, x) < \varepsilon$$

□

Outside the context of metric spaces, neither complete metrizability nor compactness always entail the other. Since $\mathbb{R}$ is completely metrizable but not compact, complete metrizability is not stronger than compactness. On the other hand, the ordinal space $\omega_1 + 1$ is compact but fails to be completely metrizable. In fact, $\omega_1 + 1$ is not metrizable at all, that is, there is no metric $d : \omega_1 + 1 \times \omega_1 + 1 \to [0, \infty)$ that generates the order topology. This is because all metric spaces are

[14]Consider the open cover $\{\alpha : \alpha < \omega_1\}$ where each α is considered as the set $\alpha = \{\gamma : \gamma < \alpha\}$ rather than as a point in ω_1. Then this open cover has no finite subcover.

[15]The countable basis of the real numbers is the set of open intervals with rational endpoints $\{(q, p) : q, p \in \mathbb{Q}\}$. A space with a countable base is called **second countable**. A space in which every point has a countable neighborhood base is called **first countable**.

first countable[16] but $\omega_1 + 1$ is not first countable, namely because the point $\omega_1 \in \omega_1 + 1$ does not have a countable neighborhood base.

So in general, compactness and complete metrizability are incomparable. We will see as we explore more completeness properties that, for the most part, these two properties will be maximal among all other topological completeness properties.

1.3 Baire Spaces

With compactness and complete metrizability established as the monarchs of the topological completeness kingdom[17], we now turn our attention to the weakest completeness property that is commonly considered.

Definition 1.3.1. A topological space X is called **Baire** if the intersection of any countable collection of open dense sets is dense.

The concept of a Baire space was first advanced by René-Louis Baire in 1899 [4]. Definition 1.3.1 is the definition which is used in modern study, but originally, Baire had a quite different definition in mind. Fortunately, the two are easily shown to be equivalent.

Definition 1.3.2 (Baire's Original Definition)**.** Let X be a topological space. A subset $A \subseteq X$ is called **nowhere dense** if $\operatorname{int}(\operatorname{cl} A) = \varnothing$. A subset A is called **meager** if there is a countable collection of nowhere dense sets $\{N_k : k \in \mathbb{N}\}$ such that $A = \bigcup_{k=1}^{\infty} N_k$. A set that is meager is also said to be **of first category**. Any set that is not of first category is **of second category**. Finally, Baire originally defined a Baire space to be a space X such that every nonempty open set is of second category in X.

Baire used this definition to prove two theorems [4] which are of great interest to us.

Theorem 1.3.3 (The First Baire Category Theorem **BCT1**)**.** *Every complete metric space is Baire.*

Proof. Let $\langle X, d\rangle$ be a complete metric space and let $\{U_n : n \in \omega\}$ be a collection of open dense subsets of X. Since we want to show that $\bigcap_{n\in\omega} U_n$ is dense, fix an open subset $W \subseteq X$ and it suffices to show that $W \cap \bigcap_{n\in\omega} U_n \neq \varnothing$. Since U_0 is dense, $W \cap U_0 \neq \varnothing$. Moreover, since both W and U_0 are open, so there is $x_0 \in X$ and $0 < r_0 < 1$ such that the closed metric ball[18]

$$\bar{B}(x_0, r_0) \subseteq W \cap U_0.$$

By the axiom of dependent choices[19], for all $n \geq 1$, there is $x_n \in X$ and $0 < r_n < \frac{1}{n}$ such that

$$\bar{B}(x_n, r_n) \subseteq B(x_{n-1}, r_{n-1}) \cap U_n.$$

Then for all $n > m$, $x_n \in B(x_m, r_m)$, making the sequence $\langle x_n : n \in \omega\rangle$ Cauchy. Since X is a complete metric space, there is some $x \in X$ such that $\langle x_n : n \in \omega\rangle \to x$. Finally by closedness,

$$x \in \bigcap_{n\in\omega} \bar{B}(x_n, r_n) \subseteq \bigcap_{n\in\omega} U_n \cap W$$

□

[16] Let $\langle X, d\rangle$ be a metric space and let $x \in X$. Consider the countable collection of open metric balls of radius $\frac{1}{n}$ centered at x:

$$\left\{B\left(x, \frac{1}{n}\right) : n \in \mathbb{N}\right\} := \left\{\left\{y \in X : d(x,y) < \frac{1}{n}\right\} : n \in \mathbb{N}\right\}.$$

This collection is a countable neighborhood base for the induced topology on X at the point x.

[17] Like all monarchs, compactness and complete metrizability are **closed-hereditary**.

[18] In a metric space, an **open metric ball** $B(x,r) = \{y \in X : d(x,y) < r\}$. Similarly, a **closed metric ball** $\bar{B}(x,r) = \{y \in X : d(x,y) \leq r\}$.

[19] See discussion following **BCT2**.

It follows that any topological space that is homeomorphic to a complete metric space (i.e. any completely metrizable space) is Baire. Interestingly, **BCT1** is logically equivalent to the axiom of dependent choices[20] over the standard set-theoretic axioms **ZF** [8].

Theorem 1.3.4 (The Second Baire Category Theorem **BCT2**). *Every locally compact Hausdorff space is Baire.*

Proof. The proof of this theorem proceeds similarly to that of **BCT1** except with compact neighborhoods rather than bounded closed metric balls.

Let X be a locally compact Hausdorff topological space and let $\{U_n : n \in \omega\}$ be a collection of open dense subsets of X. Again, fix an open subset $W \subseteq X$ and it suffices to show that $W \cap \bigcap_{n\in\omega} U_n \neq \varnothing$. Since U_0 is dense, $W \cap U_0 \neq \varnothing$. Moreover, since X is locally compact and Hausdorff, every point $x \in X$ has a compact neighborhood base. Then fix $x \in W \cap U_0$ and there is a compact neighborhood K such that

$$x \in \operatorname{int} K \subseteq K \subseteq W \cap U_0.$$

By the axiom of dependent choices, we can choose a compact K_n for all $n \geq 1$ such that

$$x \in \operatorname{int} K_n \subseteq K_n \subseteq \operatorname{int} K_{n-1} \cap U_n.$$

Since $x \in K_n$ for all n, it follows that

$$x \in \bigcap_{n\in\omega} K_n \subseteq \bigcap_{n\in\omega} U_n \cap W$$

□

The study of Baire spaces and related concepts became popular among topologists interested in the completeness. One reason for this is that the Baire Category Theorems are applicable to many areas of mathematics like functional analysis and set-theory[21]. But for topologists, Baire spaces coherently unite the two incomparable completeness properties of (local) compactness and complete metrizability. Moreover, it was shown by Bourbaki[22] in 1955 that the product of an arbitrary collection of completely metrizable spaces is Baire [10].

As mentioned previously, a space being Baire is a weak assumption as far as completeness properties go. Consequently, a great many spaces (and almost all of the spaces we consider here) satisfy this property. However, there are spaces which fail to be Baire. For instance, consider the space of rationals $\mathbb{Q}$ equipped with the usual subspace topology. Enumerate the rational numbers so that $\mathbb{Q} = \{q_n : n \in \omega\}$ and consider the countable collection of open dense sets $\{\mathbb{Q}\setminus\{q_n\} : n \in \omega\}$. Surely $\bigcap_{n\in\omega} (\mathbb{Q} \setminus \{q_n\}) = \varnothing$ so $\mathbb{Q}$ is not Baire. Consequently, $\mathbb{Q}$ also fails to be completely metrizable as well as locally compact. This argument easily extends to show that no countable space is Baire.

Henceforth, we informally define a property φ of a topological space to be a **completeness property** if φ holding in X implies X is Baire and φ of X is implied by complete metrizability and is also implied by Hausdorff local compactness. Thus completeness properties are generalizations of the Baire Category Theorems, and important lense through which this chapter can be interpreted. Section 1.7 provides a more complete overview of what is meant by the word 'complete'.

[20] The axiom of dependent choices (**DC**) is a weakened form of the axiom of choice (**AC**).

[21] One instance of the interplay between Baire's completeness property and set theory can be found in Fleissner and Kunnen's work [18].

[22] For those who are unfamiliar with the work of Nicolas Bourbaki, he is a prolific French mathematician who happens to also not exist.

1.4 Čech-Completeness and Pseudocompleteness

While Baire spaces quickly became one of the most important properties in discussing completeness, many topologists were dissatisfied with some of the property's shortcomings. For one thing, the product of Baire spaces is not guaranteed to be Baire. In fact, Oxtoby showed that assuming the continuum hypothesis, there is a completely regular Baire space whose Cartesian product with itself is of first category[23] in theorem 5 of [35]. This result was strengthened by Fleissner and Kunen in [18] where they defined a class of Baire spaces whose squares are not Baire in **ZFC**.

The first property defined which falls into the range of strength between Baire and compact/completely metrizable is Čech-completeness, first defined by Czech mathematician Eduard Čech. This definition comes in the same 1937 paper in which Čech defined the Stone-Čech compactification βX of a topological space X [12]. The definition of βX is complicated so it will not be stated here.

Definition 1.4.1. A topological space X is **Čech-complete** if there is a compact Hausdorff space B such that X is homeomorphic to a G_δ subset of B.

Čech showed that this definition is equivalent to X being homeomorphic to a completely regular G_δ subset of βX. This property was motivated in part by the fact that a metric space is complete if and only if it is a G_δ subset of its metric completion. Then we see that Čech-completeness is a natural generalization in this direction.

Since compactness is closed-hereditary and G_δ-hereditary[24], Čech was able to prove the following proposition which is stronger than being closed-hereditary or G_δ-hereditary.

Proposition 1.4.2 (Čech)**.** *For a Čech-complete space X, a subset A is Čech-complete if and only if $A = C \cap G$ where C is closed in X and G is a G_δ subset of X.*

As a result, compact spaces are Čech-complete[25].

Proposition 1.4.3 (Čech)**.** *Every Čech-complete space is Baire.*

Proposition 1.4.4 (Čech)**.** *If X is a metrizable space and X is Čech-complete, then X is completely metrizable.*

Thus completely metrizable spaces are Čech-complete. The preceding propositions unambiguously qualify Čech-completeness to be considered as a completeness property. In fact, Čech-completeness is much more restrictive than being Baire. One example witnessing this difference is $\mathbb{R}^\kappa$ with the usual product topology and κ an uncountable cardinal. This space is Baire but not Čech-complete.

Unlike Baire completeness, Čech-completeness is preserved under arbitrary products of Čech-complete spaces. However, this was not proven until sometime later by Oxtoby[26] in [35]. At this time, comparatively little wight was given to preservation of completeness under operations like infinite products. In fact, Bourbaki hadn't yet considered infinite products of complete metric spaces and little was known about the product of Baire spaces. Thus Čech can be forgiven for not considering products. He was primarily concerned with defining a property stronger than Baire but still weaker than both Hausdorff local compactness and complete metrizability, thus establishing a strengthening of Baire's influential theorems.

By the 1960's, preservation under products was becoming increasingly important. Consequently, Oxtoby set out to define a new completeness property called pseudocompleteness in order to better understand the behavior of products of Baire spaces.

[23] Otherwise known as meager, see definition 1.3.2. Consequently such a space is certainly not Baire.

[24] A topological property φ is called **hereditary** if X possessing φ implies that any open subset has φ when considered with ths subspace topology. Similarly, φ is **closed-hereditary** if any closed subspace of X has φ. The definition is analogous for $\mathbf{G}_\delta$ **-hereditary** and G_δ subsets.

[25] In fact, Hausdorff locally compact spaces are Čech-complete.

[26] See Corollary 1.4.9 later in this section.

Definition 1.4.5. For a topological space, a collection of nonempty open subsets $\mathcal{B}$ is called a **$\boldsymbol{\pi}$-base** or **pseudobase** if every nonempty open set contains at least one element of $\mathcal{B}$[27]. A topological space is **quasiregular** if every nonempty open set contains the closure of some nonempty open set. Finally, a quasiregular space X is **pseudocomplete** if there is a sequence of π-bases $\{\mathcal{B}_n : n \in \omega\}$ in X such that if, for all n, $U_n \in \mathcal{B}_n$ and $U_n \supseteq \operatorname{cl} U_{n+1}$, then $\bigcap_{n\in\omega} U_n \neq \varnothing$.

The following proposition is immediate.

Proposition 1.4.6 (Oxtoby)**.** *Any pseudocomplete space is Baire.*

Proof. Let X be a pseudocomplete space and let $\{U_n : n \in \omega\}$ be a collection of open dense subsets of X. As in previous proofs regarding the Baire property, to show density we let W be any open set in X with the intention of showing that $W \cap \bigcap_{n\in\omega} U_n \neq \varnothing$. Since X is pseudocomplete, let $\{\mathcal{B}_k : k \in \omega\}$ be a collection of π-bases with the corresponding pseudocompletness property. Since X is quasiregular, there is a $B_0 \in \mathcal{B}_0$ such that $\operatorname{cl} B_0 \in W \cap U_0$. We proceed by recursion, letting $B_n \in \mathcal{B}_n$ such that $\operatorname{cl} B_n \subseteq B_{n-1} \cap U_n$. By pseudocompleteness, $\bigcap_{n\in\omega} B_n \neq \varnothing$. Since $B_n \subseteq W \cap U_n$ so $W \cap \bigcap_{n\in\omega} U_n \neq \varnothing$. ☐

Proposition 1.4.7 (Oxtoby)**.** *Any Čech-complete space is pseudocomplete.*

This is proved in [35], but the proof is omitted here. The following theorem is the focus of Oxtoby's consideration of pseudocompleteness.

Theorem 1.4.8 (Oxtoby)**.** *The product of any family of pseudocomplete spaces is pseudocomplete.*

Corollary 1.4.9. *The product of any family of Čech-complete spaces is Baire.*

1.5 Subcompactness

Influential Dutch topologist John de Groot was also interested in devising a generalized Baire Category Theorem. In his 1963 paper [14], he attempted to do just that. He leveled two complaints against the Baire Category Theorem as it stood:

> "Firstly, its formulation deals with two important classes of spaces of a *totally different nature*; secondly, why the countability of the number of subsets?" [14]

Towards remedying the first of these complaints, de Groot defines a completeness property called subcompactness which he hopes will constitute a more natural generalization of local compactness and complete metrizability.

Definition 1.5.1. Let X be a topological space with base $\mathcal{B}$. A nonempty subset $\mathcal{F} \subseteq \mathcal{B}$ is a **regular filter base** relative to $\mathcal{B}$ if for any $F_1, F_2 \in \mathcal{F}$, there is an $F_3 \in \mathcal{F}$ such that $\operatorname{cl} F_3 \subseteq F_1 \cap F_2$. We also require that $\varnothing \notin \mathcal{F}$.

A regular filter base $\mathcal{F}$ is **preconvergent** if $\bigcap \mathcal{F} \neq \varnothing$ in X. A regular filter base $\mathcal{F}$ is **convergent** if there is an element $x \in X$ such that $\bigcap \mathcal{F} = \{x\}$. In this case we say $\lim \mathcal{F} = x$.

We say that a regular filter base $\mathcal{F}$ is a regular ultrafilter base if $\mathcal{F}$ is maximal with respect to set-inclusion, i.e. no other regular filter base contains $\mathcal{F}$ as a subset. By the ultrafilter lemma[28], any regular filter base can be extended to a regular ultrafilter base.

[27] If the topological space is considered as a complete lattice of open sets ordered by set-inclusion, $\mathcal{B}$ is a π-base if and only if $\mathcal{B}$ is order-dense in the lattice.

[28] The ultrafilter lemma states that any filter is contained in an ultrafilter, and is an immediate consequence of the Boolean Prime Ideal Theorem which states that any ideal is contained in a prime ideal. Both of these are consequences of Zorn's Lemma.

Countable sequences are a reasonable way to measure convergence in a first countable space like a metric space. However, convergence measured via countable sequences (of points or of open sets) is often contrived or downright unhelpful in larger spaces. The standard mechanism used to generalize this convergence is that of converging filters[29]. For instance, if κ is an uncountable cardinal, convergence of countable sequences of points or open sets means little in $\mathbb{R}^\kappa$, whereas convergence of filters satisfies our intuition of convergence.

Definition 1.5.2. A regular Hausdorff space $\langle X, \tau \rangle$ is called **subcompact** there is a base $\mathcal{B}$ for τ for which one of the following equivalent conditions holds:

i) every regular filter base $\mathcal{F}$ relative to $\mathcal{B}$ is preconvergent, i.e. for all regular filter bases $\mathcal{F} \subseteq \mathcal{B}$, $\bigcap \mathcal{F} \neq \varnothing$;

ii) every regular ultrafilter $\mathcal{F}$ relative to $\mathcal{B}$ is convergent, i.e. for all regular ultrafilter bases $\mathcal{F} \subseteq \mathcal{B}$, there is an $x \in X$ such that $\bigcap \mathcal{F} = \{x\}$.

Similarly, X is called **countably subcompact** if there is a base $\mathcal{B}$ such that every countable regular filter base relative to $\mathcal{B}$ is preconvergent.

De Groot remarks that if X is Hausdorff and locally compact, then a base open sets whose closures are compact will satisfy the requirement that any regular filter base is preconvergent. Thus locally compact Hausdorff spaces are subcompact.

Theorem 1.5.3 (de Groot)**.** *If X is metrizable, then the following are equivalent:*

i) X is countably subcompact

ii) X is subcompact

iii) X is completely metrizable.

The proof has been omitted here but can be found in [14]. As a result of this theorem, completely metrizable spaces are subcompact.

Thus it seems de Groot has been successful in addressing his first complaint since the class of subcompact spaces seems natural and contains both locally compact Hausdorff spaces and completely metrizable spaces. However, addressing his second concern presents more difficulty. One cannot simply extend the notion of Baire to require density of the intersection of κ-many open dense sets and retain any interesting properties. For instance, $\mathbb{R}$ does not satisfy this notion for the cardinal $2^{\aleph_0}$ for the same reason that $\mathbb{Q}$ is not Baire. To generalize the notion of Baire to arbitrary cardinality, de Groot first generalizes density and nowhere-density accordingly. It is in these terms that Baire is generalized to arbitrary cardinality. In his "Generalized Baire-Theorem", de Groot claims that subcompactness implies this newly defined generalization of Baire for all cardinals. Unfortunately, as Isidore Fleischer demonstrates in [17], there is an error in de Groot's argument which is solved by slightly strengthening the hypotheses. Fortunately, the far weaker conclusion that all subcompact spaces are Baire remains valid. Actually, the following proposition is easily verified.

Proposition 1.5.4. *Every subcompact space is pseudocomplete (and therefore Baire).*

Proof. Let X be a topological space with subcompact base $\mathcal{B}$. Then by definition, any regular filter base $\mathcal{F} \subseteq \mathcal{B}$ is preconvergent. Since $\mathcal{B}$ is a base, it certainly qualifies as a π-base. For the countable collection of π-bases needed for pseudocompleteness, we simply take the singleton $\{\mathcal{B}\}$. Let $\{U_n : n \in \omega\} \subseteq \mathcal{B}$ be a sequence of basic open sets such that for all n, $\operatorname{cl} U_{n+1} \subseteq U_n$. This countable decreasing chain surely constitutes a filter. Thus by subcompactness of $\mathcal{B}$, $\bigcap_{n\in\omega} U_n \neq \varnothing$. $\square$

[29] Compare this notion to the directed suprema of section 2.2.

In 5.1.4 of [1], Aarts and Lutzer give an example of a space that is subcompact but not Čech-complete. The reverse implication, that all Čech-complete spaces are subcompact is still not known. At any rate, this demonstrates that Čech-completeness is at least as strong as subcompactness. The Sorgenfrey line $\mathbb{S}$ is defined in [41] to be the topology on the set of real numbers generated by the base of half open intervals $\{[a, b) : a, b \in \mathbb{R}\}$.

Proposition 1.5.5 (Aarts, Lutzer)**.** *The Sorgenfrey line $\mathbb{S}$ is subcompact but not Čech-complete.*

Proof. First we show that $\mathbb{S}$ is subcompact. By a wellordering argument, it can be shown that $\mathbb{S}$ has a base $\mathcal{I}$ of bounded, half-open intervals such that for any $I_1, I_2 \in \mathcal{I}$ that are distinct, $\sup I_1 \neq \sup I_2$. We identify $\mathcal{I}$ as our candidate for subcompact base and let $\mathcal{F} \subseteq \mathcal{I}$ be a regular filter base. Since $\mathcal{F}$ consists of bounded intervals, the set $E = \bigcap\{\mathrm{cl}_{\mathbb{R}} I : I \in \mathcal{F}\} \neq \varnothing$ where $\mathrm{cl}_{\mathbb{R}}$ denotes the closure operation with respect to the Euclidean topology. Let $p \in E$ and suppose for the sake of contradiction that $\bigcap \mathcal{F} = \varnothing$. Then there would be distinct sets $I_1, I_2 \in \mathcal{F}$ neither of which contain p. But since $p \in \mathrm{cl}_{\mathbb{R}} I_1 \cap \mathrm{cl}_{\mathbb{R}} I_2$, then $\sup I_1 = p = \sup I_2$ giving a contradiction.

That $\mathbb{S}$ is not Čech-complete makes use of several propositions from the literature as well as some metrization theorems. They are not included here but can be found in 5.1.4 of [1]. $\square$

1.6 Topological Games: A Reading from the Scottish Book

Lviv, Poland was a topological hub during the interwar period. It was the site of collaboration between some of the most prominent members of the field. These mathematicians met at the Scottish Café to casually propose problems, conjectures, and proofs of interest. In 1935 after several years of the group's collaboration, Stefan Banach decided that the group should actually record their discussions. Thereafter, the café kept a book which came to be known as the Scottish Book. The staff brought it out along with coffee whenever these mathematicians gathered [31].

Problem 43 in the book was posed by Mazur[30] and the English translation is quoted here:

> "Definition of a certain game. Given is a set E of real numbers. A game between two players A and B is defined as follows: A selects an arbitrary interval d_1; B then selects an arbitrary segment d_2 contained in d_1; then A in his turn selects an arbitrary segment d_3 contained in d_2 and so on. A wins if the intersection $d_1, d_2, d_3, \ldots, d_n, \ldots$ contains a point of the set E; otherwise her loses. If E is a complement of a set of first category, there exists a method through which A can win; if E is a set of first category, there exists a method through which B will win.
>
> **Problem:** Is it true that there exists a method of winning for the play A only for those sets E whose complement is, in a certain interval, of first category; similarly, does a method of win exist for B if E is a set of first category?" [31]

Banach answered in the affirmative, and in doing so, was awarded a bottle of wine by Ulam. Finite games had been considered prior to this, but Mazur's suggestion seems to be the first instance of a game of infinite length being considered mathematically.

This question evolved in two very fruitful directions. The first grew out of a modification by Ulam for which, instead of playing intervals, the players alternate plays of 0 and 1. Given a subset $E \subseteq 2^{\omega}$, Player I wins the game[31] if and only if the sequence of plays by I and II is an element of E. The set E is said to be **determined** if either I or II has a winning strategy. This game and many of its modifications gave birth to a fascinating area of set theory. Briefly, it was found that the

[30] Problem 43 was recorded on August 4, 1935 but was likely posed several years prior.

[31] We adopt the convention that Player I is the player who goes first and Player II is the player who replies. Other terms have also been used in the literature like using β or EMPTY to refer to Player I and α or NONEMPTY to refer to Player II.

statement "every subset of 2^ω is determined"[32] contradicts the axiom of choice but is independent of the set theoretic axioms **ZF**. The theory $\mathbf{ZF}+\neg\mathbf{AC}+\mathbf{AD}$ proves some very strange facts about $\mathbb{R}$ and the cardinals. For instance, in this theory, every subset of $\mathbb{R}$ is Lebesgue measurable. Perhaps stranger still, the cardinal $\aleph_1$ is a measurable cardinal.

The other major direction originating from Problem 43, which we will focus on here, involves another generalization of Mazur's game, this one proposed by Banach. The game has gone through several formulations, but the version currently considered in the topology literature will be presented.

Definition 1.6.1. Let X be a topological space. In the **Banach-Mazur game** on X, denoted $\mathrm{BM}(X)$, has plays consisting of open sets contained in the previous play. Thus Player I plays a nonempty open set U_1, to which Player II replies a nonempty open set $V_1 \subseteq U_1$. Then Player I plays a nonempty open set $U_2 \subseteq V_1$ and Player II plays a nonempty open set $V_2 \subseteq U_2$. They continue in this fashion for ω rounds. Player II wins if and only if

$$\bigcap_{n\in\omega} U_n \cap V_n \neq \varnothing.$$

In fact,

$$\bigcap_{n\in\omega} U_n \cap V_n = \bigcap_{n\in\omega} U_n = \bigcap_{n\in\omega} V_n$$

so only one of the players' sequence of plays need be considered.

Already we have seen enough to know that completeness will play a prominent role in this game since it involves intersecting open sets being nonempty. The concept of a winning strategy in a game is hopefully an intuitive one, but for the sake of formalizing proofs, we give a formal definition. Formally, the moves of $\mathrm{BM}(X)$ are given by a tree where the nodes represent possible states of the game and the directed edges between nodes denote legal plays in the game. In this tree, every node other than the root has exactly one edge going into it represent the state of the game prior to the previous move. Let s be a node in the game representing the plays $\langle U_1, V_1, \ldots, V_{n-1}, U_n\rangle$. Then it is Player II's turn to move. Note that II must play any nonempty open set contained in U_n. Thus the nodes succeeding s compose the set

$$\{\langle U_1, V_1, \ldots, V_{n-1}, U_n, V_n\rangle : V_n \subseteq U_n \wedge V_n \in \tau^*\}.$$

By τ^*, we mean the collection of all nonempty open sets of the topology τ of the space we are considering X. Call this tree of plays $\mathbb{T}$. Then we can say

$$\mathbb{T} = \left\{s \in (\tau^*)^{<\omega} : \forall n \in \mathrm{dom}(s) \setminus \{0\}\, (s(n) \subseteq s(n-1))\right\}.$$

The order on $\mathbb{T}$ is just the usual order of extension, i.e. $s \leq s'$ if and only if for $\mathrm{dom}(s') \supseteq \mathrm{dom}(s)$ and for all $n \in \mathrm{dom}(s)$, $s'(n) = s(n)$. Oftentimes in set theory, a winning strategy is defined to be a subtree of $\mathbb{T}$ with the property that Player II can always play in the subtree and, as long as she does so, she wins the game. However, there is little need for this degree of formality for our purposes. Instead, we will use a simpler functional notation.

Definition 1.6.2. Let S be the set of legal moves that can be made by Player I in $\mathrm{BM}(X)$. For the Banach Mazur game, this is just τ^* (without accounting for which moves may legally succeed each other). Consider a function $\varsigma : S^{<\omega} \to S$ such that for any $t \in S^{<\omega}$ with $t(i+1) \subseteq \varsigma(t(1), \ldots t(i))$, $\varsigma(t(1), \ldots, t(i), t(i+1)) \subseteq t(i+1)$. Such a function ς is called a **strategy** for Player II since it always outputs a legal move in the game $\mathrm{BM}(X)$.

[32]This statement is known as the axiom of determinacy and is abbreviated **AD**. Determinacy in 2^ω is logically equivalent to determinacy in ω^ω. The modification of Ulam's game in which players each play any element of ω is equivalent to Ulam's original game.

The function ς is a **winning strategy** for Player II if for any $s \in S^\omega$ such that for any $n \in \omega$, $s(n+1) \subseteq \varsigma(s(1), \ldots, s(n)) \subseteq s(n)$, the intersection

$$\bigcap_{n\in\omega} \varsigma\left(s(1), \ldots, s(n)\right) \neq \varnothing.$$

A function $\varsigma : S \to S$ is a **stationary winning strategy** for Player II if for any $s \in S^\omega$ such that for any $n \in \omega$, $s(n+1) \subseteq \varsigma(s(n)) \subseteq s(n)$, the intersection

$$\bigcap_{n\in\omega} \varsigma\left(s(n)\right) \neq \varnothing.$$

Intuitively, a winning strategy is a procedure by which Player II can can move that always results in victory. A Stationary winning strategy is a winning strategy which does not make use of any information aside from the previous move of the opponent.

Definition 1.6.3. Finally, a space X is **weakly $\boldsymbol{\alpha}$-favorable** if Player II (nonempty player) has a winning strategy in $\mathrm{BM}(X)$. A space X is **$\boldsymbol{\alpha}$-favorable** if Player II has a stationary winning strategy.

These two properties of topological spaces will prove to be completeness properties which approximate the Baire completeness property quite nicely. The following theorem generalizes Banach's original answer to Mazur for topological spaces in general. The theorem was first proved by Krom in [27] but relied heavily on results of Oxtoby in [34].

For X a topological space:

Theorem 1.6.4 (Krom)**.** *Player I has a winning strategy in* $\mathrm{BM}(X)$ *if and only if* X *is not Baire.*

This characterization of Baire spaces in terms of topological games immediately entails that all weakly α-favorable spaces and α-favorable spaces are Baire. On the other side of the completeness spectrum, we have the following result:

Proposition 1.6.5. *Any pseudocomplete space is* α*-favorable.*

Proof. First it is convenient to note that that Player II has a stationary strategy if and only if she has a **Markov** strategy, that is, a strategy which only makes use of the opponent's most recent play as well as the number of plays that have already occurred[33].

Let X be pseudocomplete. We want to construct a Markov strategy for Player II. Since we have no other information, suppose Player I just played the open set U and n plays have been made so far. By pseudocompleteness, there is a sequence of π-bases $\{P(n) : n \in \theta\}$, such that any closure-inclusion decreasing sequence in $\prod_{n\in\omega} P(n)$ has nonempty intersection. Since X is quasiregular, there is a nonempty open set $U' \subseteq X$ such that $\mathrm{cl}\, U' \subseteq U$. In the π-base $P(n+1)$ (where n is the number of plays that have been made) there is a $V_{n+1} \in P(n+1)$ such that $V_{n+1} \subseteq U'$ so $\mathrm{cl}\, V_{n+1} \subseteq U$.

Then Player II plays V_{n+1} and by pseudocompleteness,

$$\bigcap_{m\in\mathbb{E}} V_m \neq \varnothing$$

where $\mathbb{E}$ denotes the set of positive even integers[34]. $\square$

[33]This is a result of Galvin and Telgársky [21].

[34]We consider only the even numbers since Player 1 plays during odd rounds. However, $\mathbb{E}$ and ω are order-isomorphic so relabelling the indices is not an issue.

The properties weakly α-favorable and α-favorable are given different names for a reason: having a winning strategy in $\mathrm{BM}(X)$ does not imply that one as a stationary winning strategy. This fact is witnessed by Debs's space presented in [15].

Unfortunately, weak α-favorability and α-favorability lack an important feature desirable in completeness properties: neither imply that a metric space is completely metrizable. This was unsatisfactory to Gustave Choquet, so he designed his own modification to the Banach Mazur game to remedy this issue [13].

Definition 1.6.6. The **Choquet game** on a space $\langle X, \tau \rangle$, denoted $\mathrm{Ch}(X)$, is a modification of the Banach Mazur game in which Player I's plays consist of ordered pairs of the form $\langle U, x \rangle$ where $U \in \tau^*$ and $x \in U$. Player II replies with and ordered pair $\langle V, x \rangle$ such that $x \in V \subseteq U$ and $V \in \tau^*$. Note that Player II may not change the selected point! However in the next play, Player I may by selecting $\langle U', y \rangle$ such that $y \in U' \subseteq V$ with $U' \in \tau^*$. The plays continue in this fashion with Player I selecting a (potentially different) point during his turn.

If Player II has a winning strategy in $\mathrm{Ch}(X)$, X is called **Choquet complete** and if Player II has a stationary strategy, X is **strongly $\boldsymbol{\alpha}$-favorable**.

It immediately follows that X being Choquet complete implies that X is weakly α-favorable, and likewise, X being strongly α-favorable implies that X is α-favorable. That these concepts are indeed distinct as evidenced by the following example.

Example 1.6.7. Consider the space $X = (\mathbb{R} \times [0, \infty)) \setminus (\mathbb{I} \times \{0\})$ as a subspace of $\mathbb{R}^2$ where $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$. In $\mathrm{BM}(X)$, Plalyer II has a winning strategy since on her first move, she can play an open set avoiding the copy of $\mathbb{Q}$ at $y = 0$, thereby reducing the game to one on $\mathbb{R}$, which Player II can win. On the other hand, Player I has a winning strategy in $\mathrm{Ch}(X)$:

Enumerate the rationals $\mathbb{Q} = \{q_1, q_2, \dots\}$. Then Player 1 can just play open sets that shrink to the empty set along with the first enumerated point $\langle q_n, 0 \rangle$ in the set. The shrinking open sets along with the placement of points forces the sequence of plays into the copy of $\mathbb{Q}$ at the bottom of the space. Since this part of the space is countable, Player I can eliminate each point thereby ensuring that the intersection of plays is empty.

This aligns with intuition since the space described doesn't seem like it 'should be' complete. Moreover, Choquet proved in [13] that this new game possesses what the Banach Mazur game lacked:

Theorem 1.6.8 (Choquet)**.** *If a metrizable space is Choquet complete, then it is completely metrizable.*

1.7 Evaluating Completeness

Thus far, we have introduced a web of completeness properties related to each other in numerous distinct ways, a web which will become even more complicated come chapter 3. Even after that, there are numerous properties and modifications of properties which have not been mentioned here. Bennett and Lutzer provide a solid survey of many of these properties in [7], as do Aarts and Lutzer in the older [1]. Telegársky provides a comprehensive survey of topological game properties in [43]. Despite these, we still need some criteria with which we can evaluate proposed completeness properties or at least compare their utility.

Let's return to our basic intuition regarding completeness. In the context of metric spaces, the epitome of completeness[35] is unambiguously the property of completeness wherein all Cauchy sequences converge. While it is true that compactness is a stronger property and still maintains a

[35]Or one might say 'the ideal completion'.

distinctly complete flavor, being a complete metric is much closer to our intuition. Any definition of complete which excludes $\mathbb{R}$ is too restrictive to model our conception of completeness.

The problem arises however, when we extend our scope to include topological spaces in general. Complete metrizability no longer perfectly captures the idea of completeness since it excludes seemingly 'complete' spaces like $\mathbb{R}^\kappa$ for uncountable cardinals κ. Moreover, outside the context of metric spaces, seemingly equally 'complete' properties like locally compact Hausdorff and complete metrizability become distinct. Finally, although this point is more of a stylistic preference but still holds some small weight, complete metrizability doesn't *feel* topological since any proof must involve defining a metric function or a homeomorphism to a metric space.

Some of this intuition is echoed in the so called Unification problem of Aarts and Lutzer:

> "Is there a natural class of spaces which contains all completely metrizable spaces and all locally compact Hausdorff space, and for which the conclusion of the Baire category theorem remains valid?" [1]

Another important feature of our intuitive completeness:
It is preferable for a completeness property to coincide with complete metrizability when restricted to the class of metric spaces.

Aarts and Lutzer present another avenue of evaluation via Problem III

> "To what extent do the completeness properties in the various solutions to [the Unification problem] behave[36] at least as well as the Baire spaces?" [1]

We note here that Wicke and Worrell have enumerated their own eleven axioms describing this generalized completeness, placing particular emphasis on the preservation problem [44].

We said that Baire spaces failed these requirements because they behave erratically in the construction of products. Pseudocompleteness and and weak α-favorability fail because they are weaker than complete metrizability even when restricted to metrizable spaces. Čech-completeness failed in that it is too restrictive. This of course does not mean that these properties are useless; to the contrary, some of them are indispensable in the search for a natural generalization of completeness.

Subcompactness is an interesting case since it is so far the only competeness property not essentially tied to a certain cardinality aside perhaps from local compactness. Unfortunately, the de Groot's property is often difficult to manipulate so many basic open questions surround subcompactness. In fact, it is still not known whether subcompactness is G_δ-hereditary (and therefore implies Čech-completeness) or whether the property is closed under topologically relevant constructions like images of perfect mappings or retracts [7]. Choquet completeness seems like a strong contender to be the 'natural' generalization of completeness. Unfortunately the property behaves strangely in certain spaces [43].

In the years since the search for a solution to the Unification problem began, no front-runner has emerged from the tangle of completeness properties. In chapter 3, we advance two properties which fail some of these intuitions but introduce a new perspective which could allow progress to be made towards a generalized completeness property.

[36] Here the authors are referring to behavior as preservation of completeness under usual topological operations or constructions.

Chapter 2

Introduction to Domain Theory

2.1 Denotational Semantics

We begin this chapter with a question with seemingly no relation to topology: can we give a mathematical semantics for computer languages? That is, is there some way to encode the objects of computer languages using strictly mathematical notions? At first glance, this seems trivial since high-level computer languages are feed into a compiler which mechanically translates the statement into the binary bits of computers. However, upon closer inspection, serious issues arise in this endeavour with relation to unrestricted procedures and self-application of procedures.

The framework for a solution comes via logician Dana Scott in 1970 [37]. Until then, it had often been suggested that the meaning of a computer language rests entirely in its compiler (that process of translation from high-level computer language to binary machine language), yet this cannot be since the same high-level language can have several different compilers. In light of this, Scott turns to mathematics.

Intuitively, a computational procedure can be thought of as a function from the data type of the input to the data type of the output. This is a step in the right direction but this approach has a few issues we need to deal with. Mathematical functions are certainly more abstract than procedures, making them simpler and more general while sacrificing more specific information like computational complexity. This generality may present some difficulty in the case of non-computable functions for obvious reasons, or even in the case of infinite functions since these modelled procedures eventually must be implemented in a finite machine. As a result, we must develop a notion of finite approximation and of computability within our model. Furthermore, it is commonplace to define procedures which are "unrestricted" in that they may take inputs of any data type at all and output a similarly unrestricted procedure. In mathematics, such a function domain would fail to be a set, risking the consistency of our new model. Another issue, which is of particular interest to Scott, is the self-application of procedures. To illustrate this, he accounts for a store mathematically as a function $\sigma : L \to V$ where L enumerates the possible locations of data and V is the set of all possible values, so that $\sigma(l) \in V$ is interpreted as "the current contents of location l is $\sigma(l)$". The store function constitutes a state (at a particular time). If Σ is the set of possible states, then a command is a function $Y : \Sigma \to \Sigma$ which transforms one state into another state. Suppose a command is stored in location l. Then $\sigma(l)$ is a function from Σ to Σ, so $\sigma(l)(\sigma)$ is well-defined. Each of these potential issues needs to be dealt with in Scott's semantics.

Now we begin with an outline of Scott's semantics for an arbitrary computer language as presented in [37]:

A **data type** is represented by the set D of all objects of that type. These objects may be related to each other in more ways than being unrelated or identical. Thus for $x, y \in D$, we define a relation $x \sqsubseteq y$ which we intuitively interpret as y is consistent with x and y is at least as accurate[1] as x. Scott argues that this sort of relation exists naturally (or ought to) in all computer languages. This relation can be equally well expressed in terms of approximation: $x \sqsubseteq y$ means that y is a better, more complete approximation than x. Approximation of what? That is the flaw with this method of intuition: in order to maintain the generality of our semantics, we can't really talk about what is being approximated just yet. For now a useful example to keep in mind might be decimal approximations of natural numbers where $x \sqsubseteq y$ means that the decimal string y extends x and so does a better job of approximating certain real numbers. The intuition for this relation suggests that it be reflexive, symmetric, and transitive, that is, a **partially ordered set** or poset for short[2].

Let $\langle D, \sqsubseteq \rangle$ and $\langle D', \sqsubseteq' \rangle$ be two posets representing data types. What kind of mappings should we consider between these data types? Following our intuitive picture, we want to consider mappings $f : D \to D'$ that preserve accuracy, i.e. mappings are be **monotonic**[3]. In symbols, $x \sqsubseteq y$ implies $f(x) \sqsubseteq' f(y)$. If $x, y \in D$ are approximating some value, and $x \sqsubseteq y$ (y is a more complete approximation than x), then given a monotonic function $f : D \to D'$, we have $f(x) \sqsubseteq' f(y)$, so $f(y)$ is still a better approximation in D' than $f(x)$.

Suppose we have some infinite sequence of approximations

$$x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \ldots.$$

Then our mathematical instinct should be screaming "Take the limit!" which makes sense given our guiding example of decimal approximations of real numbers. If we are finitely approximating something, the best way to interact with that something is via a limit of approximations. Naturally, we want the limit of such a sequence to exist in our model. The limit of an increasing sequence is its supremum (least upper bound). For the sake of mathematical simplicity, we generalize and say that our data types must include suprema of *all* subsets. This entails that all subsets also have an infimumso we are specifying that our data types must be **complete lattices**[4].

Notationally, for $x, y \in D$, $x \sqcup y := \sup\{x, y\} \in D$ is called the **join** of x and y in lattice theory, and $x \sqcap y := \inf\{x, y\} \in D$ is called the **meet** of x and y. Likewise for an arbitrary subsets $A \subseteq D$, the join of A is $\bigsqcup A := \sup A$ and the meet is $\bigsqcap A := \inf A$. Since suprema and infima are guaranteed for any subset, we may take $\bigsqcap D$ which must be the smallest element in D, denoted $\bot$. Likewise, there must be a largest element $\bigsqcup D =: \top$. Intuitively, $\bot$ is interpreted as having only trivial information, making it a useless approximation and $\top$ is interpreted as having inconsistent or overdetermined information, making it similarly useless by itself. Mathematically, these objects are useful since we may write $x \sqcup y = \top$ to mean that x and y possess inconsistent information and $x \sqcap y = \bot$ to mean that x and y possess only unrelated information[5]. Each of these is distinct from the weaker relation of **incomparability** which means $x \not\sqsubseteq y$ and $y \not\sqsubseteq x$.

The notion of limit-taking outlined here is our analog for measuring finite approximation. We build on this to tackle computability as well. Intuitively, a function being computable means that "getting out a 'finite' amount of information about one of its values ought to require putting in only a 'finite' amount of information about the argument" [37]. Using this intuition, we specify that we want to consider only those mappings which preserve these limits (suprema). We formalizing this by first defining $A \subseteq D$ to be **directed** if for all $x, y \in A$, there is a $z \in A$ such that $x \sqsubseteq z$ and

[1]Here, Scott means "accurate" to mean something like "possesses more information" rather "possesses information that is more true".

[2]Scott takes this intuition as "Axiom 1" in [37].

[3]Scott takes this statement to be "Axiom 2" in [37].

[4]This is Axiom 3 in [37]. Note that Axiom 3 implies Axiom 1 since any complete lattice is necessarily a partially ordered set.

[5]In terms of decimal approximations of real numbers, the former case is interpreted as x and y have no common extension, that is they disagree on a digit which is defined for both. Unfortunately the latter case is trivial since $x \sqcap y = \bot$ if and only if $x = \varnothing$ or $y = \varnothing$.

$y \sqsubseteq z$. A function $f : D \to D'$ is called **continuous** if for all directed subsets $A \subseteq D$,

$$f\left(\bigsqcup A\right) = \bigsqcup\{f(x) : x \in A\},$$

that is, continuous functions preserve directed suprema (limits of directed subsets). A subset being directed captures the idea of convergence more faithfully than an arbitrary subset. For instance, a subset might be decreasing in relation to $\sqsubseteq$ which does nothing towards the goal of approximation. Then we lose no part of our intuition and avoid some potential problems by associating computability with the preservation of directed suprema[6].

So far so good, but we have yet to account for the realization of this semantics in a finite machine, a task for which our theory is still much too abstract. To formalize the notion of finite representability which is found in machines, Scott turns to topology. This should come as no surprise given our usage of topologically sensitive words like 'limit' and 'continuous'.

Definition 2.1.1. A set $U \subseteq D$ is defined to be **Scott open** if and only if

(1) for all $x \in U$, if $x \sqsubseteq y$, then $y \in U$ (we say U is **closed upward** or U is an up-set); and

(2) for all directed subsets $A \subseteq D$ with $\bigsqcup A \in U$, $A \cap U \neq \varnothing$.

The collection of all subsets satisfying this property constitutes a topology called the **Scott topology** on D and a function $f : D \to D'$ is continuous in the sense of limit-preservation if and only if it is continuous in the topological sense with respect to the Scott topology.

Using this definition, we have characterized our previous notions of approximation and of computability in strictly topological terms. This will prove to be an immensely useful definition throughout this thesis. We define the relation $\prec$ on D such that

$$x \prec y \text{ iff } y \in \operatorname{int} \uparrow x = \operatorname{int}\{z \in D : x \sqsubseteq z\}.$$

While this seems rather irreflexive, there are data types containing certain elements x such that $x \prec x$. We define a slightly weaker relation $\preccurlyeq$ on D as follows:

$$x \preccurlyeq y \text{ iff } \bigsqcap \operatorname{int} \uparrow x \sqsubseteq y.$$

Inspired by standard methods in topology, we use these relations to define the notion of a basis.

Definition 2.1.2. A subset $B \subseteq D$ is a **basis** for D if and only if:

(1) for all $b, b' \in B$, $b \sqcup b' \in B$; and

(2) for all $x \in D$, we have $x = \bigsqcup\{b \in B : b \preccurlyeq x\}$.

Bases are intended to be a simpler subset from which all other elements of the data type can be generated. It is reassuring then that the existence of a basis for a data type implies that the meet operation $\sqcap$ is continuous.

We are now in a position to attack the problem of finite representation by making our newly defined notion of basis understandable to a computer. We say that a basis B for D is **effectively-given** if B is countable and there is some enumeration $B = \{b_0, b_1, b_2, \dots\}$ in terms of which, the relations $\prec, \preccurlyeq$, and $\sqsubseteq$ are recursive sets (when considered as sets of ordered pairs). For our denotational semantics, we allow only data types which possess such a basis[7].

[6]This constitutes Axiom 4 for Scott [37] which, it can be noted, implies Axiom 2.

[7]This gives the fifth and final axiom given by Scott in [37] to describe a mathematical model for computer languages.

Given our final definition of data type, it follows that the Scott topology on any data type D is separable since the effectively-given basis B is countable and dense in the Scott topology. Moreover, the topology is second countable since the countable collection of sets of the form $\{x \in D : b \prec x\}$ for each $b \in B$ forms a (topological) basis for the Scott topology.

With the definition of effectively-given bases, we are finally able to define a suitable notion of computability of an element $x \in D$ relative to an effectively-given basis B: x is computable with respect to B if there is an effectively-given, $\sqsubseteq$-increasing subsequence of B, the supremum of which is equal to x. This is interpreted as saying that x is computable if we can effectively give a sequence of increasingly accurate approximations of x from the basis which approaches x as a limit, which corresponds to our intuitive understanding of computability. This concludes Scott's discussion of the foundations of denotational semantics.

In that same 1970 publication, Scott defines three operations which can be done on data types which would generate new data types in an interesting way. The first of which is the Cartesian product. Let $\langle D, \sqsubseteq\rangle, \langle D, \sqsubseteq'\rangle$ be data types as defined throughout this section. Then the product data type $D \times D'$ is given by the set product $D \times D'$ as elements along with the relation $\sqsubseteq_\times$ where $\langle x, x'\rangle \sqsubseteq_\times \langle y, y'\rangle$ if and only if $x \sqsubseteq y$ and $x' \sqsubseteq' y'$. Let D^n be the product data type on n-tuples of D ordered coordinatewise.

The sum $D + D'$ has elements consisting of the disjoint union of D and D' with bottom elements identified and a new top element added. More specifically, let $\perp_D$ and $\perp_{D'}$ be the minimum elements of D and D' respectively. Then the elements of $D + D'$ are exactly those in the set

$$((D \setminus \{\perp_D\}) \times 1) \cup ((D' \setminus \{\perp_{D'}\}) \times \{1\}) \cup \{\perp\} \cup \{\top''\}.$$

The relation $\sqsubseteq_+$ coincides exactly with $\sqsubseteq$ on $D \setminus \{\perp_D\}$, with $\sqsubseteq'$ on $D' \setminus \{\perp_{D'}\}$, and with $\perp$ defined to be $\sqsubseteq_+$-below all other elements and $\top''$ defined to be $\sqsubseteq_+$-above all other elements.

Given a data type D, we can define the data type of finite lists of D as follows:

$$D^* = D^0 + D^1 + D^2 + D^3 + \dots.$$

Nothing is stopping us from repeating the process to form $(D^*)^*$, the data type of finite lists of finite lists of D objects. Let D^∞ be the limit of this process[8]. Then

$$D^\infty = D + (D^\infty)^*,$$

that is, each element in D^∞ is either an element in D or a list of other elements in D^∞. Observe that for $a \in D$, the function

$$f_a : D^\infty \to D^\infty,\ x \mapsto \langle a, x\rangle$$

is continuous. Since f_a is continuous and therefore monotonic, we apply Tarksi's Fixed Point Theorem [42] to give the existence of a fixed point x_a for this function. Then

$$x_a = f_a(x_a) = \langle a, x_a\rangle = \langle a, \langle a, \langle a, \dots\rangle\rangle\rangle.$$

Is this a problem? Scott says no and goes on to assert that "One might say that D^∞ gives us the topological completion of the space of finite lists" [37].

We define the third operation on data types: that of function spaces. The data type $D \to D'$ has as elements all continuous mappings from D to D' ordered so that $f \sqsubseteq g$ iff $f(x) \sqsubseteq' g(x)$ for all $x \in D$. For any data type D, let $D_0 = D$ and for all $n \in \omega$, recursively define

$$D_{n+1} = D_n \to D_n$$

[8] Scott did not rigorously define D^∞ in this paper, but merely suggests that it would per possible if one took the limit of the process "in the right way" [37].

Analogously to the product/sum construction, we are defining higher and higher types of functions; D_1 is the data type of functions of D and D_2 contains the functions of functions of D and so on. Scott argues that there is a "natural way of isomorphically embedding each D_n successively into the next space D_{n+1}" using constant functions. Using these embeddings, we may define the limit space D_∞ where for all $n \in \omega$, there is an embedding of D_n into D_∞. Furthermore,

$$D_\infty \cong D_\infty \to D_\infty$$

This limit space gives a useful framework for dealing with the problem of self-application since each element of D_∞ can be considered as a continuous function from D_∞ to itself and vice versa. Within this data type, applying a procedure (continuous function) to itself is routine rather than pathological. The space D_∞ also serves as the first mathematically defined semantics for the Church-Curry λ-calculus [9] [37].

Finally, let's situate these concepts in the storage situation outlined at the beginning of the section. As before let L be the space of locations and V the space of possible values which is to be defined via limiting methods. If Σ is the set of states, then

$$\Sigma = L \to V.$$

The space Γ of commands would then be

$$\Gamma = \Sigma \to \Sigma.$$

The space of procedures P is given by

$$P = V \to (\Sigma \to (V \times \Sigma)),$$

that is, a procedure is a function which takes as input a value, followed by a state of the system and then outputs a 'computed' value along with the corresponding change in state. But what sort of values can we consider? They could be numbers (real or natural), locations, lists of other values, commands, or procedures, giving us the following equation:

$$V = \mathbb{N} + \mathbb{R} + L + V^* + \Gamma + P$$

Using our previous characterizations of Σ, Γ, and P as function spaces, we can represent V so that it is similar in flavor to our previous limit spaces D^∞ and D_∞, thereby giving us excellent footing for defining such an object mathematically.

This marks the end of the content presented in Scott's 1970 paper. If there are concepts which seem hastily defined or lines of thought which seem only half pursued, that is because this was the beginning of a field which would perfect and generalize the ideas presented here. Such finished projects will be given throughout this chapter. The reason there is a section included here following Scott's paper so closely is not merely historical; here is where the deep connection between topology and continuous lattices[10] and one of the clearest instances in which this connection develops mathematically within its original context of computer science. As we will see, this deep connection has spawned an area of mathematics called Domain Theory which seeks to research such structures as the data types defined here for their own (mathematical) sake. A more detailed, computer science-oriented treatment of the discussion here can be found in Scott and Strachey's 1971 publication [39].

In 1982, Scott published another paper [38] intended to motivate the study of domains, this time by defining the notion of information systems.

[9] Briefly, the λ-calculus is the pure language of unrestricted procedures, thereby solving another issue raised at the beginning of this section.

[10] See section 2.2.

Definition 2.1.3. An **information system** is an ordered quadruple $\langle D, \triangle, \mathrm{Con}, \vdash\rangle$ where D is a set, $\triangle \in D$, $\mathrm{Con} \subseteq \operatorname{fin} D$ and $\vdash \subseteq \mathrm{Con} \times D$ satisfying the following properties for all $u, v \in \operatorname{fin} D$ and $x, y \in D$:

i) $u \in \mathrm{Con}$ whenever $u \subseteq v \in \mathrm{Con}$;

ii) $\{x\} \in \mathrm{Con}$ whenever $x \in D$;

iii) $u \cup \{x\} \in \mathrm{Con}$ whenever $u \vdash x$;

iv) $u \vdash \triangle$ whenever $u \in \mathrm{Con}$;

v) $u \vdash x$ whenever $x \in u \in \mathrm{Con}$; and

vi) for $u, v \in \mathrm{Con}$ if $v \vdash y$ for all $y \in u$ and $u \vdash x$, then $v \vdash x$.

In this definition, D is intended to be a set of propositions with $\triangle$ being a trivial or tautological proposition, Con being the set of consistent subsets of D, and $\vdash$ being the usual logical entailment relation for consistent sets and propositions. The point of Scott's 1982 paper regards not information systems of themselves, but rather Scott focuses on the set of objects which may satisfy the propositions of given information system. We associate such objects with the set of propositions which are true for that object, i.e. the for the object x, $x = \{d \in D : d \text{ is true of } x\}$. We could just as easily use terms from logic to say that a (semantic) object x is identified with its full theory.

Definition 2.1.4. The set of objects satisfying an information system $\langle D, \triangle, \mathrm{Con}, \vdash\rangle$ are those subsets $x \subseteq D$ such that:

i) $\operatorname{fin} x \subseteq \mathrm{Con}$; and

ii) whenever $u \subseteq x$ and $u \vdash d \in D$, then $d \in x$.

Scott proves that this set of objects is a domain in a similar[11] sense as the data types defined in [37]. It is clear that such a definition has deep connections to mathematical logic[12].

2.2 An Order-Theoretic Introduction to Domains

So far we have outlined the origins and motivations of domain theory as a means of providing a semantics for computer programming languages. However, as was alluded to earlier, the mathematical objects discussed are interesting in their own right and have become the object of study of the now-developed field of domain theory. In addition to the already evident applications to computer science, the mathematical study of domains has rich overlap with order theory, point-set topology, pointless topology, and category theory. In this section we define what we mean by domain from the most general perspective available, that is, via order theory. Throughout the rest of this chapter, we rely heavily on the indispensable texts by Abramsky and Jung [2] and by Gierz, Hofmann, Keimel, Lawson, Mislove, and Scott [23].

Definition 2.2.1. We begin with a poset $\langle P, \leq\rangle$. Recall that a subset $D \subseteq P$ is **directed** if for all $x, y \in D$, there is a $z \in D$ such that $x \leq z$ and $y \leq z$. We call the poset P **directed complete** if every directed subset of P has a supremum in P. If this property holds, we call P a **dcpo** as an abbreviation for directed complete partially ordered set. Such ordered sets will form the foundation of our study here, and we will return to them shortly.

[11] The set of objects generated from an information system is a domain in the sense of section 2.2 while data types are continuous lattice in the sense of 2.2.

[12] For one example of a logical consideration that grew from Scott's information systems, see [11].

It is often beneficial to define certain auxiliary relations on partially ordered set. For instance, the strictly below order is useful to consider on linearly ordered sets, but more or less useless in considering topologies as ordered sets. For such topologies the auxiliary relation of closure inclusion or relative compactness does a much better job of illuminating the structure of the ordered set.

Definition 2.2.2. We define our own auxiliary relation, called the **way below relation** $\ll$, for a poset $\langle P, \leq \rangle$. For $p, q \in P$, $p \ll q$ if and only if for all directed subsets $D \subseteq P$ with $\sup D \in P$, $\sup D \geq q$ implies that there is an element $d \in D$ such that $d \geq p$.

Since an ideal is merely a directed set which is closed downwards, this can be equivalently phrased as $p \ll q$ if and only if for any ideal $I \subseteq P$ with $\sup I \geq q$, then $x \in I$. We define

$$\twoheaduparrow x = \{y \in P : x \ll y\} \text{ and } \twoheaddownarrow x = \{y \in P : y \ll x\}.$$

The way below relation will be the focus around which the rest of this chapter will revolve. The following proposition regarding this relation is found in Proposition I-1.2 in [23].

Proposition 2.2.3 (Gierz, et al.)**.** *For the poset P the following hold for all elements x, y, u, z:*

i) $\ll \subseteq \leq$;

ii) $u \leq x \ll y \leq z$ *implies* $u \ll z$;

iii) If P has a minimum element 0, then $0 \ll x$ for all $x \in P$;

iv) If $x \ll z$ and $y \ll z$, then $x \vee y \ll z$ when $x \vee y$ exists; and

v) For any $z \in P$, $\twoheaddownarrow z$ is directed.

Proof. For (i), it is not difficult to see that if $x \ll y$, then $x \leq y$ by taking the principal ideal $\downarrow y$ which is then must contain x by our ideal definition of $\ll$, so $x \leq y$. The converse need not hold in general[13], and will never hold in the ordered sets considered here. For (ii), suppose $u \leq x \ll y \leq z$ and let D be directed with $\sup D \geq z$. Then $\sup D \geq y$ so there is $d \in D$ with $x \leq d$ so $u \leq d$. Therefore $u \ll z$. The proof of (iii) is immediate. Finally for (iv), suppose $x \ll z$ and $y \ll z$ and let D be directed with $\sup D \geq z$. Then there are $d_x, d_y \in D$ such that $x \leq d_x$ and $y \leq d_y$. Then by directedness, there is $d \in D$ with $x \leq d$ and $y \leq d$. Then if $x \vee y$ exists, $x \vee y \leq d$ so $x \vee y \ll z$. In fact, even if $x \vee y$ does not exist, we still have that $x \leq d$ and that $y \leq d$ and $d \in \twoheaddownarrow z$, proving (v). □

By parts (i) and (ii), $\ll$ is antisymmetric and transitive. However, the way below relation is typically rather irreflexive. We call an element x **compact**[14] if $x \ll x$.

Example 2.2.4. Consider the poset $P = [0,1] \times [0,1]$, ordered coordinatewise[15]. In this case, $\ll$ behaves similarly to $<$.

Claim. *For points $\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \in [0,1]^2$, $\langle x_1, y_1 \rangle \ll \langle x_2, y_2 \rangle$ if and only if $x_1 < x_2$ and $y_1 < y_2$ or $\langle x_1, y_1 \rangle = \langle 0, 0 \rangle$.*

Proof. Suppose $\langle x_1, y_1 \rangle \ll \langle x_2, y_2 \rangle$. Let $\{a_n : n \in \omega\} \subset [0,1]$ such that $\sup\{a_n : n \in \omega\} < x_2$. Then $D = \{\langle a_n, y_2 \rangle : n \in \omega\}$ is directed in P. Then there is some $N \in \omega$ such that $\langle a_N, y_2 \rangle \geq \langle x_1, y_1 \rangle$. More specifically, $x_1 \leq a_n < x_2$ so $x_1 < x_2$. A parallel construction using $\{b_n : n \in \omega\} \subset [0,1]$ with $\sup\{b_n : n \in \omega\} < y_2$ shows that $y_1 < y_2$.

[13] In fact, $\ll = \leq$ if and only if P has the ascending chain condition.

[14] Elsewhere in the literature, such elements are called finite, but compact suggests additional generality that suits our purposes.

[15] That is, let P be ordered so that $\langle x, y \rangle \leq \langle x', y' \rangle$ if and only if $x \leq x'$ and $y \leq y'$.

The case where $\langle x_1, y_1\rangle = \langle 0, 0\rangle$ is trivial by the part 3 of the previous proposition. Conversely, suppose $x_1 < x_2$ and $y_1 < y_2$. Let D be directed with $\sup D \geq \langle x_2, y_2\rangle$. Then the images of D under each projection map, i.e. $\pi_1[D], \pi_2[D]$ are each directed subsets of $[0, 1]$ with $\sup \pi_1[D] = \pi_1(\sup D) \geq x_2$ and $\sup \pi_2[D] = \pi_2(\sup D) \geq y_2$. Since the function domain and codomain of each projection is a subset of $[0, 1]$, a linearly ordered set, we have $x_1 < x_2 \leq \pi_1(\sup D)$ and $y_1 < y_2 \leq \pi_2(\sup D)$. Then there is a $d \in \pi_1[D]$ with $x_1 < d$ and $e \in \pi_2[D]$ with $y_1 < e$. Then there is a $d' \in \pi_2[D]$ and $e' \in \pi_1[D]$ such that $\langle d, d'\rangle \in D$ and $\langle e', e\rangle \in D$. Since D is directed, there exists $\langle a, b\rangle \in D$ such that $\langle d, d'\rangle \leq \langle a, b\rangle$ and $\langle e', e\rangle \leq \langle a, b\rangle$, so $\langle x_1, y_1\rangle \leq \langle a, b\rangle$. Therefore $\langle x_1, y_1\rangle \ll \langle x_2, y_2\rangle$. □

Example 2.2.5. Let X be a topological space and let $P = \tau(X)$ ordered by set inclusion. The following claim is proved in Proposition I-1.4 [23].

Claim. *For open sets U, V, if there is a compact set $Q \subseteq X$ with $U \subseteq Q \subseteq V$, then $U \ll V$. If X is locally compact, the converse also holds.*

Proof. First we remark that, in general for complete lattices, $x \ll y$ if and only if for every $R \subseteq P$ with $y \leq \sup R$, there is a finite $S \subseteq R$ such that $\sup S \geq x$. This applies to our topological question since topological spaces constitute complete lattices. Any open cover of V is an open cover of Q and thus has a finite subcover of Q and therefore a finite subcover of $U \subseteq Q$. Then the union (supremum) of these finitely many open sets contains U so by our characterization of way below, $U \ll V$.

On the other hand, let X be locally compact and suppose $U \ll V$. Then for each $v \in V$, there is a compact neighborhood $Q_v \subseteq V$ with $x \in \operatorname{int} Q_v$. Then

$$V = \bigcup\{\operatorname{int} Q_v : v \in V\}.$$

Since $U \ll V$, there are finitely many $v_1, \ldots v_k$ such that

$$U \subset \operatorname{int} Q_{v_1} \cup \cdots \cup \operatorname{int} Q_{v_k} \subseteq Q_{v_1} \cup \cdots \cup Q_{v_k} \subseteq V.$$

The set $Q_{v_1} \cup \cdots \cup Q_{v_k}$ is compact, giving our desired result. □

In both of these examples, P had a very important property with respect to $\ll$.

Definition 2.2.6. We call a poset P **continuous** if for all $x \in P$, $\twoheaddownarrow x$ is directed and P satisfies the **axiom of approximation**:

$$\forall x \in P \left(x = \sup \twoheaddownarrow x\right)$$

Finally we define a **domain** to be a continuous dcpo.

The observant reader will notice that a domain is exactly what we defined in our discussion of information systems[16] in section 2.1. However, during the discussion in that same section regarding data types, we didn't close our space under suprema of directed sets. Instead, the space was closed it under arbitrary suprema, making it a complete lattice complete lattices.

Definition 2.2.7. A complete lattice that is also continuous is called a **continuous lattice** and was actually the object of study of domain theory prior to continuous dcpos[17].

[16] Actually, we defined an algebraic dcpo in section 2.1 which is stronger than a continuous dcpo and we'll get to the definition in section 2.3.

[17] This is evidenced by the fact that the previous version of *Continuous Lattices and Domains* was titled *A Compendium of Continuous Lattices* [22]. In fact, the article by Dana Scott discussed at the end of section 2.1 [38] was actually the first occurrence in the literature to consider continuous dcpos as a semantics for computer languages instead of using continuous lattices.

By considering dcpos, the theory of domains is made more general without sacrificing any significant structure encoded by continuous lattices that is lost in dcpos. And the translation between the two concepts is often not difficult. Clearly, every complete lattice is a dcpo. Moreover, the removal of the maximum element from a continuous lattice characterizes a special kind of continuous dcpo called a Scott domain.

Definition 2.2.8. A **Scott domain** is a domain for which all bounded subsets have suprema. Equivalently within the context of dcpos, a Scott domain is a domain for which any two elements with an upper bound have a least upper bound. Scott domains are also referred to as **bounded complete domains**. This constitutes a special kind of domain which we will consider in more detail in chapter 3.

While we are on the subject of types of domains, there are several other special classes of domains worth mentioning.

Definition 2.2.9. A domain which is also a join semilattice is called a **continuous semilattice** and a domain in which every principal ideal is a complete lattice in its induced order is called an **L-domain**.

These classes of domains are ordered from strongest to weakest as follows:

$$\text{continuous lattice} \implies \text{bounded complete domain} \implies \begin{Bmatrix} \text{continuous semilattice} \\ \text{L-domain} \end{Bmatrix}$$

$$\implies \text{domain} \implies \text{continuous poset.}$$

There is still one more continuity-related property which is useful particularly in lemmas and propositions to follow:

Definition 2.2.10. A semilattice P is **meet-continuous** if P is a dcpo and satisfies the following equation for all $x \in P$ and directed $D \subseteq P$:

$$x \wedge \sup D = \sup\{x \wedge d : d \in D\}.$$

Meet-continuity is not defined for arbitrary posets like continuity is, but for directed complete semilattices, continuity implies meet-continuity, which is shown in Proposition I-1.8 of [23].

With these definitions in mind, we prove that continuous posets (and therefore domains) have the interpolation property. The following lemma and theorem can be found in Theorem I-1.9 of [23].

Lemma 2.2.11 (Gierz, et al.). *Let P be a continuous poset. If $x \ll z$ and $z \leq \sup D$ for some directed $D \subseteq P$, then there is a $d \in D$ such that $x \ll d$.*

Proof. Let D be as specified and let $I = \bigcup \left\{ \twoheaddownarrow d : d \in D \right\}$. By (v) of Proposition 2.2.3, $\twoheaddownarrow d$ is directed and by (ii), $\twoheaddownarrow d$ is an ideal. Since the directed union of ideals is an ideal, I is an ideal. By continuity of P, $\sup I = \sup D$. Then since $x \ll z$, we have that $x \in I$, so there is some $d \in D$ such that $x \in \twoheaddownarrow d$, or in other words, $x \ll d$. □

Theorem 2.2.12 (Gierz, et al.). *In all continuous posets, the way below relation has the interpolation property:*

$$x \ll z \textit{ implies } \exists y \in P \, (x \ll y \ll z) \, .$$

Proof. This theorem follows immediately from the previous lemma by choosing D to be $\twoheaddownarrow z$. □

This theorem can strengthened via the discussion in [23][18]

Definition 2.2.13. An **auxiliary order** $\prec$ on a poset P such that $\prec$ satisfies conditions (i)-(iii) of Proposition 2.2.3, that is

i $\prec \subseteq \leq$.

ii $u \leq x \prec y \leq z$ implies $u \prec z$.

iii If P has a minimum element 0, then $0 \prec x$ for all $x \in P$.

Consider the set of all such relations $\mathrm{Aux}(P)$ partially ordered by set inclusion. Then the maximum element of $\mathrm{Aux}(P)$ is $\leq$. The minimum element is $\varnothing$ if P does not have a minimum element and $\bigcirc$ where $x \bigcirc y$ iff $x = 0$. The intersection of any subset of $\mathrm{Aux}(P)$ is also an auxiliary relation so $\mathrm{Aux}(P)$ is a complete lattice.

This lattice is isomorphic to the set of monotone functions $s : P \to \mathrm{Low}(P)$[19] satisfying $s(x) \subseteq \downarrow x$ for all $x \in P$, with the order on this set of monotone functions defined by $s \leq t$ iff $s(x) \subseteq t(x)$ for all $x \in P$. This isomorphism is given by the mapping

$$\prec \mapsto s_{\prec} := (x \mapsto \{y : y \prec x\}).$$

For I an ideal in P, we define

$$m_I(x) = \begin{cases} \downarrow x \cap I \text{ if } x \leq \sup I \\ \downarrow x \text{ otherwise.} \end{cases}$$

This function m_I is monotone and clearly for all x, $m_I(x) \subseteq \downarrow x$. Thus $m_I \in M$ so we may associate to it an auxiliary function via the isomorphism.

Definition 2.2.14. If P is a dcpo, we call an auxiliary relation $\prec$ **approximating** if the set $s_{\prec}(x) = \{u \in P : u \prec x\}$ is directed (and hence an ideal) and for all $x \in P$,

$$x = \sup\{u \in P : u \prec x\} = \sup s_{\prec}(x).$$

Clearly $\leq$ is approximating. The way below relation $\ll$ is an auxiliary relation and is approximating exactly in continuous posets. In fact, in a meet continuous semilattice (and therefore in a continuous semilattice lattice) , any auxiliary relation associated to one of the functions m_I is approximating by Lemma I-1.14 in [23]. Let $\mathrm{App}(P)$ denote the set of approximating auxiliary relations on P. This leads us to Proposition I-1.15 in [23]:

Proposition 2.2.15 (Gierz, et al.). *In a dcpo P, the way below relation $\ll \subseteq \prec$ for all $\prec \in \mathrm{App}(P)$. Additionally, if P is also a meet continuous semilattice, then*

$$\ll = \bigcap \mathrm{App}(P).$$

Proof. Suppose $y \ll x$ and let $\prec \in \mathrm{App}(P)$. The set $s_{\prec}(x)$ is directed with $\sup s_{\prec}(x) = x$ by an argument analogous to that of 2.2.3 (iv) and (v). Then there is a $u \in s_{\prec}(x)$ such that $y \leq u \prec x$ so $y \prec x$ as desired.

Next, assume L is also meet continuous. Then

$$\Downarrow x = \bigcap\{m_I(x) : I \in \textbf{Idl}\, P\} \supseteq \bigcap\{s_{\prec}(x) : \prec \in \mathrm{App}(P)\}.$$

Thus we conclude that $\ll = \bigcap \mathrm{App}(P)$. $\square$

It is an easy consequence of the first statement of proposition 2.2.15 that P is a domain if and only if $\ll$ is the minimum element of $\mathrm{App}(P)$. Moreover, if P is a meet continuous semilattice, then the existence of a minimum element of $\mathrm{App}(P)$ is also equivalent to P being a domain.

Finally we get to the point of this discussion regarding $\mathrm{Aux}\, P$:

[18] See pages 59-62.

[19] $\mathrm{Low}(P)$ denotes the set of all downward-closed subsets of P.

Definition 2.2.16. An auxiliary relation $\prec$ has the **strong interpolation property** if the following statement holds for all $z, x \in P$:

$$(x \prec z \wedge x \neq z) \rightarrow \exists y(x \prec y \prec z \wedge x \neq y).$$

Clearly the strong interpolation property implies the interpolation property.

Lemma 2.2.17 (Gierz, et al.). *For any approximating auxiliary relation $\prec$ on a dcpo P, for all $x, z \in P$,*

$$(x \prec z \wedge x \neq z) \rightarrow \exists y\,(x \leq y \prec z \wedge x \neq y)\,.$$

Proof. Since $z = \sup\{u : u \prec z\}$ and $x < z$, there is a u such that $u \prec z$ and $u \not\leq x$. By directedness of $\{u : u \prec z\}$, there is a $y \prec z$ with $x \leq y$ and $u \leq y$. Since $u \not\leq x$, it follows that $x \leq y \prec z$ and $x \neq y$. □

Lemma 2.2.18 (Gierz, et al.). *For any approximating auxiliary relation $\prec$ on a dcpo P, for all $x, z \in P$, if $x \ll z$ and $x \neq z$ and there is a directed set D with $z \leq \sup D$, then there is a $d \in D$ with $x \prec d$, and $x \neq d$.*

Proof. Let D be as specified and let $I = \bigcup\{s_\prec(d) : d \in D\}$. Since $\prec$ is approximating and since I is itself an ideal, it follows taht $\sup I = \sup\sup\{s_\prec(d) : d \in D\} = \sup D \geq z$. Since $x \ll z$, then $x \in I$ so there must be some specific $d \in D$ with $x \in s_\prec(d)$.

So far this is merely a repetition of the proof of Theorem 2.2.12. However, we have yet to find such a d so that $x \neq d$. Since $x \not\geq z$, and $z \leq \sup D$, then there must be a $c \in D$ with $c \not\leq x$. By directedness of D, there is a common upper bound $b \in D$ so $d \leq b$ and $c \leq b$. Finally since $c \not\leq x$ and $c \leq b$, $d \neq b$ and since $x \prec d \leq b$, $x \prec b$. □

Finally we have the desired strengthening of Theorem 2.2.12:

Theorem 2.2.19 (Gierz, et al.). *For any approximating auxiliary relation $\prec$ on a dcpo P, for all $x, z \in P$, if $x \ll z$ and $z \neq x$, then there is a y such that $x \prec y \prec z$ and $x \neq y$.*

Proof. This is proved by letting our directed set from the previous proposition be $D = s_\prec(z)$. □

It quickly follows that if P is a domain which entails that $\ll$ is approximating, by setting $\prec = \ll$, this theorem concludes that the way below relation $\ll$ possesses the strong interpolation property in every domain. This conclusion along with the two preceding lemmas and theorem 2.2.19 can be found in Lemma I-1.18, Proposition I-1.19, and Corollary I-1.20 in [23].

2.3 Domain Bases and Continuous Functions

Domain Bases

With the way below relation and the notion of domain defined, we turn to study of specific facets and properties of domains which are particularly foundational or useful. As they stand, domains can at times be very large or exceedingly complex, so we introduce the notion of domain basis to simplify defining and manipulating domains.

Definition 2.3.1. For a dcpo D, we say that a subset $B \subseteq D$ is a **basis**[20] for D if for all $x \in D$, the set $B_x := \mathop{\downarrow\!\!\downarrow} x \cap B$ contains a directed subset with supremum x.

[20]Recall definition 2.1.2.

Intuitively, a basis represents some set of elements from which all other elements in a dcpo may be approximated (by $\ll$). Then it makes sense that we be on the lookout for the smallest basis possible.

Recall that an element $x \in D$ is compact if $x \ll x$. Let $K(D)$ denote the set of compact elements of D. Then the smallest possible basis for D is $K(D)$ and the largest possible is D itself. This presents a useful characterization.

Proposition 2.3.2. *For a dcpo D, the following are equivalent:*

i) D is continuous;

ii) D has a basis;

iii) D is a basis for itself.

Proof. That (iii)$\Rightarrow$(ii) is trivial. Then we first endeavor to prove the converse, that is, (ii)$\Rightarrow$(iii). In doing so, we will actually prove a more general statement:

Claim. *Let $B \subseteq D$ be a basis and let $B' \subseteq D$ such that $B \subseteq B'$. Then B' is a basis for D as well.*

If B is a basis for D, then by definition, for all $x \in D$, there is a directed subset $A_x \subseteq \twoheaddownarrow x \cap B$. Then since $B \subseteq B'$, it follows that $\twoheaddownarrow x \cap B \subseteq \twoheaddownarrow x \cap B'$ so the directed set $A_x \subseteq \twoheaddownarrow x \cap B'$ for all $x \in D$, making B' a basis for D.

It easily follows that if D has a basis, then D is a basis for itself. We use this fact to prove (i)$\Leftrightarrow$(iii). If D is continuous, then $\twoheaddownarrow x$ is itself directed and $\sup \twoheaddownarrow x = x$. Then clearly $\twoheaddownarrow x$ satisfies the conditions necessary for the desired directed subset of $\twoheaddownarrow x \cap D$ so D is a basis for itself.

On the other hand, if D is a basis for itself, then by definition, for every $x \in D$, $\twoheaddownarrow x \cap D = \twoheaddownarrow D$ has a directed subset converging to x. Fix an arbitrary x and let $A \subseteq \twoheaddownarrow x$ be directed with $\sup A = x$. As it happens, directedness here matters little. Since $A \subseteq \twoheaddownarrow x$, $x = \sup A \leq \sup \twoheaddownarrow x$. If $x \neq \sup \twoheaddownarrow x$, then $x < \sup \twoheaddownarrow x$ which means there is a $y \ll x$ with $y \not\leq x$. This is contradictory since $y \ll x$ implies $y \leq x$. Thus $x = \sup \twoheaddownarrow x$ so D is continuous. $\square$

Lemma 2.2.15 in [2] shows that domain bases are interpolative.

Proposition 2.3.3. *For a domain D with basis B, for all finite subsets $M \subseteq D$, and for all $y \in D$, if $m \ll y$ for all $m \in M$, then there is a $y' \in B$ such that $m \ll y' \ll y$ for all $m \in M$.*

Using bases, we may define still more special classes of domains:

Definition 2.3.4. A dcpo D is called **algebraic** if $K(D)$ is a basis for D. By the previous proposition, an algebraic dcpo, sometimes called an algebraic domain, is necessarily continuous. The dcpo D is **ω-continuous** if D has a countable basis and D is **ω-algebraic** if it is both algebraic and ω-continuous, i.e. if $K(D)$ is a countable basis for D. A dcpo D is an **algebraic lattice** if $K(D)$ is a basis for D and D is a complete lattice. Historically, the concept of algebraic lattices preceded that of algebraic dcpos.

Let's consider examples of bases in examples 2.2.4 and 2.2.5. Once consequence of our investigation in 2.2.4 is that for $x, y \in [0,1]$, $x \ll y$ if and only if $x < y$ or $x = 0$. Then for all $z \in [0,1]$,

$$\twoheaddownarrow z = (\downarrow z \setminus \{z\}) \cup \{0\}.$$

It follows that $[0,1]$ is continuous so the domain has a basis. But we are interested in finding the smallest possible basis so first we consider $K([0,1])$. In this case, the only compact element is 0 and one can obviously find an $x \in [0,1]$ such that there is no directed set $A \subseteq \twoheaddownarrow x \cap \{0\} = \{0\}$ with $\sup A = x$. In fact, any element other than 0 has this property. So much for $K([0,1])$. The problem extends slightly deeper: whenever $K(D)$ is not a basis for D, then D has no minimum basis as shown in Corollary 2.2.5 (2) of [2].

Let $B \subseteq [0,1]$ be a (topologically) dense subset, i.e. for all $x, y \in [0,1]$ with $x < y$, there is a $b \in B$ such that $x < b < y$.

Claim. *A subset $B \subseteq [0,1]$ is a basis if and only if B is dense in $[0,1]$ and $0 \in B$.*

Proof. Suppose B is a basis for $[0,1]$. Since 0 is compact, $0 \in B$. Now let $x, y \in [0,1]$ such that $x < y$. Then there are directed sets $A_x \subseteq \twoheaddownarrow x \cap B$ and $A_y \subseteq \twoheaddownarrow y \cap B$ such that $\sup A_x = x$ and $\sup A_y = y$. Then $x = \sup A_x < \sup A_y = y$. Then there is a $b \in A_y$ such that $b \not\leq x$ so by linearity, $b > x$. Furthermore, $b \in B$ and $b \ll y$ so $b < y$. Thus $x < b < y$.

On the other hand, suppose B is dense and $0 \in B$. Clearly $\twoheaddownarrow 0 \cap B = \{0\}$ contains the vacuously directed subset $\{0\}$ and $\sup\{0\} = 0$. Now fix $x \in (0,1]$. We define

$$A_x = \twoheaddownarrow x \cap B = [0,x) \cap B.$$

Clearly $A_x \subseteq \twoheaddownarrow x \cap B$. Directedness of A_x is also easily verified. Then we need only show that $\sup A_x = x$. Since $A_x = [0,x) \cap B$, $\sup A_x = \sup[0,x) \wedge \sup B = x \wedge \sup B$. However, clearly $x \leq \sup B$ since either $x = 1 = \sup B$ or by density we can find an element $b \in B$ with $x < b \leq \sup B$. Thus B is a basis for $[0,1]$. $\square$

In particular, $\mathbb{Q} \cap [0,1]$ is a basis for $[0,1]$ which is countable. Thus $[0,1]$ is ω-continuous but not algebraic.

In example 2.2.5 we considered locally compact topologies. For X a locally topological space, $\tau(X)$ when considered as a poset (complete lattice) ordered by set-inclusion, for open sets U and V, the way below relation is given by $U \ll V$ if and only if there is a compact set $Q \subseteq X$ with $U \subseteq Q \subseteq V$.

Claim. *First, let's verify that $\tau(X)$ is indeed a domain.*

Proof. Let $V \in \tau(X)$. Then $\twoheaddownarrow V = \{U \in \tau(X) : \exists \, \text{compact}\, Q(U \subseteq Q \subseteq V)\}$. Since the supremum of any subset in the lattice $\tau(X)$ is the union of that subset, we have $\sup \twoheaddownarrow V = \bigcup\{U \in \tau(X) : \exists \, \text{compact}\, Q(U \subseteq Q \subseteq V)\}$. Clearly $\sup \twoheaddownarrow V \subseteq V$. On the other hand, let $x \in V$. By local compactness, there is an open set U and a compact set V such that $x \in U \subseteq Q \subseteq V$. Therefore $x \in \sup \twoheaddownarrow V$ so $\sup \twoheaddownarrow V = V$. $\square$

We are assured that $\tau(X)$ has a domain basis[21]. In fact, since $\tau(X)$ is also a complete lattice, this set is an example of a continuous lattice.

Claim. *If $\mathcal{B}$ is a domain basis for $\tau(X)$, then $\mathcal{B}$ is a topological base for $\tau(X)$.*

Proof. It is easy to show that $\mathcal{B}$ covers X: let $x \in X$ and let U be an open neighborhood of x. Then there is a directed family $\mathcal{A} \subseteq \twoheaddownarrow U \cap \mathcal{B}$ such that $\bigcup \mathcal{A} = U$ so there is a set $A \in \mathcal{A} \subseteq \mathcal{B}$ with $x \in A$.

Let $B_0, B_1 \in \mathcal{B}$ and let $x \in B_0 \cap B_1$. Then there is a directed family $\mathcal{A} \subseteq \twoheaddownarrow(B_0 \cap B_1) \cap \mathcal{B}$ with $\bigcup \mathcal{A} = B_0 \cap B_1$. Then there is a set $A \in \mathcal{A}$ such that $x \in A$. Clearly $A \subseteq B_0 \cap B_1$ and $A \in \mathcal{B}$. $\square$

[21] Since we are dealing with a set which is acting as both a topology and as an ordered space, the words 'basis/base' and 'compact' are ambiguous. For the time being, we will simply attempt to specify at each occurrence where 'basis' refers to a domain basis or a topological base and likewise for 'compact'.

Then to find a domain basis, we need to consider topological bases. Naturally, we first look to domain-compact elements $K(\tau(X))$. As it happens, an open set being topologically compact and domain compact are equivalent. Then, $K(\tau(X)) = \{U \in \tau(X) : U \text{ is (topologically) compact}\}$ which would make for a rather odd topological space if such a collection of sets were to form its basis.

Scott-Continuous Functions

When considering a mathematical object or class of objects, studying its behavior under certain types of order-preserving functions is usually of the utmost importance.

Definition 2.3.5. Let D and E be posets. A function $f : P \to Q$ is **Scott-continuous** if f preserves suprema of directed subsets.

It is a consequence then that f is monotone as well. When no confusion will arise, such functions will merely be called continuous. As we will see, continuous functions interact with domains and domain bases in the way one would expect them to for the most part. Proposition 2.2.11 of [2] give the following result:

Proposition 2.3.6 (Abramsky and Jung)**.** *A function f between domains D and E with bases B and C respectively. Then f is Scott-continuous if and only if for all $x \in D$ and for all $e \in \twoheaddownarrow f(x) \cap C$, there is a $d \in \twoheaddownarrow x \cap B$ such that $f[\uparrow d] \subseteq \uparrow e$.*

Definition 2.3.7. For a partially ordered set X, a **projection** $p : X \to X$ is a self-map that is both monotone and idempotent.

Then Theorem I-2.2 in [23] gives the following result regarding Scott-continuous projections:

Theorem 2.3.8 (Gierz, et al.)**.** *Let L be a continuous poset and $p : L \to L$ a Scott-continuous projection. The the image $p(L)$ with the order induced from L is a continuous poset as well. For $x, y \in p(L)$, we have*

$$x \ll_{p(L)} y \text{ iff there is an element } u \in L \text{ such that } x \leq p(u) \text{ and } u \ll_L y.$$

Corollary 2.3.9. *Let M be the image of a Scott-continuous projection on L. If L is a domain, L-domain, Scott domain, continuous lattice, or continuous semilattice, then the same is true of M.*

Abstract Bases

It will be useful to consider domain bases as objects in-themselves so that we may study the basis and generate the corresponding domain later. As such, we offer the following definition:

Definition 2.3.10. An **abstract basis** is a set B together with a transitive relation $\prec \subseteq B \times B$ such that

$$M \prec x \to \exists y \in B(M \prec y \prec x)$$

holds for all $x \in B$ and for all $M \in \operatorname{fin}(B)$[22].

Concrete domain bases satisfy this definition with $\prec$ being the restriction of $\ll$. However, it is important that such structures generate domains (and nothing else). In the discussion to follow, context will differentiate whether we are referring to a basis as abstract or concrete.

Let **Idl**(B) denote the ideal completion of B and define $i : B \to \textbf{Idl}(B)$, $x \mapsto \downarrow x$. Then we have the following result of Proposition 2.2.22 in [2]

[22] This definition first appeared in [40] under the name 'R-structures'.

Proposition 2.3.11 (Abramsky and Jung)**.** *Let B be a set and let $\prec$ be a transitive order*[23] *on B.*

i) ***Idl***(B) *is a dcpo.*

ii) $I \ll J$ *in* ***Idl***(B) *if and only if there are* $x \prec y \in B$ *such that* $I \subseteq i(x) \subseteq i(y) \subseteq J$ *if and only if there is* $x \in J$ *with* $I \subseteq i(x)$.

iii) ***Idl***(B) *is a domain and* $i(B)$ *is a basis for* ***Idl***(B).

iv) If $\prec$ *is reflexive, then* ***Idl***(B) *is an algebraic domain.*

v) If $\langle B, \prec \rangle$ *is a poset, then* $B \cong K(\textbf{\textit{Idl}}(B)) \cong i(B)$.

Proof. Clearly **Idl**(B) is a poset since its ordered by set-inclusion. Then **Idl**(B) is a dcpo since the directed union of ideals is an ideal, proving (i).

We next note that since ideals are closed downwards, we have

$$\forall I \in \textbf{Idl}(B) \left(I = \bigcup_{x \in I} \downarrow x \right).$$

Given this fact, we prove the set of equivalences in (ii).

a) Suppose $I \ll J$. Then consider the set $\{\downarrow x : x \in J\}$. Since J is directed, so is $\{i(x) : x \in J\}$ and $\bigcup\{\downarrow x : x \in J\} = J$ as we have already shown. Then since $I \ll J$, there exists $x \in J$ such that $I \subseteq i(x)$. Since J is directed, there is a $y \in J$ with $x \prec y$. Then $I \subseteq i(x) \subseteq i(y) \subseteq J$ as desired.

b) Suppose there are $x, y \in B$ with $x \prec y$ and $I \subseteq i(x) \subseteq i(y) \subseteq J$. Since $x \prec y$, $x \in i(y)$ so $x \in J$ and $i(x) \supseteq I$.

c) Suppose there is an $x \in J$ such that $I \subseteq i(x)$. Let $\mathcal{D} \subseteq \textbf{Idl}(B)$ such that $\bigcup \mathcal{D} \supseteq J$. Then $x \in \bigcup \mathcal{D}$ so there is an ideal $K \in \mathcal{D}$ such that $x \in K$. Since ideals are closed downwards, $I \subseteq i(x) \subseteq K \in \mathcal{D}$. This proves part (ii) of the theorem.

To show that **Idl**(B) is a domain and $i(B)$ is a basis, we need to show that for each $I \in \textbf{Idl}(B)$, there is a directed subset of $\twoheaddownarrow I \cap i(B)$ that sups (unions) to I. Fix $I \in \textbf{Idl}(B)$. If $x \in I$, then by directedness of I, there is a $y \in I$ such that $x \prec y$. Then $i(x) \subseteq i(y)$ which by (ii), indicates that $i(x) \ll I$. Then $\twoheaddownarrow I \cap i(B) = \{i(x) : x \in I\}$. This is directed because I is directed and we have already shown that $I = \bigcup\{i(x) : x \in I\}$. Therefore **Idl**$(B)$ is a domain and $i(B)$ is a concrete basis for **Idl** B, proving (iii).

For (iv), suppose $\prec$ is reflexive. Then for any $x \in B$, $x \prec x$ which means $x \in i(x)$ so by (ii), $i(x) \ll i(x)$. Then $i(B) \subseteq K(\textbf{Idl}(B))$ and since $i(B)$ is a basis by (iii), $K(\textbf{Idl}(B))$ is also a basis. In fact, $i(B)$ and $K(\textbf{Idl}(B))$ are equal since there is only one basis for a domain composed entirely of compact elements.

Suppose $\langle B, \prec \rangle$ is a poset. Now for (v) we need only show that $B \cong i(B)$. Clearly $i : B \to i(B)$ is a surjection. If $i(x) = i(y)$, then $x \prec y$ and $y \prec x$ so by antisymmetry, $x = y$ making the mapping i bijective. Let $x, y \in B$ and suppose $x \prec y$. Then $x \in i(y)$ and by transitivity of $\prec$, $i(x) \subseteq i(y)$. Conversely, suppose $i(x) \subseteq i(y)$. Then by reflexivity, $x \in i(x)$ so $x \in i(y)$ allowing us to conclude that $x \prec y$. Thus i is an order-isomorphism, completing our proof. $\square$

This proposition completes our discussion of abstract bases for the time being, but the conclusions will be of vital importance later in this text, specifically in section 3.2.

[23] As originally stated, $\prec$ was specified to form an abstract base by being interpolative and transitive. Upon careful reflection of the proof, the requirement that $\prec$ be interpolative seems extraneous.

2.4 A Topological Approach to Domains

As we have seen in Section 2.3, domains have bases which can be considered concretely within a domain or abstractly and used to generate a domain. Mappings which preserve domain-like structure are called continuous. All this should be reminiscent of basic point-set topology. All of the relevant properties discussed so far can be defined and made sense of using a very interesting topology.

Definition 2.4.1. For a poset P a subset $U \subseteq P$ is called **Scott-open**[24] if U satisfies both of the following properties:

- For all $x \in U$ if $y \in P$ such that $x \leq y$ then $y \in U$.
- For all (nonempty) directed subsets $D \subseteq P$, if $\sup D \in U$, then $D \cap U \neq \varnothing$.

Proposition 2.4.2. *The collection of Scott-open sets $\sigma(P) := \{U \subseteq P : U \text{ is Scott-open}\}$ is a topology on P.*

Proof. Firstly, it is easy to see that $\varnothing$ satisfies these properties trivially since there is no directed set with supremum an element of the empty set. That $P \in \sigma(P)$ is also immediate since all nonempty directed subsets have nonempty intersection with P regardless of their suprema.

Secondly, is $\sigma(P)$ closed under finite intersections? Let $U, V \in \sigma(P)$. To show that $U \cap V$ is an upset we let $x \in U \cap V$ and let $x \leq y$. Then since both U and V are upsets, $y \in U$ and $y \in V$ so $y \in U \cap V$. Now, let $D \subseteq P$ be directed and nonempty with $\sup D \in U \cap V$. Then since U and V are open, there are elements $d_1, d_2 \in D$ such that $d_1 \in U$ and $d_2 \in V$. Since D is directed, there is a $e \in D$ with $d_1 \leq e$ and $d_2 \leq e$. Then since U and V are both upsets, $e \in U \cap V$.

Thirdly, is $\sigma(P)$ closed under unions? Let $\mathcal{U} \subseteq \sigma(P)$. Clearly $\bigcup \mathcal{U}$ is closed upwards since each $U \in \mathcal{U}$ is. Then let $D \subseteq P$ be directed and nonempty with $\sup D \in \bigcup \mathcal{U}$. Then there is some specific $U \in \mathcal{U}$ such that $\sup D \in U$. Since U is Scott-open, there is a $d \in D$ with $d \in U$. Clearly $d \in \bigcup \mathcal{U}$ as well. □

We naturally call this topology of Scott-open sets the **Scott topology** on a poset P. We will always denote this topology using the notation $\sigma(P)$.

Proposition 2.4.3. *A set $C \subseteq P$ is closed in $\sigma(P)$ if and only if C is a down-set that is closed under directed suprema*[25]*. Then $\mathrm{cl}\{x\} = \downarrow x$ for all $x \in P$. As a result, $\sigma(P)$ is T_0, and unless $\sigma(P) = \mathcal{P}(P)$, the topology fails to be T_1.*

Proof. Suppose C is closed in $\sigma(P)$. There there is some $U \in \sigma(P)$ such that $C = P \setminus U$. Let $x \in C$ and $y \in P$ such that $y \leq x$. If $y \in U$, then $x \in U$ since U is an upset, but this contradicts the fact that C is the complement of U. Thus $y \in C$ and C is a downset. Let $D \subseteq C$ be directed such that $\sup D \in P$. If $\sup D \in U$, then there would be $d \in D$ such that $d \in U$. But this contradicts $D \subseteq C$ so it must be that $\sup D \in C$.

Conversely, suppose C is closed downwards and closed under (existing) directed suprema. Then we want to show that $U = P \setminus C \in \sigma(P)$. That U is closed upward is immediate. Then let $D \subseteq P$ such that $\sup D \in U$. If $D \subseteq C$, we have a contradiction since $\sup D$ exists and C is closed under suprema that have suprema in P, so $\sup D \in C$. Then we conclude that $D \nsubseteq C$. So there is an element $d \in D$ with $d \notin C$ so $d \in P \setminus C = U$. Thus U is Scott-open so C is Scott-closed.

For any $x \in P$, it is easy to see that $\downarrow x$ is Scott-closed. Let $C \subseteq P$ be Scott-closed such that $x \in C$. Then since C is closed downward, $\downarrow x \subseteq C$. Thus $\mathrm{cl}\{x\} = \downarrow x$.

[24] See definition 2.1.1.

[25] Since we haven't assumed P to be a dcpo, we cannot be guaranteed that the supremum of a directed set exists in P at all, let alone in C. We can side-step this by claiming that C is closed iff C is closed downwards and for any directed subset $D \subseteq C$ if $\sup D \in P$, then $\sup D \in C$.

We can use this last fact to prove that $\sigma(P)$ is T_0. Let $x, y \in P$ be distinct. Then since P is a poset, it is not the case that both $x \leq y$ and $y \leq x$. By De Morgan's law, this is equivalent to $x \not\leq y$ or $y \not\leq x$. Without loss of generality, assume $y \not\leq x$. Then $y \notin \downarrow x$ so $y \in P \setminus \downarrow x$. However, we know that $\downarrow x$ is Scott-closed so $P \setminus \downarrow x$ is Scott-open and $x \notin P \setminus \downarrow x$. Then $\sigma(P)$ is T_0.

If the order $\leq$ on P is trivial[26], then $\sigma(P)$ is the discrete topology on P since singletons would be open. On the other hand, suppose $\sigma(P) \neq \mathcal{P}(P)$. Thus there are distinct $x, y \in P$ such that $x \leq y$. Since Scott-open sets are directed upwards, for any $U \in \sigma(P)$ where $x \in U$, it must be that $y \in U$. Therefore $\sigma(P)$ fails to be T_1. ☐

From what we've seen so far, the idea of a set being Scott-open seems similar in flavor to the way-below relation. To flesh out this intuitive connection, we consider continuous domains D rather than arbitrary posets[27].

Proposition 2.4.4. *In a domain D, $x \ll y$ if and only if $y \in \operatorname{int} \uparrow x$, so $\twoheaduparrow x$ is Scott open. Equivalently, for all $x \in D$, $\twoheaduparrow x = \operatorname{int} \uparrow x$.*

Proof. Suppose $x \ll y$. Then we want to find an open neighborhood of y that is contained in $\uparrow x$. The open neighborhood that meets these requirements is $\twoheaduparrow x$. By proposition 2.2.3(ii), $\twoheaduparrow x$ is closed upward. Let $A \subseteq D$ be directed with $\sup A \in \twoheaduparrow x$. Then $x \ll \sup A$ and by interpolation, there is $z \in D$ with $x \ll z \ll \sup A$. By the definition of $\ll$, there is $a \in A$ such that $z \leq a$. Thus $x \ll a$, so $\twoheaduparrow x$ is Scott-open. Surely then $y \in \operatorname{int} \uparrow x$ and in fact, $\twoheaduparrow x \subseteq \operatorname{int} \uparrow x$.

Conversely, suppose $y \in \operatorname{int} \uparrow x$. Then y has an open neighborhood $U \subseteq \uparrow x$. Let $A \subseteq D$ be directed with $\sup A \geq y$. Then $\sup A \in U$ and by the definition of openness, there is $a \in A$ such that $a \in U$. Of course, since $U \subseteq \uparrow x$, $a \geq x$. Therefore $x \ll y$ and $\twoheaduparrow x = \operatorname{int} \uparrow x$. ☐

It is worth mentioning that continuity of D is not required for the proof that if $y \in \operatorname{int} \uparrow x$, then $x \ll y$. Proposition II-1.10 in [23] gives the following result:

Proposition 2.4.5 (Gierz, et al.)**.** *For a domain D, an upset U is Scott-open if and only if for all $x \in U$, there is a $u \in U$ such that $u \ll x$.*

Proof. Let $U \in \sigma(D)$ and $x \in U$. Since D is continuous, $\twoheaddownarrow x$ is directed and $\sup \twoheaddownarrow x = x$. Since $\sup \twoheaddownarrow \in U$, there must be some $u \in \twoheaddownarrow x$ with $u \in U$.

On the other hand, suppose U is closed upwards and for all $x \in U$, there is a $u \in U$ with $u \ll x$. Let $A \subseteq D$ be directed with $\sup A \in U$. Then there is $u \in U$ with $u \ll \sup A$. This implies that there is $a \in A$ with $u \leq a$, and since U is closed upwards, $a \in U$. Thus $U \in \sigma(D)$. ☐

Now we prove a proposition which justifies our use of the word "basis" to describe domain bases.

Proposition 2.4.6. *Let B be a basis for domain D. Then $\{\twoheaduparrow x : x \in B\}$ is a base for $\sigma(D)$.*

Proof. It is not difficult to show that $\{\twoheaduparrow x : x \in B\}$ covers D. Let $b_1, b_2 \in B$ and let $x \in \twoheaduparrow b_1 \cap \twoheaduparrow b_2$. Then we want to find another basis element $a \in B$ such that $x \in \twoheaduparrow a \subseteq \twoheaduparrow b_1 \cap \twoheaduparrow b_2$. By the definition of domain bases, there is a directed subset $A \subseteq \twoheaddownarrow x \cap B$ with $\sup A = x$. We know that $\twoheaduparrow b_1 \cap \twoheaduparrow b_2$ is Scott-open, and since $\sup A \in \twoheaduparrow b_1 \cap \twoheaduparrow b_2$, there is an $a \in A$ such that $a \in \twoheaduparrow b_1 \cap \twoheaduparrow b_2$. Then $a \in B$ and $x \in \twoheaduparrow a \subseteq \twoheaduparrow b_1 \cap \twoheaduparrow b_2$ as desired. ☐

If concrete bases behave so well in the Scott topology, perhaps abstract ones do as well. Can the Scott topology of a domain be easily derived from an abstract basis? As in Proposition 2.3.7 in [2], the following proposition answers in the affirmative.

[26] By the order being trivial, it is meant that $\leq = \{(x, x) : x \in P\}$.

[27] Mashburn gives a more detailed treatment of the behavior of the Scott topology and the way below relation in the absence of continuity in [30].

Proposition 2.4.7 (Abramsky and Jung)**.** *Let $\langle B, \prec\rangle$ be an abstract basis and let $M \subseteq B$. Then the set $\{I \in \textbf{\textit{Idl}}(B) : M \cap I \neq \varnothing\}$ is Scott-open and all sets in $\sigma(\textbf{\textit{Idl}}(B))$ are of this form.*

Finally, Proposition II-2.1 in [23] justifies our use of the terminology "continuous" in referring to Scott-continuous functions.

Proposition 2.4.8 (Gierz, et al.)**.** *Let S and T be dcpos with $f : S \to T$. Then the following are equivalent:*

i) f is topologically continuous with respect to the Scott topologies $\sigma(S)$ and $\sigma(T)$;

ii) f is Scott-continuous;

And if S and T are domains, then the following are also equivalent to (i) and (ii):

iii) $y \ll f(x)$ if and only if for some $w \ll x$, $y \ll f(w)$ for all $x \in S$ and $y \in T$;

iv) $f(x) = \sup\{f(w) : w \ll x\}$ for all $x \in S$.

Furthermore, if S and T are algebraic domains, then the following conditions are also equivalent to the preceding ones:

v) $k \leq f(x)$ if and only if for some $j \leq x$ with $j \in K(S)$, $k \leq f(j)$ for all $x \in S$ and $k \in K(T)$;

vi) $f(x) = \sup\{f(j) : j \leq x \wedge j \in K(S)\}$, for all $x \in S$.

Proof. (i) $\Longrightarrow$ (ii):

First we show that f is monotone by contrapositive. Suppose that $f(x) \not\leq f(y)$. Then the Scott-open set $V = T \setminus \downarrow f(y)$ contains $f(x)$. Since f is continuous, $U = f^{-1}(V)$ is Scott-open and $y \notin U$. Thus, since U is closed upwards, $x \not\leq y$.

Now let $D \subseteq S$ be directed. We want to show that $\sup f(D) = f(\sup D)$. Since f is monotone, $f(D)$ is directed in T and $\sup f(D) \leq f(\sup D)$. We set $x = \sup D$ and $t = \sup f(D)$. Suppose for the sake of contradiction that $f(x) \not\leq t$. Then $f(x) \in T \setminus \downarrow t \in \sigma(T)$ so $x \in U = f^{-1}(T \setminus \downarrow t) \in \sigma(S)$. Then there is a $d \in D$ such that $d \in U$ by the definition of openness. Then $f(d) \in T \setminus \downarrow t$ so $f(d) \not\leq t = \sup f(D)$, contradicting $d \in D$.

(ii) $\Longrightarrow$ (i):

Let A be a Scott-closed subset of T. To show that f is continuous, we need to show that $f^{-1}(A)$ is closed in $\sigma(S)$. Let $D \subseteq f^{-1}(A)$ be directed. Since f is Scott-continuous, $f(\sup D) = \sup f(D)$ and $\sup f(D) \in A$ by proposition 2.4.3. Then clearly $f(\sup D) \in A$ so $\sup D \in f^{-1}(A)$ so $f^{-1}(A)$ is closed under directed suprema. Since f is monotone, $f^{-1}(A)$ is also closed downwardsso by 2.4.3, $f^{-1}(A)$ is closed in $\sigma(S)$ as desired.

For the remainder of the proof, we assume that S and T are domains.

(ii) $\Longrightarrow$ (iv):

Since S is a domain, $\twoheaddownarrow x$ is directed and $\sup \twoheaddownarrow x = x$. By Scott-continuousness, f preserves directed suprema so $f(\sup \twoheaddownarrow x) = f(x) = \sup \twoheaddownarrow f(x)$.

(iv) $\Longrightarrow$ (iii):

Let $x \leq y$ in S. Then $\twoheaddownarrow x \subseteq \twoheaddownarrow y$ so by (iv), $f(x) = \sup f(\twoheaddownarrow x) \leq \sup f(\twoheaddownarrow y) = f(y)$. Therefore f is monotone. Now suppose $y \ll f(x) = \sup f(\twoheaddownarrow x)$. Since f is monotone, $f(\twoheaddownarrow x)$ is directed. Then by interpolation, there is a $w \in S$ such that $w \ll x$ and $y \ll f(w)$. Conversely, if $y \ll f(w)$ for some $w \ll x$, then $y \ll f(x)$ by monotonicity of f and proposition 2.2.3(ii).

(iii) $\implies$ (i):

Let $U \in \sigma(T)$ and $x \in f^{-1}(U)$. By definition of Scott-open, there is a $y \in U$ with $y \ll f(x)$. By (iii), there is a $w \in S$ such that $w \ll x$ and $f(y) \ll w$. Now we let $z \in \mathord{\Uparrow} w$. For all $y' \ll f(w)$, we have $y' \ll f(z)$ by (iii). Then $f(w) = \sup \mathord{\Downarrow} f(w) \leq f(z)$. By 2.2.3(i), $y \leq f(w)$. Since $y \in U$, we have $f(z) \in U$.

That the remaining statements are equivalent to the preceding ones follow from the definition of algebraic domains. □

In this section we have shown that the way below relation, continuity of posets, bases, and Scott-continuous functions can all be encoded within the Scott topology[28]. One may ask why we don't abandon order theory altogether in lieu of a purely topological approach to domains. Recall that every nontrivial Scott topology is T_0 but not T_1. To a topologist, this amount of separation seems very low as many in the field rarely consider spaces that fail to be Hausdorff or even Tychonoff. So topologically-speaking, domains seem highly pathological. But to order theory and pointless topology, separation matters little, if at all, and such barely-separated spaces can be of tremendous utility[29]. Thus we will continue to take from both approaches to domains as is expedient.

2.5 Product Domains

Originally this section was titled "The Category of Domains" but was changed because any fair account of the interactions between category theory and domain theory would constitute an entire textbook. The richness of this interaction likely comes at no surprised to a seasoned mathematician as category theory is remarkably adept at showing up whenever several disciplines of mathematics meet. Indeed, with the algebra, logic, topology, pointless topology, order theory, and computer science all interacting in some way in this chapter, the connection is bound to be strong. But as already mentioned, we will only account for one of the most basic domain-theoretic constructions: domain products[30]. However, Geirz, et al. discuss the categorical side of this construction in section I-2 [23] as do Abramsky and Jung in section 3.2.1 [2] and will content ourselves with the knowledge that universal properties are working behind the scenes.

Definition 2.5.1. Let D and E be dcpos. The **Cartesian product** $D \times E$ is the set product $D \times E$ ordered coordinatewise, i.e. $\langle x, y\rangle \leq \langle x', y'\rangle$ if and only if $x \leq x'$ in D and $y \leq y'$ in E.

This is the most natural definition for the product of dcpos. However, before proceeding, it will be advantageous to generalize this construction to infintiary products.

Definition 2.5.2. Let $\{D_i : i \in I\}$ be a family of domains. The product is given by the set

$$\prod_{i \in I} D_i$$

ordered coordinatewise, i.e. for $x, y \in \prod_{i \in I} D_i$, $x \leq y$ if and only if for all $i \in I$, $x(i) \leq y(i)$ in the domain D_i.

This construction applies to far more order-theoretic structures than just domains, but we need only treat domains here. The following is proposition I-2.1 of [23]:

[28] In fact, the analogy goes deeper still with section II-3 of [23] being entirely dedicated to characterizing continuous lattices using the Scott topology in exclusively topological terms.

[29] For an idea of the versatility of T_0 spaces in order theory and pointless topology, the reader is directed to section O-5 in [23].

[30] See the products of data types in section 2.1.

Theorem 2.5.3 (Gierz, et al.). *For a family of domains $\{D_i : i \in I\}$ each having a minimum element 0_i, the product $\prod_{i\in I} D_i$ is a domain. Furthermore, the way below relation is given by $x \ll y$ if and only if $x(i) \ll y(i)$ for all $i \in I$ and $x(i) = 0_i$ for all but finitely many $i \in I$.*

Proof. Let $\{D_i : i \in I\}$ be a family of domains each having a minimum element 0_i. We refer to the partial order and the way below relation on each domain with $\leq_i$ and $\ll_i$ respectively, reserving $\leq$ and $\ll$ for the partial order and way below relation on the product. For ease, we denote the product $\prod_{i\in I} D_i$ with $\mathcal{D}$. It is not difficult to show that suprema and infima are calculated coordinatewise since they are only influenced by the partial order, which is also calculated coordinatewise. Thus we are already guaranteed that $\langle \mathcal{D}, \leq \rangle$ constitutes a dcpo.

Let $x, y \in \mathcal{D}$ and suppose $x \ll y$. For any finite $F \subseteq I$, define $y^F \in \mathcal{D}$ such that

$$y^F(i) = \begin{cases} y(i) \text{ if } i \in F \\ 0_i \text{ if } i \notin F \end{cases}$$

Then $\{y^F : F \in \operatorname{fin}(I)\}$ is directed and $y = \sup\{y^F : F \in \operatorname{fin}(I)\}$. Consequently, by the fact that $x \ll y$, there is a finite F such that $x(i) \leq y^F$. Thus $x(i) = 0_i$ for all $i \in I \setminus F$.

Now we need to show that for $i \in F$, $x(i) \ll_i y(i)$. Fix $i \in F$. Let $A \subseteq D_i$ be directed such that $\sup A \geq_i y(i)$. For each $a \in A$ define $\bar{a} \in \mathcal{D}$ such that $\bar{a}(i) = a$ and for all $j \neq i$, $\bar{a}(j) = y(j)$. Then the family $\{\bar{a} : a \in A\}$ is directed with $y \leq \sup\{\bar{a} : a \in A\}$. Then since $x \ll y$, there is some $a \in A$ such that $x \leq \bar{a}$ so $x(i) \leq_i a$. Therefore $x(i) \ll_i y(i)$.

Conversely, suppose for all $i \in I$, $x(i) \ll_i y(i)$ and there is a finite subset $F \subseteq I$ such that for all $i \in I \setminus F$, $x(i) = 0_i$. Let $\mathcal{A} \subseteq \mathcal{D}$ be directed with $\sup \mathcal{A} \geq y$. Then for all $i \in I$, $y(i) \leq_i \sup\{a(i) : a \in \mathcal{A}\}$. Since $x(i) \ll_i y(i)$, there is a $b_i \in \mathcal{A}$ such that $x(i) \leq_i b_i(i)$. Since $\mathcal{A}$ is directed, there is a $b \in \mathcal{A}$ such that for all $i \in F$, $b_i \leq b$. Thus for $i \in F$, $x(i) \leq_i b(i)$. Since $x(i) = 0_i$ for all $i \in I \setminus F$, we can conclude that $x \leq b$. Therefore $x \ll y$.

Notice that until this point, we have made no assumption of continuity for any of the factor spaces D_i. Thus the proof so far goes through under the assumption that each D_i is a lifted[31] dcpo. From this characterization of $\ll$, directedness of $\twoheaddownarrow x$ for $x \in \mathcal{D}$ follows immediately from the assumption that each D_i is continuous. That $x = \sup \twoheaddownarrow x$ is similarly proven. Thus $\mathcal{D}$ is a domain. □

Corollary 2.5.4. *If each D_i has a basis B_i then*

$$\bigcup \left\{ \prod_{j\in F} B_j \times \prod_{k \in I\setminus F} \{0_k\} : F \in \operatorname{fin} I \right\}$$

is a base for the product $\prod\{D_i : i \in I\}$.

This line of argument also shows that the product of a family of lifted continuous semilattices is a continuous semilattice, and likewise for continuous lattices, L-Domains, and Scott domains.

Thus concludes our treatment of domain products as well as that of domain theory proper.

[31] For a poset P ordered by $\leq$, the **lift** of P is the poset $P_\perp = P \cup \perp$ where $x \leq_\perp y$ if and only if $x \leq y$ or $x = \perp$. Thus the term lifted dcpo merely refers to a dcpo with a bottom element.

Chapter 3

Domain Representability

In the preceding treatment of domain theory, specifically in section 2.4, topological definitions, theorems, and intuitions were employed in service of the study of domains. In this chapter, this is reversed and the rich structure of domains will be a useful tool in studying the topological completeness properties introduced in the first chapter.

As noted in proposition 2.4.3, any Scott topology generated on a nontrivial poset is T_0 but not T_1 which is an unacceptable degree of separation for many topologists. However, there are subspaces which are of topological interest in their own right. Specifically, for a continuous domain D, the subspace of the maximal elements $\max D$ can be Hausdorff, regular, or normal, or $\max D$ could satisfy any other separation axiom without necessarily generating the discrete Scott topology.

Definition 3.0.5. A topological space X is **domain representable**, sometimes shortened to DR, if there is a continuous domain $\langle D, \leq \rangle$ such that X is homeomorphic to $\max D$ equipped with the subspace topology of $\sigma(D)$.

Recall that a domain D is a Scott domain if any subset of D that is bounded above has a least upper bound.

Definition 3.0.6. A space X is **Scott domain representable**, sometimes just SDR, if there is a Scott domain $\langle D, \leq \rangle$ such that X is homeomorphic to the subspace $\max D$.

3.1 Domain Representability and Topological Games

The connection between the maximal space of a domain and complete metric spaces was hinted at by several mathematicians working in domain theory but first entered into the topological consciousness thanks to Keye Martin's 2003 paper [29]. Here, Martin proved two foundational theorems in the study of domain representability, securing its status as a completeness property.

Theorem 3.1.1 (Martin)**.** *All domain representable spaces are Choquet complete.*

Proof. Let $\langle D, \leq \rangle$ be a domain with maximal space $\max D \cong X$. Let $S = \{\langle U \cap X, x \rangle : x \in X \wedge U \in \sigma(D)$ be the set of plays in the Choquet game $\mathrm{Ch}(X)$. Then a sequence of n plays is in S^n. We define a winning strategy ς by recursion on n the number of plays.

For the initial case of $n = 1$, let $\langle U_1 \cap X, x_1 \rangle \in S$. Since $x_1 \in D$, we have $x_1 = \sup \mathord{\downarrow\!\!\downarrow} x_1 \in U_1$. Since $\mathord{\downarrow\!\!\downarrow} x_1$ is directed, there is a $d_1 \in U_1$ with $d_1 \ll x_1$. Set

$$\varsigma(U_1 \cap X, x_1) = \langle \mathord{\uparrow\!\!\uparrow} d_1 \cap X, x_1 \rangle.$$

Suppose ς is defined on $\bigcup_{1\le i\le n} S^i$ for some $n \geq 1$. The next step is to define $\varsigma(\langle U_1 \cap X, x_1\rangle, \ldots, \langle U_{n+1} \cap X, x_{n+1}\rangle \in S^{n+1}$. Note that we need only consider sequences of plays that could have possibly occurred legally. Thus for all $i \leq n$,

$$U_{i+1} \cap X \subseteq (\pi_1 \circ \varsigma)(\langle U_1 \cap X, x_1\rangle, \ldots, \langle U_i \cap X, x_i\rangle).$$

By the induction hypothesis, we have $\varsigma(\langle U_1 \cap X, x_1\rangle, \ldots, \langle U_n \cap X, x_n\rangle) = \langle \twoheaduparrow d_n \cap X, x_n\rangle$. Then we set

$$\varsigma(\langle U_1 \cap X, x_1\rangle, \ldots, \langle U_{n+1} \cap X, x_{n+1}\rangle = \langle \twoheaduparrow b_{n+1} \cap X, x_{n+1}\rangle$$

where $b_{n+1} \in U_{n+1} \cap \twoheaduparrow b_n$ and $b_{n+1} \ll x_{n+1}$.

With Player II's strategy defined, it must now be shown that the strategy is winning. Let $s \in S^\omega$ be a legal sequence of plays given by Player I so that

$$\pi_1 \circ s(n+1) \cap X \subseteq (\pi_1 \circ \varsigma)\,(s|_n)$$

for $n \in \omega$. This defines the sequence

$$\left\langle d_n \in D : \twoheaduparrow d_n \cap X = (\pi_1 \circ \varsigma)\,(s|_{n-1}) \wedge n \geq 1 \right\rangle.$$

Clearly $d_n \ll d_{n+1}$ and $d_n \in U_n$ for all $n \geq 1$. Since D is a dcpo, $d := \sup\{d_n : 1 \leq n\} \in D$. Scott open sets U_n are closed upwards, so $d \in U_n$ for all n. Thus

$$\uparrow d \cap X \subseteq \bigcap_{n\geq 1} (U_n \cap X)$$

and $\uparrow d \cap X \neq \varnothing$ by the Hausdorff maximality principle. $\square$

It is immediate consequence that domain representable spaces are Baire and that domain representable metric spaces are complete metric spaces. It follows that there are pseudocomplete spaces that are not domain representable. In fact, pseudocompleteness and domain representability are incomparable, with the converse presented in the next section.

In that same paper, Martin proved the following theorem relation Scott domain representability to topological games:

Theorem 3.1.2 (Martin). *Any Scott domain representable space is strongly α-favorable.*

Proof. Let $\langle \mathcal{D}, \leq\rangle$ be a Scott domainWe consider the game $\mathrm{Ch}(\max \mathcal{D})$. As before, let $S = \{\langle U, x\rangle : U \in \sigma(\mathcal{D}) \cap \max \mathcal{D} \wedge x \in U\}$ denote the set of possible plays of the game. Since we want to define a stationary strategy for Player II in this game, for a strategy ς, we need only consider $\operatorname{dom}\varsigma = X$, so $\varsigma : \max \mathcal{D} \to \max \mathcal{D}$. For the play $\langle U \cap \max \mathcal{D}, x\rangle$ we know that $x = \sup \twoheaddownarrow x \in U$. Since U is Scott open, there exists $a \in U$ with $a \ll x$. By interpolation, there is a $b \in U$ with $a \ll b \ll x$. Set $\varsigma(U \cap \max \mathcal{D}, x) = \langle \twoheaduparrow b \cap \mathcal{D}, x\rangle$.

Now we must show that ς defined in this way is a winning strategy. Let $s \in S^\omega$ be a run of the game composed of Player I's plays. Then that $s = \langle U_n \cap \max \mathcal{D}, x_n : n \in \omega\rangle$. For each n, $\varsigma(U_n \cap \max \mathcal{D}, x_n) = \langle \twoheaduparrow b_n \cap \max \mathcal{D}, x_n\rangle$ where $b_n \in \twoheaddownarrow x_n \cap U_n$. Since this sequence must correspond to legal moves in the game,

$$U_{n+1} \cap \max \mathcal{D} \subseteq \twoheaduparrow b_n \cap \max \mathcal{D} \subseteq \uparrow a_n \cap \max \mathcal{D} \subseteq U_n \cap \max \mathcal{D}.$$

Then $a_i \leq x_{n+1}$ for all $1 \leq i \leq n$. Since $\mathcal{D}$ is a dcpo, let $y_n = \sup\{a_i : 1 \leq i \leq n\} \in \mathcal{D}$. The sequence $\langle y_n : n \in \omega\rangle$ is increasing so $y = \sup\{y_n : n \in \omega\} \in \mathcal{D}$. Moreover, $y \in \uparrow a_n \subseteq U_n$. By the Hausdorff maximality principle, there is an $m \in \max \mathcal{D}$ such that $y \leq m$, so $m \in U_n$ for all $n \in \omega$. Consequently,

$$m \in \bigcap_{n\in\omega} (U_n \cap X) \neq \varnothing$$

so ς is a stationary winning strategy for Player II. $\square$

3.2 Domain Representability and Subcompactness

Following the usual pattern when defining new topological completeness properties, we now turn to the study of the properties which are stronger than domain representability. Specifically, we focus on the relationship between subcompactness and domain representability. That subcomactness implies domain representability was first proven by Bennett and Lutzer in [6]. However, our discussion will more closely follow that of Fleissner and Yengulalp in [19]. Firstly, it is convenient to give the following redundant definition of subcompactness.

Definition 3.2.1. A T_1 regular space $\langle X, \tau \rangle$ is subcompact when there is a set $\mathcal{B}$ satisfying:

i) $\mathcal{B} \subseteq \tau^*$ is a base for τ[1];

ii) $\prec_{\mathrm{cl}}$ is an antisymmetric, transitive relation on $\mathcal{B}$[2];

iii) $B \prec_{\mathrm{cl}} B'$ implies $B \subseteq B'$;

iv) if $x \in X$, then $\{B \in \mathcal{B} : x \in B\}$ is downward-directed[3] by $\prec_{\mathrm{cl}}$; and

v) if $\mathcal{F} \subseteq \mathcal{B}$ and $\langle \mathcal{F}, \prec_{\mathrm{cl}} \rangle$ is downward-directed, then $\bigcap \mathcal{F} \neq \varnothing$.

By replacing the specific closure inclusion relation $\prec_{\mathrm{cl}}$ with any sufficiently similar binary relation $\prec$, the following definition is obtained.

Definition 3.2.2. A T_1 regular space $\langle X, \tau \rangle$ is **generalized subcompact**, sometimes abbreviated to GSC, when there is a set $\mathcal{B}$ satisfying:

i) $\mathcal{B} \subseteq \tau^*$ is a base for τ;

ii) $\prec$ is an antisymmetric, transitive relation on $\mathcal{B}$;

iii) $B \prec B'$ implies $B \subseteq B'$;

iv) if $x \in X$, then $\{B \in \mathcal{B} : x \in B\}$ is downward-directed by $\prec$; and

v) if $\mathcal{F} \subseteq \mathcal{B}$ and $\langle \mathcal{F}, \prec \rangle$ is downward-directed, then $\bigcap \mathcal{F} \neq \varnothing$.

It is clear that any subcompact space is generalized subcompact. We make use of one additional definition before moving on to the main theorem of the section.

Definition 3.2.3. Let $\langle X, \tau \rangle$ be a T_1 space and let Q be a set with $\ll \subseteq Q \times Q$ and $B : Q \to \tau^*$. Then the triple $\langle Q, \ll, B \rangle$ **represents** X if the following are satisfied satisfying:

i) $\{B(q : q \in Q\} \subseteq \tau^*$ is a base for τ;

ii) $\ll$ is an antisymmetric, transitive relation on Q;

iii) for all $p, q \in Q$, $p \ll q$ implies $B(q) \subseteq B(p)$;

iv) if $x \in X$, then $\{q : x \in B(q)\}$ is upward-directed by $\ll$; and

v) if $D \subseteq Q$ and $\langle D, \ll \rangle$ is upward-directed, then $\bigcap \{B(q) : q \in D\} \neq \varnothing$.

We relate generalized subcompactness and being represented by a triple with the following lemma.

[1] In general we define $\tau^* = \tau \setminus \{\varnothing\}$.

[2] The relation $\prec_{\mathrm{cl}} \subseteq \mathcal{B} \times \mathcal{B}$ is defined by $B_1 \prec_{\mathrm{cl}} B_2$ if and only if $\mathrm{cl}\, B_1 \subseteq B_2$.

[3] A poset X is downward-directed, if for any $x, y \in X$, there is $z \in X$ such that $z \leq x$ and $z \leq y$. The definition of directedness that we ahve been using is upward-directedness, which will usually just be referred to as directedness.

Lemma 3.2.4 (Fleissner and Yengualp)**.** *A space X is generalized subcompact if and only if there is a triple $\langle Q, \ll, B\rangle$ representing X with B an injective function.*

Proof. Suppose X is generalized subcompact and let $\mathcal{B}$, $\prec$ be as in definition 3.2.2. Define $Q = \mathcal{B}$ and let B be the identity map on $\mathcal{B}$. For $V, V' \in \mathcal{B} = Q$, let $V \ll V'$ if and only if $V' \prec V$.

Conversely, suppose $\langle Q, \ll, B\rangle$ represents X and suppose that B injective. Then set

$$\mathcal{B} = \{B(q) : q \in Q\}$$

and $B(q) \prec B(q')$ if and only if $q' \ll q$. Since B is injective, $\prec$ is well-defined. □

As we will see, these two notions are equivalent. But for now, we have the following chain of implications.

$$\text{subcompact} \implies \text{GSC} \implies \text{representable}$$

thereby motivating the next theorem.

Theorem 3.2.5 (Fleissner and Yengulalp)**.** *A space X is domain representable if and only if X is representable by a triplet $\langle Q, \ll_Q, B\rangle$.*

Proof. Suppose $\langle X, \tau\rangle$ is domain representable by the domain $\langle \mathcal{D}, \leq\rangle$ and the homeomorphism $\phi : X \to \max \mathcal{D}$. Let Q be a basis for $\mathcal{D}$. Define $B : Q \to \tau^*$ such that $B(q) = \{x \in X : q \ll \phi(x)\}$ for all $q \in Q$. For $p, q \in Q$, let $p \ll_Q q$ if and only if $p \ll q$[4] We verify that the five criteria of definition 3.2.3 are satisfied:

i) Since Q is a basis for $\mathcal{D}$, $\{\Uparrow q : q \in Q\}$ is a base for $\sigma(\mathcal{D})$, thereby providing a base for the subspace $\max \mathcal{D} \cong X$.

ii) Clearly $\ll_Q$ is antisymmetric and transitive since $\ll_Q$ is the restriction of $\ll$ to Q.

iii) If $p \ll_Q q$ then $p \ll q$ so $B(q) = \Uparrow q \subseteq \Uparrow p = B(p)$.

iv) Let $x \in X$ and $p_1, p_2 \in \{q \in Q : x \in B(q)\}$. Then $p_1 \ll \phi(x)$ and $p_2 \ll \phi(x)$. Then by the interpolation property in definition 2.3.10, there is $q \in Q$ such that $p_1, p_2 \ll q \ll \phi(x)$. Thus $p_1 \ll_Q q$ and $p_2 \ll_Q q$ so $\{q \in Q : x \in B(q)\}$ is directed.

v) Let $P \subseteq Q$ be directed by $\ll_Q$. Since $\mathcal{D}$ is a dcpo, let $x \in X$ such that $\sup P \leq \phi(x) \in \max \mathcal{D}$. By interpolation of $\ll$, for all $p \in P$, $p \ll \phi(x)$. Therefore $x \in \bigcap\{B(q) : q \in P\}$.

Conversely, suppose $\langle Q, \ll_Q, B\rangle$ represents X. By theorem 2.3.11, it follows that $\mathbf{Idl}(Q, \ll_Q)$ is a domain and for the order-preserving map $i : Q \to i(Q)$, $x \mapsto \downarrow x$, the set $i(Q)$ is a domain basis for $\mathbf{Idl}(Q, \ll_Q)$. Going forward, we refer to $\mathbf{Idl}(Q, \ll_Q)$ simply as $\mathbf{Idl}(Q)$.

Observe that for any $I \in \mathbf{Idl}(Q)$, I is directed upwards by $\ll_Q$ and by (v) of definition 3.2.3, $\bigcap\{B(q) : q \in I\} \neq \varnothing$. Define the map $N : X \to \mathbf{Idl}(Q)$ such that $N(x) = \{q \in Q : x \in B(q)\}$.

Claim. *N is a homeomorphism from X to $\max(\mathbf{Idl}(Q))$.*

Before we show that the range is as specified, we show that N is surjective, i.e. that $\max(\mathbf{Idl}(Q)) \subseteq \{N(x) : x \in X\}$. Let $M \in \max(\mathbf{Idl}(Q))$. For any x, by (iii) and (iv) of 3.2.3, $N(x) = \{q \in Q : x \in B(q)\}$ is an ideal. If $x \in \bigcap\{B(p) : p \in M\}$, then $M \subseteq N(x)$. Then $M = N(x)$ because M is maximal. Injectivity follows from a short argument: let $x, y \in X$ be distinct. Since $\{B(q) : q \in Q\}$ is a base for a T_1 topology on X, there is a $q \in Q$ with $x \in B(q)$ and $y \notin B(q)$. Then $N(x) \not\subseteq N(y)$. Therefore N is bijective.

[4]Here we mean $\ll$ to refer to the way-below relation on $\mathcal{D}$.

Next, we need to show that N actually maps to $\max(\mathbf{Idl}(Q))$. Let $x \in X$ and we want to show that $N(x)$ is a maximal ideal. Clearly $N(x)$ is an ideal on Q and by the prime ideal theorem, there is $M \in \max(\mathbf{Idl}(Q))$ such that $N(x) \subseteq M$. By the surjectivity argument, there is a $y \in \bigcap\{B(q) : q \in M\}$ such that $N(x) \subseteq M \subseteq N(y)$. Since X is T_1, we have $x = y$ so $M = N(x)$. Therefore $\max(\mathbf{Idl}(Q)) = \{N(x) : x \in X\}$.

Finally, we show that N is a homeomorphism. Observe that $\{B(q) : q \in Q\}$ is a base for X and

$$\left\{\Uparrow(\downarrow q) \cap \max(\mathbf{Idl}(Q)) : q \in Q\right\}$$

is a base for the subspace topology on $\max(\mathbf{Idl}(Q))$ where $\Uparrow$ refers to the way below relation on the domain $\mathbf{Idl}(Q)$. Then it suffices to show that for all $q \in Q$, $\{N(x) : x \in B(q)\} = \Uparrow(\downarrow q) \cap \max(\mathbf{Idl}(Q))$. Fix $q \in Q$. Then

$$\{N(x) : x \in B(q)\} = \{N(x) : q \in N(x)\} = \{M \in \max(\mathbf{Idl}(Q)) : q \in M\} = \Uparrow(\downarrow q) \cap \max(\mathbf{Idl}(Q))$$

where the final equality follows from theorem 2.3.11(ii).

Therefore X is domain representable. $\square$

Corollary 3.2.6. *All subcompact spaces are domain representable.*

In a more recent paper, Yengulalp showed that any regular domain representable space is generalized subcompact, thereby showing the equivalence of those two notions for regular spaces [45].

The concept of representation by a triple also extends to Scott domains.

Proposition 3.2.7 (Fleissner and Yengulalp)**.** *A space X is Scott domain representable if and only if X is represented by a triple $\langle Q, \ll_Q, B\rangle$ with the property that any doubleton subset of Q that is bounded above has a least upper bound.*

Proof. That any Scott domain representable space is also represented by such a triple is immediate. Suppose then that X is represented by $\langle Q, \ll_Q, B\rangle$ such that for all $q_1, q_2 \in Q$, if there is a $p \in Q$ such that $q_1 \ll_Q p$ and $q_2 \ll_Q p$, then $q_1 \vee q_2 \in Q$.

Let $I_1, I_2 \in \mathbf{Idl}(Q)$ such that $J \supseteq I_1 \cup I_2$. If $q_1 \in I_1$ and $q_2 \in I_2$, then $\{q_1, q_2\}$ has an upper bound in J, so $q_1 \vee q_2 \in J$ since J is closed downwards. Set

$$K = \bigcup\{\downarrow q_1 \vee q_2 : q_1 \in I_1 \text{ and } q_2 \in I_2\}.$$

Then K is the least upper bound of I_1 and I_2 in $\mathbf{Idl}(Q)$. $\square$

Furthermore, Fleissner and Yengulalp defined an extension of Debs space in [19] which is domain representable but not α-favorable. As a result, this space is domain representable but is neither subcompact nor pseudocomplete. Thus subcompactness is strictly stronger than domain representability and domain representability and pseudocompleteness are incomparable. This also implies that the space fails to be Scott domain representable since any Scott domain representable space is strongly α-favorable.

In [5], Bennett and Lutzer proved that any Čech-complete space is domain representable as well. As we will see somewhat in the next section, less is known about Scott domain representability. For instance, the connection between subcompactness and Scott domain representability is still unknown.

3.3 Domain Representability and Completeness

Domain representability and Scott domain representability have several desirable features, one of which is that they are closed under arbitrary products.

Proposition 3.3.1. *Let $\{X_\alpha : \alpha \in I\}$ be a family of (Scott) domain representable spaces. Then the product $\prod_{\alpha \in I} X_\alpha$ with the usual finite support topology is (Scott) domain representable.*

Proof. If X_α is domain representable for each $\alpha \in I$, let D_α be a domain with $\max D_\alpha \cong X_\alpha$. Then we can lift each D_α by adding a minimum element 0_α without changing the topology on the maximal space. Subsequently, we may apply theorem 2.5.3 and corollary 2.5.4. Since domain bases align with topological bases by proposition 2.4.6 $\max\left(\prod_{\alpha \in I} D_\alpha\right) \cong \prod_{\alpha \in I} X_\alpha$.

The same procedure goes through for Scott domains except Scott domains necessarily possess a minimum element so they do not even need to be lifted to apply proposition 2.5.3. $\square$

Next we move on to a construction rarely considered in the context of completeness, which will require the use of Fleissner and Yengulalps notion of representability by a triple [19].

Proposition 3.3.2 (DeVilbiss)**.** *The box product of any family of domain representable spaces is domain representable.*

Proof. Let $\{X_\alpha\}$ be a collection of domain representable spaces for $\alpha \in I$, an index set. As shown in [cite Yengulalp], since X_α is domain representable for each $\alpha \in I$, there is a set Q_α, a relation $\ll_\alpha$, and a function $B_\alpha : Q_\alpha \to \tau(X_\alpha)$ with the following properties:

i) The relation $\ll_\alpha$ is anti-symmetric and transitive on Q_α.

ii) The set $\{B_\alpha(p) | p \in Q_\alpha\}$ is a basis for the topology on X_α.

iii) If $p \ll q$ then $B(q) \subseteq B(p)$.

iv) For all $x \in X_\alpha$, $\{p | x \in B_\alpha(q)\}$ is directed by $\ll_\alpha$.

v) If $D \subseteq Q_\alpha$ is directed by $\ll_\alpha$, then $\bigcap\{B_\alpha(p) | p \in D\} \neq \varnothing$.

We will use this simplified characterization of domain representability to prove that the box product topology $\square\{X_\alpha : \alpha \in I\}$ is domain representable. To do so, we define $Q = \prod\{Q_\alpha : \alpha \in I\}$, define $\ll$ on Q such that for $p, q \in Q$, $p \ll q$ if and only if $p(\alpha) \ll_\alpha q(\alpha)$ for all $\alpha \in I$, and define $B : Q \to \square\{X_\alpha\}$ such that $B(p) = \prod\{B_\alpha(p(\alpha)) : \alpha \in I\}$. We will now prove that these choices for Q, $\ll$, and B satisfy the five criteria for simplified domain representability:

i) We want to show that $\ll$ is anti-symmetric and transitive on Q. Let $p, q \in Q$ and suppose $p \ll q$ and $q \ll p$. Therefore $p(\alpha) \ll_\alpha q(\alpha)$ and $q(\alpha) \ll_\alpha p(\alpha)$ for all $\alpha \in I$. Since $\ll_\alpha$ is anti-symmetric, it follows that $p(\alpha) = q(\alpha)$ for all $\alpha \in I$. Therefore $p = q$ so $\ll$ is anti-symmetric on Q. Now suppose $p, q, r \in Q$ such that $p \ll q$ and $q \ll r$. Therefore $p(\alpha) \ll_\alpha q(\alpha)$ and $q(\alpha) \ll_\alpha r(\alpha)$ for all $\alpha \in I$. Since $\ll_\alpha$ is transitive, $p(\alpha) \ll_\alpha r(\alpha)$ for all $\alpha \in I$. Therefore, $p \ll r$ so $\ll$ is transitive, satisfying the first property.

ii) Next we will show that $\{B(p) | p \in Q\}$ is a basis for $\square\{X_\alpha\}$. Thus we will consider $\{B(p) | p \in Q\}$. Observe that

$$\{B(p) | p \in Q\} = \{\prod\{B_\alpha(p(\alpha))\} | p(\alpha) \in Q_\alpha\} = \prod\{B_\alpha(p(\alpha)) | p(\alpha) \in Q_\alpha\}.$$

Since $\{B_\alpha(p(\alpha)) | p(\alpha) \in Q_\alpha\}$ is a basis for X_α for all $\alpha \in I$, it follows that $\prod\{B_\alpha(p(\alpha)) | p(\alpha) \in Q_\alpha\}$ is a basis for $\square\{X_\alpha\}$.

iii) Suppose $p \ll q$. Then $p(\alpha) \ll_\alpha q(\alpha)$ for each $\alpha \in I$. Then $B_\alpha(q(\alpha)) \subseteq B_\alpha(p(\alpha))$. Therefore

$$\prod_{\alpha \in I} B_\alpha(q(\alpha)) \subseteq \prod_{\alpha \in I} B_\alpha(p(\alpha))$$

as desired.

iv) Next we will show that for all $x \in \prod\{X_\alpha\}$, $\{p|x \in B(q)\}$ is directed by $\ll$. Thus we fix $x \in \prod\{X_\alpha\}$. Consider the set $Q_x = \{p|x \in B(p)\}$. Since $x \in B(p)$ if and only if $x(\alpha) \in B_\alpha(p(\alpha))$ for all $\alpha \in I$, it follows that $Q_x = \prod\{p(\alpha)|x(\alpha) \in B_\alpha(p(\alpha))\}$. Observe that the set $(Q_\alpha)_x = \{p(\alpha)|x(\alpha) \in B_\alpha(p(\alpha))\}$ is directed by $\ll_\alpha$ by our hypothesis above. Therefore, for $p(\alpha), q(\alpha) \in (Q_\alpha)_x$, there is an $r(\alpha) \in (Q_\alpha)_x$ such that $p(\alpha) \ll_\alpha r(\alpha)$ and $q(\alpha) \ll_\alpha r(\alpha)$. Fix $p, q \in Q_x$ and construct $r \in Q_x$ such that $x \in B(r)$ and $p(\alpha) \ll_\alpha r(\alpha)$ and $q(\alpha) \ll_\alpha r(\alpha)$ for all $\alpha \in I$. Therefore $p \ll r$ and $q \ll r$ so $\{p|x \in B(p)\}$ is directed by $\ll$.

v) Finally, we will show that if $D \subseteq Q$ is directed by $\ll$, then $\bigcap\{B(p)|p \in D\} \neq \varnothing$. Let $D \subseteq Q$ be directed by $\ll$ and let $p, q \in D$. Therefore there is $r \in D$ such that $p \ll r$ and $q \ll q$. By definition, we have $p(\alpha) \ll_\alpha r(\alpha)$ and $q(\alpha) \ll_\alpha r(\alpha)$ for all $\alpha \in I$. Define $D_\alpha = \{p(\alpha)|p \in D\}$. It follows that D_α is directed by $\ll_\alpha$. By our initial hypothesis, we have $\bigcap\{B_\alpha(p(\alpha))|p(\alpha) \in D_\alpha\} \neq \varnothing$. Finally, construct the function x such that $x(\alpha) \in \bigcap\{B_\alpha(p(\alpha))|p(\alpha) \in D_\alpha\}$. Therefore $x \in \prod\{\bigcap\{B_\alpha(p(\alpha))|p(\alpha) \in D_\alpha\}\} = \bigcap\{B(p)|p \in D\}$. Therefore $\bigcap\{B(p)|p \in D\} \neq \varnothing$.

Since our choice of Q, $\ll$, and B satisfy the five above requirements, it follows that the box product topology on $\prod\{X_\alpha\}$ is domain representable. □

Aside from being preserved under various topological products, domain representability is also preserved by retracts [33]. The same question is still open for Scott domain representability. Both DR and SDR are open-hereditary and neither are closed hereditary. Interestingly, domain representability is G_δ-hereditary while Scott domain representability is not [7].

From just these few properties, we can see that domain representability (and Scott domain representability to a lesser extent) certainly possess desirable properties, but perhaps not enough to warrant the title generalized completeness property[5]. However, domain representability and Scott domain representability, like subcompactness, lack the seemingly superfluous use of cardinality in the definition of the convergence mechanism, unlike most completeness properties discussed here. In light of this, further study of domain representability, Scott domain representability, and subcompactness may be promising avenues of research in the area of topological completeness.

On the other hand, completeness properties that do not make essential use of cardinality can easily

[5]See section 1.7.

Bibliography

[1] J. M. Aarts and D. J. Lutzer. Completeness properties designed for recognizing Baire spaces. *Dissertationes Math. (Rozprawy Mat.)*, 116:48, 1974.

[2] Samson Abramsky and Achim Jung. Domain theory. In *Handbook of Logic in Computer Science*, pages 1–168. Clarendon Press, 1994.

[3] P. S. Alexandrov and P. Urysohn. *Mémoire sur les espaces topologiques compacts: Dedié à monsieur D. Egoroff*. 1929.

[4] R. Baire. Sur les fonctions de variables réelles. *Annali di Matematica Pura ed Applicata Annali di Matematica, Serie III*, 3(1):1–123, 1899.

[5] Harold Bennett and David Lutzer. Domain-representable spaces. *Fund. Math.*, 189(3):255–268, 2006.

[6] Harold Bennett and David Lutzer. Domain-representability of certain complete spaces. *Houston J. Math.*, 34(3):753–772, 2008.

[7] Harold Bennett and David Lutzer. Domain representable spaces and completeness. *Topology Proc.*, 34:223–244, 2009.

[8] Charles E. Blair. The Baire category theorem implies the principle of dependent choices. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 25(10):933–934, 1977.

[9] Bernard Bolzano. *Rein analytischer Beweis des Lehrsatzes: dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewahren, wenigstens eine reelle Wurzel der Gleichung liege*. Gedruckt bei Gottlieb Haase, 1817.

[10] Nicolas Bourbaki. *Elements de mathematique. Espaces vectoriels topologiques*. 1955.

[11] Michael A. Bukatin. Logic of fixed points and Scott topology. In *Proceedings of the 16th Summer Conference on General Topology and its Applications (New York)*, volume 26, pages 433–468, 2001/02.

[12] Eduard Čech. On bicompact spaces. *Ann. of Math. (2)*, 38(4):823–844, 1937.

[13] Gustave Choquet. *Lectures on analysis. Vol. I: Integration and topological vector spaces*. Edited by J. Marsden, T. Lance and S. Gelbart. W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[14] J. de Groot. Subcompactness and the Baire category theorem. *Nederl. Akad. Wetensch. Proc. Ser. A 66=Indag. Math.*, 25:761–767, 1963.

[15] Gabriel Debs. Stratégies gagnantes dans certains jeux topologiques. *Fund. Math.*, 126(1):93–105, 1985.

[16] Ryszard Enelking. Kazimierz kuratowski (1896-1980) - his life and work in topology. In Charles E. Aull and R.Editors Lowen, editors, *Handbook of the history of general topology*, volume 2, chapter 3, pages 431–452. Kluwer Academic Publishers, 1997.

[17] Isidore Fleischer. On: "Subcompactness and the Baire category theorem" [Indag. Math. **25** (1963), 761–767; MR **28** #2520] by J. de Groot. *Nederl. Akad. Wetensch. Indag. Math.*, 41(1):9–11, 1979.

[18] W. G. Fleissner and K. Kunen. Barely Baire spaces. *Fund. Math.*, 101(3):229–240, 1978.

[19] William Fleissner and Lynne Yengulalp. From subcompact to domain representable. *Topology Appl.*, 195:174–195, 2015.

[20] M. Maurice Fréchet. Sur quelques points du calcul fonctionnel. *Rendiconti del Circolo Matematico di Palermo Rend. Circ. Matem. Palermo*, 22(1):1–72, 1906.

[21] Fred Galvin and Ratislav Telgársky. Stationary strategies in topological games. *Topology Appl.*, 22(1):51–69, 1986.

[22] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, 1980.

[23] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous lattices and domains*. Cambridge University Press, 2003.

[24] F. Hausdorff. *Grundzüge der Mengenlehre*. 1914.

[25] F. Hausdorff. Gestufte räume. *Fund. Math.*, 28:486–502, 1935.

[26] R. E. Hodel. Some results in metrization theory, 1950–1972. In *Topology Conference (Virginia Polytech. Inst. and State Univ., Blacksburg, Va., 1973)*, pages 120–136. Lecture Notes in Math., Vol. 375. Springer, Berlin, 1974.

[27] M. R. Krom. Infinite games and special Baire space extensions. *Pacific J. Math.*, 55:483–487, 1974.

[28] Casimir Kuratowski. Sur l'opération $\bar{a}$ de l'analysis situs. *Fundamenta Mathematicae*, 3(1):182–199, 1922.

[29] Keye Martin. Topological games in domain theory. *Topology Appl.*, 129(2):177–186, 2003.

[30] Joe Mashburn. A comparison of three topologies on ordered sets. *Topology Proc.*, 31(1):197–217, 2007.

[31] R. Daniel Mauldin. *The Scottish book: mathematics from the Scottish Café*. Birkhäuser, 1981.

[32] James R. Munkres. *Topology; a first course*. Prentice-Hall, 2 edition, 1974.

[33] Süleyman Önal and Çetin Vural. Domain representability of retracts. *Topology Appl.*, 194:1–3, 2015.

[34] John C. Oxtoby. The Banach-Mazur game and Banach category theorem. In *Contributions to the theory of games, vol. 3*, Annals of Mathematics Studies, no. 39, pages 159–163. Princeton University Press, Princeton, N. J., 1957.

[35] John C. Oxtoby. Cartesian products of Baire spaces. *Fund. Math.*, 49:157–166, 1960/1961.

[36] Gerhard Preuß. Felix hausdorff (1868-1942). In Charles E. Aull and R.Editors Lowen, editors, *Handbook of the history of general topology*, volume 1, chapter 1, pages 1–20. Kluwer Academic Publishers, 1997.

[37] Dana S. Scott. *Outline of a Mathematical Theory of Computation*. Oxford University Computing Laboratory, Oxford, 1970.

[38] Dana S. Scott. Domains for denotational semantics. In *Proceedings of the 9th Colloquium on Automata, Languages and Programming*, pages 577–613, London, UK, UK, 1982. Springer-Verlag.

[39] Dana S. Scott and Christopher Strachey. *Toward a mathematical semantics for computer languages*. Oxford University Computing Laboratory, Programming Research Group, 1971.

[40] M. B. Smyth. Effectively given domains. *Theoretical Computer Science*, 5(3):257–274, 1977.

[41] R. H. Sorgenfrey. On the topological product of paracompact spaces. *Bull. Amer. Math. Soc.*, 53:631–632, 1947.

[42] Alfred Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics Pacific J. Math.*, 5(2):285–309, 1955.

[43] Rastislav Telgársky. Topological games: on the 50th anniversary of the Banach-Mazur game. *Rocky Mountain J. Math.*, 17(2):227–276, 1987.

[44] H. H. Wicke and J. M. Worrell, Jr. Completeness and topologically uniformizing structures. In *TOPO 72—general topology and its applications (Proc. Second Pittsburgh Internat. Conf., Pittsburgh, Pa., 1972; dedicated to the memory of Johannes H. de Groot)*, pages 557–585. Lecture Notes in Math., Vol. 378. Springer, Berlin, 1974.

[45] Lynne Yengulalp. Coding strategies, the choquet game and domain representability. *Topology and its Applications*, (202):384–396, 2016.