

2011

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POSITIVE SOLUTIONS OF NONLOCAL BOUNDARY VALUE PROBLEM FOR HIGHER ORDER FRACTIONAL DIFFERENTIAL SYSTEM

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ABSTRACT. In this paper, we study existence and multiplicity results for a coupled system of nonlinear nonlocal boundary value problems for higher order fractional differential equations of the type

$$\begin{cases} {}^c\mathcal{D}_{0+}^{\alpha} u(t) = \lambda a(t)f(u(t), v(t)), & {}^c\mathcal{D}_{0+}^{\beta} v(t) = \mu b(t)g(u(t), v(t)), \\ u'(0) = u''(0) = u'''(0) = \dots = u^{(n-1)}(0) = 0, & u(1) = \xi_1 u(\eta_1), \\ v'(0) = v''(0) = v'''(0) = \dots = v^{(n-1)}(0) = 0, & v(1) = \xi_2 v(\eta_2), \end{cases}$$

where $\lambda, \mu > 0$, $n - 1 < \alpha, \beta \leq n$ for $n \in \mathbb{N}$; $\xi_i, \eta_i \in (0, 1)$ for $i = 1, 2$ and ${}^c\mathcal{D}_{0+}^{\alpha}$ is Caputo fractional derivative. We employ the Guo-Krasnosel'skii fixed point theorem to establish existence and multiplicity results for positive solutions. We derive explicit intervals for the parameters λ and μ for which the system possess the positive solutions or multiple positive solutions. Examples are included to show the applicability of the main results.

1. INTRODUCTION

Fractional differential equations is rapidly growing both in theory and in applications to real world problems. Many problems in engineering, physics and other applied sciences can be modeled as differential equations of fractional order. It has been established that, in many situations, these models provide more suitable results than analogous models with integer derivatives.

The attention drawn to boundary value problems for nonlinear fractional differential equations is evident from an increased number of recent publications. Existence and uniqueness of solutions of fractional differential equations is well studied, see for

example [1, 2, 3, 6, 12, 13, 15] and references therein. Recently, existence and multiplicity results for positive solutions of boundary value problems involving fractional order derivatives have received much attention, for example, [4, 5, 7, 8, 9, 14, 16, 18]. In contrast, boundary value problems for coupled systems of fractional differential equations have received less attention; only a few results can be found in the literature concerning existence and multiplicity of positive solutions of boundary value problems for coupled systems of fractional differential equations. See for example [4, 9, 18].

In this paper, we study existence and multiplicity results for positive solutions to nonlinear three-point boundary value problems corresponding to a higher order coupled system of fractional differential equations of the type

$$(1.1) \quad \begin{cases} {}^c\mathcal{D}_{0+}^\alpha u(t) = \lambda a(t)f(u(t), v(t)), & {}^c\mathcal{D}_{0+}^\beta v(t) = \mu b(t)g(u(t), v(t)), \\ u'(0) = u''(0) = u'''(0) = \dots = u^{(n-1)}(0) = 0, & u(1) = \xi_1 u(\eta_1), \\ v'(0) = v''(0) = v'''(0) = \dots = v^{(n-1)}(0) = 0, & v(1) = \xi_2 v(\eta_2), \end{cases}$$

where $\lambda, \mu > 0$, $n - 1 < \alpha, \beta \leq n$ for $n \in \mathbb{N}$; $\xi_i, \eta_i \in (0, 1)$ for $i = 1, 2$ and ${}^c\mathcal{D}_{0+}^\alpha$ is Caputo fractional derivative. We employ the Guo-Krasnosel'skii fixed point theorem to study existence and multiplicity results. We point out that the results in this article extend those obtained in [3] as we study a system of higher order equations and more importantly, our results extend those obtained in [3, 9] as we consider $f(u(t), v(t))$, $g(u(t), v(t))$ instead of $f(v(t))$, $g(u(t))$.

The paper is organized as follows: In Section 2, we recall the definitions of fractional integral and fractional derivative and some basic lemmas. In Section 3, properties of the Green's function for the associated linear problem are studied. By employing the Guo-Krasnosel'skii fixed point theorem, some sufficient conditions for the existence or multiplicity of positive solutions to the system (1.1) are established. We obtain explicit intervals for λ and μ such that for any value of λ and μ in the intervals, existence or multiplicity of positive solution is guaranteed. Examples are included to demonstrate the application the main results.

2. PRELIMINARIES

We recall some basic definitions and lemmas from fractional calculus.

Definition 2.1. [17] The fractional integral of order $\alpha > 0$ of a function $g : (a, \infty) \rightarrow \mathbb{R}$ is defined by

$$\mathcal{I}_{a+}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} ds,$$

provided the integral converges.

Definition 2.2. [17] The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $g : (0, \infty) \rightarrow \mathbb{R}$ is defined by

$${}^c\mathcal{D}_{a+}^\alpha g(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{g^{(n)}(s)}{(t - s)^{\alpha - n + 1}} ds,$$

where $n = [\alpha] + 1$, (the notation $[a]$ stands for the largest integer not greater than α).

Remark 2.3. Under the natural conditions on $g(t)$ the Caputo fractional derivative becomes conventional integer order derivative of function $g(t)$ as $\alpha \rightarrow n$.

Lemma 2.4. For $\alpha > 0$, $g(t) \in C(0, 1) \cap L(0, 1)$, the homogenous fractional differential equation ${}^c\mathcal{D}_{0+}^\alpha g(t) = 0$, has a solution $g(t) = c_1 + c_2t + c_3t^2 + \dots + c_nt^{n-1}$, where, $c_i \in \mathbb{R}$, $i = 0, \dots, n$ and $n = [\alpha] + 1$.

Lemma 2.5. Assume that $g(t) \in C(0, 1) \cap L(0, 1)$, with derivative of order n that belongs to $C(0, 1) \cap L(0, 1)$, then $\mathcal{I}_{0+}^\alpha {}^c\mathcal{D}_{0+}^\alpha g(t) = g(t) + c_1 + c_2t + c_3t^2 + \dots + c_nt^{n-1}$, where, $c_i \in \mathbb{R}$, $i = 0, \dots, n$ and $n = [\alpha] + 1$.

Lemma 2.6. [10] If $\alpha > \beta > 0$, then ${}^c\mathcal{D}_{0+}^\beta \mathcal{I}_{0+}^\alpha \phi(t) = \mathcal{I}_{0+}^{\alpha - \beta} \phi(t)$. In particular, if m is positive integer and $\delta > m$, then $\frac{d^m}{dt^m} (\mathcal{I}_{0+}^\delta \phi(t)) = \mathcal{I}_{0+}^{\delta - m} \phi(t)$.

The proofs of our main results are based on the following fixed point theorem.

Theorem 2.7 ([11] Guo-Krasnosel'skii Fixed Point Theorem). Let \mathcal{B} be Banach space and $\mathcal{E} \subset \mathcal{B}$ be a cone. Assume that Ω_1, Ω_2 are open disks in \mathcal{B} such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Let $\mathcal{A} : \mathcal{E} \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow \mathcal{E}$ be completely continuous such that either

- (i) $\|\mathcal{A}u\| \leq \|u\|$, for $u \in \mathcal{E} \cap \partial\Omega_1$ and $\|\mathcal{A}u\| \geq \|u\|$, for $u \in \mathcal{E} \cap \partial\Omega_2$ or
- (ii) $\|\mathcal{A}u\| \geq \|u\|$, for $u \in \mathcal{E} \cap \partial\Omega_1$ and $\|\mathcal{A}u\| \leq \|u\|$, for $u \in \mathcal{E} \cap \partial\Omega_2$.

Then, \mathcal{A} has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. MAIN RESULTS

Lemma 3.1. Let $h \in C[0, 1]$, then the three-point linear boundary value problem

$$(3.1) \quad \begin{aligned} & {}^c\mathcal{D}_{0+}^\alpha u(t) + h(t) = 0, \quad t \in (0, 1), \quad n - 1 < \alpha \leq n, \\ & u'(0) = u''(0) = u'''(0) = \dots = u^{(n-1)}(0) = 0, \quad u(1) = \xi_1 u(\eta_1), \end{aligned}$$

has a unique unique solution given by

$$u(t) = \int_0^1 \mathcal{H}_\alpha(t, s) h(s) ds,$$

where

$$(3.2) \quad \mathcal{H}_\alpha(t, s) = \begin{cases} \frac{(1-s)^{\alpha-1} - (1-\xi_1)(t-s)^{\alpha-1} - \xi_1(\eta_1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)}, & s \leq t, \eta_1 \geq s, \\ \frac{(1-s)^{\alpha-1} - (1-\xi_1)(t-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)}, & \eta_1 \leq s \leq t \leq 1, \\ \frac{(1-s)^{\alpha-1} - \xi_1(\eta_1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)}, & 0 \leq t \leq s \leq \eta_1, \\ \frac{(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)}, & t \leq s, s \geq \eta_1. \end{cases}$$

Proof. In view of Lemma 2.5, the general solution of differential equation in (3.1) is given by

$$(3.3) \quad u(t) = -\mathcal{I}_{0+}^\alpha h(t) + \sum_{k=1}^n c_k t^{k-1}, c_k \in \mathbb{R}, k = 1, 2, \dots, n.$$

By Lemma 2.6 and equation (3.3), we obtain

$$(3.4) \quad u^{(m)}(t) = -\mathcal{I}_{0+}^{\alpha-m} h(t) + \sum_{k=m+1}^n \frac{(k-1)!c_k}{(k-m-1)!} t^{k-m-1},$$

where $m = 1, 2, \dots, n-1$. The boundary conditions imply that $c_k = 0, k = 2, 3, \dots, n$, and

$$c_1 = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\xi_1)} h(s) ds - \int_0^{\eta_1} \frac{\xi_1(\eta_1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\xi_1)} h(s) ds.$$

Substitute c_1 into (3.3) to obtain

$$\begin{aligned} u(t) &= \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\xi_1)} h(s) ds \\ &\quad - \int_0^{\eta_1} \frac{\xi_1(\eta_1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\xi_1)} h(s) ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds. \end{aligned}$$

Thus,

$$u(t) = \int_0^1 \mathcal{H}_\alpha(t, s) h(s) ds$$

where $\mathcal{H}_\alpha(t, s)$ is given by (3.2). \square

Similarly, the general solution of ${}^c\mathcal{D}_{0+}^\beta v(t) + h(t) = 0, t \in (0, 1), 1 < \alpha < 2, v'(0) = v''(0) = v'''(0) = \dots = v^{(n-1)}(0) = 0, v(1) = \xi_2 u(\eta_2)$, is given by $v(t) = \int_0^1 \mathcal{H}_\beta(t, s) h(s) ds$, where

$$\mathcal{H}_\beta(t, s) = \begin{cases} \frac{(1-s)^{\beta-1} - (1-\xi_2)(t-s)^{\beta-1} - \xi_2(\eta_2-s)^{\beta-1}}{(1-\xi_2)\Gamma(\beta)}, & s \leq t, \eta_2 \geq s, \\ \frac{(1-s)^{\beta-1} - (1-\xi_2)(t-s)^{\beta-1}}{(1-\xi_2)\Gamma(\beta)}, & \eta_2 \leq s \leq t \leq 1, \\ \frac{(1-s)^{\beta-1} - \xi_2(\eta_2-s)^{\beta-1}}{(1-\xi_2)\Gamma(\beta)}, & 0 \leq t \leq s \leq \eta_2, \\ \frac{(1-s)^{\beta-1}}{(1-\xi_2)\Gamma(\beta)}, & t \leq s, s \geq \eta_2. \end{cases}$$

Lemma 3.2. *The Green's function $\mathcal{H}_\alpha(t, s)$ defined by (3.2) satisfies the following properties:*

(P1) $\mathcal{H}_\alpha(t, s) > 0$ for all $t, s \in (0, 1)$;

(P2) For each $s \in [0, 1]$, $\mathcal{H}_\alpha(t, s)$ is nonincreasing in t ;

(P3) For $\ell \in (0, 1)$, $\Phi_\alpha(s) \geq \mathcal{H}_\alpha(t, s) \geq \gamma_1 \min_{\ell \leq t \leq 1} \mathcal{H}_\alpha(t, s) \geq \gamma_1 \Phi_\alpha(s)$ where

$$\gamma_1 = \xi_1(1 - \eta_1^{\alpha-1}), \quad \Phi_\alpha(s) = \frac{(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)}.$$

Proof. (P1): For $0 \leq t \leq 1$, $\eta_1 \geq s$,

$$\mathcal{H}_\alpha(t, s) = \frac{(1-s)^{\alpha-1} - (1-\xi_1)(t-s)^{\alpha-1} - \xi_1(\eta_1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)}.$$

For $t < \eta_1$,

$$\mathcal{H}_\alpha(t, s) > \frac{(1-s)^{\alpha-1} - (\eta_1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} > 0$$

and for $t \geq \eta_1$,

$$\mathcal{H}_\alpha(t, s) \geq \frac{(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} > 0.$$

For $\eta_1 \leq s \leq t \leq 1$,

$$\mathcal{H}_\alpha(t, s) = \frac{(1-s)^{\alpha-1} - (1-\xi_1)(t-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} \geq \frac{(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} > 0.$$

Thus, for each case, $\eta_1 \leq s \leq t \leq 1$ and $0 \leq t \leq s$, $s \geq \eta_1$, $\mathcal{H}_\alpha(t, s) > 0$.

(P2) : From (3.2), if $t > s$,

$$\frac{\partial}{\partial t} \mathcal{H}_\alpha(t, s) = -\frac{(\alpha-1)}{\Gamma(\alpha)}(t-s)^{\alpha-2} = -\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \leq 0.$$

If $t \leq s$, then

$$\frac{\partial}{\partial t} \mathcal{H}_\alpha(t, s) \equiv 0.$$

Thus, $\mathcal{H}_\alpha(t, s)$ is nonincreasing in t .

(P3): Clearly, $\mathcal{H}_\alpha(t, s) \leq \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\xi_1)} = \Phi_\alpha(s)$.

Case (i): $s \leq \eta_1$: For $t \geq s$, apply (P2) so that

$$\mathcal{H}_\alpha(t, s) \geq \mathcal{H}_\alpha(1, s) = \frac{\xi_1(1-\eta_1^{\alpha-1})(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} = \gamma_1 \Phi_\alpha(s).$$

For $t \leq s$, we have

$$\mathcal{H}_\alpha(t, s) \geq \frac{(1-s)^{\alpha-1} - \xi_1 \eta_1^{\alpha-1} (1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} \geq \xi_1(1-\eta_1^{\alpha-1})\Phi_\alpha(s) = \gamma_1 \Phi_\alpha(s).$$

Case (ii): $s > \eta_1$: For $t \geq s$,

$$\mathcal{H}_\alpha(t, s) \geq \mathcal{H}_\alpha(1, s) = \frac{\xi_1(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} \geq \gamma_1 \Phi_\alpha(s).$$

For $t < s$, we have $\mathcal{H}_\alpha(t, s) = \frac{(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} = \Phi_\alpha(s) \geq \gamma_1 \Phi_\alpha(s)$.

Hence, $\min_{\ell \leq t \leq 1} \mathcal{H}_\alpha(t, s) \geq \gamma_1 \Phi_\alpha(s)$ where $\ell \in (0, 1)$, $\gamma_1 = \xi_1(1 - \eta_1^{\alpha-1})$ and $\Phi_\alpha(s) =$

$$\frac{(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)}.$$

□

Similarly, we can prove that $\mathcal{H}_\beta(t, s) > 0$ for all $t, s \in (0, 1)$ and for $\ell \in (0, 1)$, $\min_{\ell \leq t \leq 1} \mathcal{H}_\beta(t, s) \geq \gamma_2 \Phi_\beta(s)$ where $\gamma_2 = \xi_2(1 - \eta_2^{\beta-1})$ and $\Phi_\beta(s) = \frac{(1-s)^{\beta-1}}{(1-\xi_2)\Gamma(\beta)}$. Furthermore $\mathcal{H}_\beta(t, s)$ is decreasing in t .

Now we write the system of boundary value problem (1.1) as an equivalent system of integral equations

$$(3.5) \quad \begin{cases} u(t) = \lambda \int_0^1 \mathcal{H}_\alpha(t, s) a(s) f(u(s), v(s)) ds \\ v(t) = \mu \int_0^1 \mathcal{H}_\beta(t, s) b(s) g(u(s), v(s)) ds, \end{cases}$$

and consider the Banach space $C[0, 1]$ equipped with the norm

$$\|(u, v)\| = \|u\| + \|v\| = \max_{0 \leq t \leq 1} |u(t)| + \max_{0 \leq t \leq 1} |v(t)|.$$

Define operators $\mathcal{A}_\lambda, \mathcal{A}_\mu : C[0, 1] \times C[0, 1] \rightarrow C[0, 1] \times C[0, 1]$ by

$$\mathcal{A}_\lambda(u, v)(t) = \lambda \int_0^1 \mathcal{H}_\alpha(t, s) a(s) f(u(s), v(s)) ds,$$

$$\mathcal{A}_\mu(u, v)(t) = \mu \int_0^1 \mathcal{H}_\beta(t, s) b(s) g(u(s), v(s)) ds,$$

respectively, and define an operator $\mathcal{A} : C[0, 1] \times C[0, 1] \rightarrow C[0, 1] \times C[0, 1]$ by

$$\mathcal{A}(u, v) = (\mathcal{A}_\lambda(u, v), \mathcal{A}_\mu(u, v)).$$

By a solution of (1.1), we mean a fixed point of the operator \mathcal{A} .

Let $\gamma = \min\{\gamma_1, \gamma_2\}$, fix $\ell \in (0, 1)$, and define a cone in $C[0, 1] \times C[0, 1] = (C[0, 1])^2$ by

$$P = \left\{ (u, v) \in (C[0, 1])^2 : u(t) \geq 0, v(t) \geq 0, \min_{\ell \leq t \leq 1} (u(t) + v(t)) \geq \gamma \|(u, v)\| \right\}.$$

For the remainder of the paper, we shall employ the following assumptions when appropriate:

(H1) $f, g \in C([0, 1] \times \mathbb{R}^+, \mathbb{R})$, and the limits

$$\begin{aligned} f_0 &= \lim_{u+v \rightarrow 0} \frac{f(u, v)}{u+v}, & f_\infty &= \lim_{u+v \rightarrow \infty} \frac{f(u, v)}{u+v}, \\ g_0 &= \lim_{u+v \rightarrow 0} \frac{g(u, v)}{u+v}, & g_\infty &= \lim_{u+v \rightarrow \infty} \frac{g(u, v)}{u+v} \end{aligned}$$

exist and $f_0, f_\infty, g_0, g_\infty \in [0, \infty)$;

(H2) $a, b \in C([0, 1], (0, \infty))$ such that $\mathcal{I}_{0+}^\alpha a(1)$ and $\mathcal{I}_{0+}^\beta b(1)$ exist and are finite;

(H3)

$$\begin{aligned} (1 - \xi_1) \mathcal{I}_{0+}^\alpha a(1) f_0 &< \gamma_1 \gamma \mathcal{I}_{\ell+}^\alpha a(1) f_\infty, \\ (1 - \xi_2) \mathcal{I}_{0+}^\beta b(1) g_0 &< (1 - \gamma) \gamma_2 \mathcal{I}_{\ell+}^\beta b(1) g_\infty; \end{aligned}$$

(H4)

$$(1 - \xi_1)\mathcal{I}_{0+}^\alpha a(1)f_\infty < \gamma\gamma_1\mathcal{I}_{l+}^\alpha a(1)f_0,$$

$$(1 - \xi_2)\mathcal{I}_{0+}^\beta b(1)g_\infty < (1 - \gamma)\gamma_2\mathcal{I}_{0+}^\beta b(1)g_0;$$

(H5) there exist constants r, Λ, \mathcal{N} satisfying

$$(1 - \xi_1)\mathcal{N}\mathcal{I}_{0+}^\alpha a(1) < \gamma^2\Lambda\mathcal{I}_{l+}^\alpha a(1), \quad (1 - \xi_2)\mathcal{N}\mathcal{I}_{0+}^\beta b(1) < \gamma(1 - \gamma)\Lambda\mathcal{I}_{l+}^\beta b(1),$$

and such that

- (i) $f_0 = 0, g_0 = 0, f_\infty = 0, g_\infty = 0;$
- (ii) $f(u, v) \geq \Lambda r, g(u, v) \geq \Lambda r,$ for $\|(u, v)\| \in [\gamma r, r];$

(H6) there exist constants r, Λ, \mathcal{N} satisfying

$$(1 - \xi_1)\Lambda\mathcal{I}_{0+}^\alpha a(1) < \gamma^2\mathcal{N}\mathcal{I}_{l+}^\alpha a(1) \quad (1 - \xi_2)\Lambda\mathcal{I}_{0+}^\beta b(1) < \gamma(1 - \gamma)\mathcal{N}\mathcal{I}_{l+}^\beta b(1),$$

and such that

- (i) $f_0 = \infty, g_0 = \infty, f_\infty = \infty, g_\infty = \infty;$
- (ii) $f(u, v) \leq \Lambda r, g(u, v) \leq \Lambda r,$ for $\|(u, v)\| \in [0, r].$

Lemma 3.3. *Assume that (H1), (H2) hold. Then $\mathcal{A} : P \rightarrow P$ is completely continuous.*

Proof. First, we prove that $\mathcal{A}P \subset P$. For any $(t, s) \in [l, 1] \times [0, 1]$, by Lemma 3.2, we have

$$\begin{aligned} & \min_{l \leq t \leq 1} (\mathcal{A}_\lambda(u, v)(t) + \mathcal{A}_\mu(u, v)(t)) \\ &= \min_{l \leq t \leq 1} \left(\lambda \int_0^1 \mathcal{H}_\alpha(t, s)a(s)f(u(s), v(s))ds, + \mu \int_0^1 \mathcal{H}_\beta(t, s)b(s)g(u(s), v(s))ds \right) \\ &\geq \lambda\gamma_1 \int_0^1 \Phi_\alpha(s)a(s)f(u(s), v(s))ds + \mu\gamma_2 \int_0^1 \Phi_\beta(s)b(s)g(u(s), v(s))ds \\ &= \max_{0 \leq t \leq 1} (\gamma_1\mathcal{A}_\lambda(u, v)(t) + \gamma_2\mathcal{A}_\mu(u, v)(t)) = \gamma_1\|\mathcal{A}_\lambda(u, v)\| + \gamma_2\|\mathcal{A}_\mu(u, v)\|. \end{aligned}$$

Hence $\min_{l \leq t \leq 1} (\mathcal{A}_\lambda(u, v)(t) + \mathcal{A}_\mu(u, v)(t)) \geq \gamma\|\mathcal{A}(u, v)\|$, and $\mathcal{A}P \subset P$.

Next we prove that \mathcal{A} maps bounded sets into uniformly bounded sets. For fixed $M > 0$, consider a bounded subset \mathcal{M} of P defined by $\mathcal{M} = \{(u, v) \in P : \|(u, v)\| \leq M\}$. Define

$$L_1 = \max \left\{ f(u(t), v(t)) : \|(u, v)\| \leq M \right\}, \quad L_2 = \max \left\{ g(u(t), v(t)) : \|(u, v)\| \leq M \right\}.$$

Then for $(u, v) \in \mathcal{M}$, employ (H2) and Lemma 3.2 to obtain

$$\begin{aligned} |\mathcal{A}_\lambda(u, v)| &= \left| \lambda \int_0^1 \mathcal{H}_\alpha(t, s)a(s)f(u(s), v(s))ds \right| \\ &\leq \lambda L_1 \int_0^1 \frac{(1-s)^{\alpha-1}a(s)}{\Gamma(\alpha)(1-\xi_1)} ds = \frac{L_1\lambda}{1-\xi_1}\mathcal{I}_{0+}^\alpha a(1) < +\infty. \end{aligned}$$

Similarly, $|\mathcal{A}_\mu(u, v)| < +\infty$. Therefore $|\mathcal{A}_\lambda(u, v)| + |\mathcal{A}_\mu(u, v)| < +\infty$, which implies that $\mathcal{A}(\mathcal{M})$ is uniformly bounded.

Finally, we show that $\mathcal{A}(\mathcal{M})$ is equicontinuous. By (P2) of Lemma 3.2, we have following estimate

$$\begin{aligned} \left| \frac{d}{dt} (\mathcal{A}_\lambda(u, v)(t)) \right| &\leq \lambda \int_0^1 \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} a(s) |f(u(s), v(s))| ds \\ &\leq L_1 \lambda \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} a(s) ds = L_1 \lambda \mathcal{I}_{0+}^{\alpha-1} a(1) < +\infty. \end{aligned}$$

Define $\delta = \left(L_1 \lambda \mathcal{I}_{0+}^{\alpha-1} a(1) + L_2 \mu \mathcal{I}_{0+}^{\beta-1} b(1) \right)^{-1}$, and choose $t_1, t_2 \in [0, 1]$ such that $t_1 < t_2$ and $t_2 - t_1 < \delta$. Then for all $\varepsilon > 0$ and $(u, v) \in \mathcal{M}$, we obtain

$$\begin{aligned} |\mathcal{A}_\lambda(u, v)(t_2) - \mathcal{A}_\lambda(u, v)(t_1)| &= \left| \int_{t_1}^{t_2} \frac{d}{ds} (\mathcal{A}_\lambda(u, v)(s)) ds \right| \\ &\leq L_1 \lambda \mathcal{I}_{0+}^{\alpha-1} a(1) (t_2 - t_1). \end{aligned}$$

Similarly,

$$|\mathcal{A}_\mu(u, v)(t_2) - \mathcal{A}_\mu(u, v)(t_1)| \leq L_2 \mu \mathcal{I}_{0+}^{\beta-1} b(1) (t_2 - t_1).$$

Hence, it follows that

$$\|\mathcal{A}(u, v)(t_2) - \mathcal{A}(u, v)(t_1)\| \leq \left(L_1 \lambda \mathcal{I}_{0+}^{\alpha-1} a(1) + L_2 \mu \mathcal{I}_{0+}^{\beta-1} b(1) \right) (t_2 - t_1) < \varepsilon.$$

Therefore, by means of Arzela-Ascoli Theorem, $\mathcal{A} : P \rightarrow P$ is completely continuous. \square

3.1. Existence of at least one positive solution.

Theorem 3.4. (i) *Assume that (H1)-(H3) hold. Then for every*

$$\lambda \in \left(\frac{1 - \xi_1}{\gamma_1 \mathcal{I}_{l_+}^\alpha a(1) f_\infty}, \frac{\gamma}{\mathcal{I}_{0+}^\alpha a(1) f_0} \right), \quad \mu \in \left(0, \frac{1 - \gamma}{\mathcal{I}_{0+}^\beta b(1) g_0} \right),$$

or

$$\lambda \in \left(0, \frac{\gamma}{\mathcal{I}_{0+}^\alpha a(1) f_0} \right), \quad \mu \in \left(\frac{1 - \xi_2}{\gamma_2 \mathcal{I}_{l_+}^\beta b(1) g_\infty}, \frac{1 - \gamma}{\mathcal{I}_{0+}^\beta b(1) g_0} \right),$$

the boundary value problem (1.1) has at least one positive solution.

- (ii) *Assume that (H2), (H3) hold and assume $f_0 = 0$, $g_0 = 0$, $f_\infty = \infty$, $g_\infty = \infty$. Then for each $\lambda, \mu \in (0, \infty)$ the boundary value problem (1.1) has at least one positive solution.*

Proof. Assume that (H2), (H3) hold and assume either $f_\infty < \infty$ and $g_\infty < \infty$ or $f_\infty = \infty$ and $g_\infty = \infty$. Choose $\varepsilon > 0$ such that

$$\frac{1 - \xi_1}{\gamma_1 \mathcal{I}_{l_+}^\alpha a(1) (f_\infty - \varepsilon)} \leq \lambda \leq \frac{\gamma}{\mathcal{I}_{0+}^\alpha a(1) (f_0 + \varepsilon)}, \quad 0 < \mu \leq \frac{1 - \gamma}{\mathcal{I}_{0+}^\beta b(1) (g_0 + \varepsilon)}.$$

By the definition of f_0, g_0 , there exists $r > 0$ such that

$$f(u, v) \leq (f_0 + \varepsilon)(u + v), \quad g(u, v) \leq (g_0 + \varepsilon)(u + v), \quad \text{for } u + v \in [0, r].$$

Define $\Omega_r = \{(u, v) \in P : \|(u, v)\| \leq r\}$. For any $(u, v) \in P \cap \partial\Omega_r$, by Lemma 3.2, we have

$$\begin{aligned} \mathcal{A}_\lambda(u, v)(t) &= \lambda \int_0^1 H_\alpha(t, s)a(s)f(u(s), v(s))ds \\ &\leq \lambda \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)(f_0 + \varepsilon)(u+v)ds \\ &\leq \lambda(f_0 + \varepsilon)\mathcal{I}_{0+}^\alpha a(1)\|(u, v)\| \leq \gamma\|(u, v)\|, \end{aligned}$$

$$\begin{aligned} \mathcal{A}_\mu(u, v)(t) &= \mu \int_0^1 H_\beta(t, s)b(s)f(u(s), v(s))ds \\ &\leq \mu \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}b(s)(g_0 + \varepsilon)(u+v)ds \\ &\leq \mu(g_0 + \varepsilon)\mathcal{I}_{0+}^\beta b(1)\|(u, v)\| \leq (1-\gamma)\|(u, v)\|. \end{aligned}$$

Hence,

$$\begin{aligned} (3.6) \quad \|\mathcal{A}(u, v)\| &\leq \gamma\|(u, v)\| + (1-\gamma)\|(u, v)\| \\ &= \|(u, v)\|, \text{ for all } (u, v) \in P \cup \partial\Omega_r. \end{aligned}$$

Now we consider two different cases:

Case 1. Assume $f_\infty < \infty$ and $g_\infty < \infty$. Choose $\varepsilon_1 > 0$ such that $0 < \frac{1-\xi_1}{\gamma_1\mathcal{I}_{l+}^\alpha a(1)(f_\infty - \varepsilon_1)} \leq \lambda$. By the definition of f_∞, g_∞ there exists a constant $r^* > r$, such that

$$f(u, v) \geq (f_\infty - \varepsilon_1)(u + v), \quad g(u, v) \geq (g_\infty - \varepsilon_1)(u + v), \text{ for } u + v \geq \gamma r^*.$$

Define $\Omega_{r^*} = \{(u, v) \in P : \|(u, v)\| \leq r^*\}$. Then for $(u, v) \in P \cap \partial\Omega_{r^*}$, we have

$$u(t) + v(t) \geq \min_{l \leq t \leq 0} (u(t) + v(t)) \geq \gamma\|(u, v)\| = \gamma r^*.$$

Therefore, by Lemma 3.2, we have

$$\begin{aligned} \mathcal{A}_\lambda(u, v)(t) &= \lambda \int_0^1 H_\alpha(t, s)a(s)f(u(s), v(s))ds \\ &\geq \lambda\gamma_1 \int_l^1 \frac{(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)}a(s)(f_\infty - \varepsilon)(u+v)ds \\ &\geq \frac{\lambda\gamma_1}{1-\xi_1}\mathcal{I}_{l+}^\alpha a(1)(f_\infty - \varepsilon)\|(u, v)\| \geq \|(u, v)\|. \end{aligned}$$

Therefore, we have

$$(3.7) \quad \|\mathcal{A}(u, v)\| \geq \|\mathcal{A}_\lambda(u, v)\| \geq \|(u, v)\|, \text{ for all } (u, v) \in P \cap \partial\Omega_{r^*}.$$

Hence, by Theorem 2.7 and equations (3.6), (3.7), the operator \mathcal{A} has a fixed point $(u, v) \in P \cap \bar{\Omega}_{r^*} \setminus \Omega_r$ such that $r \leq \|(u, v)\| \leq r^*$.

Case 2. Assume $f_\infty = \infty$ and $g_\infty = \infty$. Let $\lambda > 0$, Choose a constant $\mathcal{N} > 0$ such that $\mathcal{N} \geq \min \left\{ \frac{1-\xi_1}{\lambda\gamma_1\mathcal{I}_{l^+}^\alpha a(1)}, \frac{1-\xi_2}{\mu\gamma_2\mathcal{I}_{l^+}^\beta b(1)} \right\}$. Since $f_\infty = \infty$, $g_\infty = \infty$, there exists $r^* > r$, such that

$$f(u, v) \geq \mathcal{N}(u + v), \text{ for } u + v \geq \gamma r^*.$$

For $(u, v) \in P \cap \partial\Omega_{r^*}$, we have $u(t) + v(t) \geq \min_{l \leq t \leq 0} (u(t) + v(t)) \geq \gamma \|(u, v)\| = \gamma r^*$. Hence,

$$f(u, v) \geq \mathcal{N}(u + v), \text{ for any } (u, v) \in P \cap \partial\Omega_{r^*}.$$

By Lemma 3.2, we have

$$\begin{aligned} \mathcal{A}_\lambda(u, v)(t) &= \lambda \int_0^1 H_\alpha(t, s) a(s) f(u(s), v(s)) ds \\ &\geq \lambda \gamma_1 \mathcal{N} \int_l^1 \frac{(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} a(s) (u+v) ds \\ &\geq \frac{\lambda \gamma_1 \mathcal{N}}{1-\xi_1} \mathcal{I}_{l^+}^\alpha a(1) \|(u, v)\| \geq \|(u, v)\|. \end{aligned}$$

Thus, we conclude that

$$(3.8) \quad \|\mathcal{A}(u, v)\| \geq \|\mathcal{A}_\lambda(u, v)\| \geq \|(u, v)\|, \text{ for all } (u, v) \in P \cap \partial\Omega_{r^*}.$$

Hence, Theorem 2.7 and equations (3.6), (3.8), the operator \mathcal{A} has a fixed point $(u, v) \in P \cap \bar{\Omega}_{r^*} \setminus \Omega_r$ such that $r \leq \|(u, v)\| \leq r^*$. \square

Theorem 3.5. (i) Assume that (H1), (H2) and (H4) hold. Then for every

$$\lambda \in \left(\frac{1-\xi_1}{\gamma_1 \mathcal{I}_{l^+}^\alpha a(1) f_0}, \frac{\gamma}{\mathcal{I}_{0^+}^\alpha a(1) f_\infty} \right), \quad \mu \in \left(0, \frac{1-\gamma}{\mathcal{I}_{0^+}^\beta b(1) g_\infty} \right),$$

or

$$\lambda \in \left(\frac{\gamma}{\mathcal{I}_{0^+}^\alpha a(1) f_\infty} \right), \quad \mu \in \left(\frac{1-\xi_2}{\gamma_2 \mathcal{I}_{0^+}^\beta b(1) g_0}, \frac{1-\gamma}{\mathcal{I}_{0^+}^\beta b(1) g_\infty} \right),$$

the boundary value problem (1.1) has at least one positive solution.

(ii) Assume (H2), (H4) hold and assume $f_0 = \infty$, $g_0 = \infty$, $f_\infty = 0$, $g_\infty = 0$. Then for each $\lambda, \mu \in (0, \infty)$ the boundary value problem (1.1) has at least one positive solution.

Proof. The proof is similar to the Theorem 3.4, so we omit it. \square

3.2. Multiplicity Results.

Theorem 3.6. Assume that (H1), (H2) and (H5) hold. Then for any

$$\lambda \in \left[\frac{1-\xi_1}{\gamma \Lambda \mathcal{I}_{l^+}^\alpha a(1)}, \frac{\gamma}{\mathcal{N} \mathcal{I}_{0^+}^\alpha a(1)} \right], \quad \mu \in \left(0, \frac{1-\gamma}{\mathcal{N} \mathcal{I}_{0^+}^\beta b(1)} \right],$$

or

$$\lambda \in \left(0, \frac{\gamma}{\mathcal{N} \mathcal{I}_{0^+}^\alpha a(1)} \right], \quad \mu \in \left[\frac{1-\xi_2}{\gamma \Lambda \mathcal{I}_{l^+}^\beta b(1)}, \frac{1-\gamma}{\mathcal{N} \mathcal{I}_{0^+}^\beta b(1)} \right],$$

the boundary value problem (1.1) has at least two positive solutions $(u_1, v_1), (u_2, v_2)$ such that $0 < \|(u_1, v_1)\| < r < \|(u_2, v_2)\|$.

Proof. Since $f_0 = 0, g_0 = 0$, there exists $r_1 \in (0, r)$ such that

$$f(u, v) \leq \mathcal{N}(u + v), \quad g(u, v) \leq \mathcal{N}(u + v), \quad \text{for } u + v \in (0, r_1).$$

Define $\Omega_{r_1} = \{(u, v) \in P : \|(u, v)\| < r_1\}$. For any $(u, v) \in P \cap \partial\Omega_{r_1}$, by Lemma 3.2 we have

$$\begin{aligned} \mathcal{A}_\lambda(u, v)(t) &= \lambda \int_0^1 H_\alpha(t, s)a(s)f(u(s), v(s))ds \\ &\leq \lambda \mathcal{N} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)(u+v)ds \\ &\leq \lambda \mathcal{N} \mathcal{I}_{0+}^\alpha a(1) \|(u, v)\| \leq \theta \|(u, v)\|. \end{aligned}$$

$$\begin{aligned} \mathcal{A}_\mu(u, v)(t) &= \lambda \int_0^1 H_\beta(t, s)b(s)g(u(s), v(s))ds \\ &\leq \mu \mathcal{N} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} b(s)(u+v)ds \\ &\leq \mu \mathcal{N} \mathcal{I}_{0+}^\beta b(1) \|(u, v)\| \leq (1-\theta) \|(u, v)\|, \end{aligned}$$

which implies that

$$(3.9) \quad \|\mathcal{A}(u, v)\| \leq \|(u, v)\|, \quad \text{for all } (u, v) \in P \cap \partial\Omega_{r_1}.$$

Since $f_\infty = 0, g_\infty = 0$, there exists $r_2 > r$ such that for some positive constant \mathcal{N} , we have

$$(3.10) \quad f(u, v) \leq \mathcal{N}(u + v), \quad g(u, v) \leq \mathcal{N}(u + v), \quad \text{for } u + v \geq r_2.$$

Set $\Omega_{r_2} = \{(u, v) \in P : \|(u, v)\| < r_2\}$. For any $(u, v) \in P \cap \partial\Omega_{r_2}$, by Lemma 3.2 we have

$$(3.11) \quad \|\mathcal{A}(u, v)\| \leq \|(u, v)\|, \quad \text{for all } (u, v) \in P \cap \partial\Omega_{r_2}.$$

Next define $\Omega_r = \{(u, v) \in P : \|(u, v)\| < r\}$. For any $(u, v) \in P \cap \partial\Omega_r$, by Lemma 3.2 we have

$$u(t) + v(t) \geq \|(u, v)\| \geq \gamma r, \quad \text{for all } t \in [l, 1].$$

Therefore we have following estimate

$$\begin{aligned} \mathcal{A}_\lambda(u, v)(t) &= \lambda \int_0^1 H_\alpha(t, s)a(s)f(u(s), v(s))ds \\ &\geq \lambda \gamma \int_l^1 \frac{(1-s)^{\alpha-1}}{(1-\xi_1)\Gamma(\alpha)} a(s)\Lambda r ds \\ &= \lambda \gamma \mathcal{I}_l^\alpha a(1)\Lambda r = r = \|(u, v)\|. \end{aligned}$$

Finally, we have

$$(3.12) \quad \|\mathcal{A}(u, v)\| \geq \|(u, v)\|, \text{ for all } (u, v) \in P \cap \partial\Omega_r.$$

Hence, by (3.9), (3.11), (3.12) and Theorem 2.7, it follows that the operator \mathcal{A} has two fixed points $u_1 \in \bar{\Omega}_r \setminus \Omega_{r_1}$ and $u_2 \in \bar{\Omega}_{r_2} \setminus \Omega_r$. \square

Theorem 3.7. *Assume that (H1), (H2) and (H6) hold. Then for any*

$$\lambda \in \left[\frac{1 - \xi_1}{\gamma \mathcal{I}_{l+}^\alpha a(1)}, \frac{\gamma}{\Lambda \mathcal{I}_{0+}^\alpha a(1)} \right], \quad \mu \in \left(\frac{1 - \gamma}{\Lambda \mathcal{I}_{0+}^\beta b(1)} \right],$$

or

$$\lambda \in \left(0, \frac{\gamma}{\Lambda \mathcal{I}_{0+}^\alpha a(1)} \right], \quad \mu \in \left[\frac{1 - \xi_2}{\gamma \mathcal{I}_{l+}^\beta b(1)}, \frac{1 - \gamma}{\Lambda \mathcal{I}_{0+}^\beta b(1)} \right],$$

the boundary value problem (1.1) has at least two positive solutions (u_1, v_1) , (u_2, v_2) such that $0 < \|(u_1, v_1)\| < r < \|(u_2, v_2)\|$.

Proof. The proof is similar to the Theorem 3.6, so we omit it. \square

Example 3.8. Consider the system of fractional differential equations

$$(3.13) \quad \begin{cases} {}^c\mathcal{D}_{0+}^{\frac{5}{2}} u(t) = \lambda \left(\frac{243}{578} + e^{\pi t} \right) \left(\pi - 6 \left(u + v + \frac{\pi}{2} \right)^{-\frac{3}{2}} \right), \\ {}^c\mathcal{D}_{0+}^{\frac{27}{10}} v(t) = \mu (\sin t + 256 e^{\pi t}) \left(\left(1 + \frac{23}{\sqrt{u+v}} \right)^{-\frac{5}{2}} + \frac{\pi}{257} \right), \\ u'(0) = u''(0) = 0, \quad u(1) = \frac{1}{2} u\left(\frac{2}{3}\right), \\ v'(0) = v''(0) = 0, \quad v(1) = \frac{3}{4} u\left(\frac{3}{5}\right), \end{cases}$$

where $\alpha = \frac{5}{2}$, $\beta = \frac{27}{10}$, $a(t) = \frac{243}{578} + e^{\pi t}$, $b(t) = \sin t + 256 e^{\pi t}$, $f(u, v) = \pi - 6 \left(u + v + \frac{\pi}{2} \right)^{-\frac{3}{2}}$, $g(u, v) = \left(1 + \frac{23}{\sqrt{u+v}} \right)^{-\frac{5}{2}} + \frac{\pi}{257}$, $\xi_1 = \frac{1}{2}$, $\xi_2 = \frac{3}{4}$, $\eta_1 = \frac{2}{3}$, $\eta_2 = \frac{3}{5}$. Obviously $f_0 = \pi - 6 \left(\frac{2}{\pi} \right)^{\frac{3}{2}}$, $f_\infty = \pi$, $g_0 = \frac{\pi}{257}$ and $g_\infty = 1 + \frac{\pi}{257}$. Choose $l = \frac{1}{4}$. By direct calculations $\gamma_1 \approx 0.227834$, $\gamma_2 \approx 0.435284$, $\gamma \approx 0.227834$; $\mathcal{I}_{0+}^\alpha a(1) \approx 1.079424$, $\mathcal{I}_{l+}^\alpha a(1) \approx 0.786560$, $\mathcal{I}_{0+}^\beta b(1) \approx 182.201363$ and $\mathcal{I}_{l+}^\beta b(1) \approx 133.391627$.

Since, $(1 - \xi_1) \mathcal{I}_{0+}^\alpha a(1) f_0 \approx 0.050678 < \gamma_1 \gamma \mathcal{I}_{l+}^\alpha a(1) f_\infty \approx 0.128268$ and $(1 - \xi_2) \mathcal{I}_{0+}^\beta b(1) g_0 \approx 0.556812 < (1 - \gamma) \gamma_2 \mathcal{I}_{l+}^\beta b(1) g_\infty \approx 45.382788$. Therefore for each $\lambda \in (0.888114, 2.247865)$ and $\mu \in (0, 0.346692)$ or $\lambda \in (0, 2.247865)$ and $\mu \in (0.004254, 0.346692)$, by Theorem 3.4, the system of fractional differential equations (3.13) has at least one positive solution.

Example 3.9. Consider the system of fractional differential equations

$$(3.14) \quad \begin{cases} {}^c\mathcal{D}_{0+}^{\frac{14}{5}} u(t) = \frac{19^4 \lambda t^2}{1+t} \left(\frac{12\pi}{u+v+12\pi} - e^{-\pi(u+v)} \right), \\ {}^c\mathcal{D}_{0+}^{\frac{12}{5}} v(t) = \frac{78125 \mu t^2}{1+t^2} \left(\frac{1}{\sqrt{u+v}} - e^{-\pi(u+v)} \right), \\ u'(0) = u''(0) = 0, \quad u(1) = \frac{37}{50} u\left(\frac{11}{25}\right), \\ v'(0) = v''(0) = 0, \quad v(1) = \frac{11}{20} u\left(\frac{19}{50}\right), \end{cases}$$

where $\alpha = \frac{14}{5}$, $\beta = \frac{12}{5}$, $a(t) = \frac{t^2}{1+t}$, $b(t) = \frac{t^2}{1+t^2}$, $f(u, v) = 19^4 \left(\frac{12\pi}{u+v+12\pi} - e^{-\pi(u+v)} \right)$, $g(u, v) = 78125 \left(\frac{12\pi}{u+v+12\pi} - e^{-\pi(u+v)} \right)$, $\xi_1 = \frac{37}{50}$, $\xi_2 = \frac{11}{20}$, $\eta_1 = \frac{11}{25}$, $\eta_2 = \frac{19}{50}$. Obviously $f_0 = 0$, $f_\infty = 0$, $g_0 = 0$ and $g_\infty = 0$. Choose constants $l = \frac{1}{2}$, $r = 5$, $\mathcal{N} = 12$, $\Lambda = 2449$, then by computations

$$\begin{aligned} \mathcal{I}_{0+}^\alpha a(1) &\approx 0.015652, & \mathcal{I}_{l+}^\alpha a(1) &\approx 0.007547, \\ \mathcal{I}_{0+}^\beta b(1) &\approx 0.034126, & \mathcal{I}_{l+}^\beta b(1) &\approx 0.018676, \\ (1 - \xi_1)\mathcal{N}\mathcal{I}_{0+}^\alpha a(1) &\approx 0.326817 < \gamma^2\Lambda\mathcal{I}_{l+}^\alpha a(1) &\approx 0.959355 \end{aligned}$$

and

$$(1 - \xi_2)\mathcal{N}\mathcal{I}_{0+}^\beta b(1) \approx 0.184282 < \gamma(1 - \gamma)\Lambda\mathcal{I}_{l+}^\beta b(1) \approx 8.046569.$$

Also $f(u, v) > 12245$, for $\|(u, v)\| \in (1.1397, 5)$ and $g(u, v) > 12245$, for $\|(u, v)\| \in (1.1397, 5)$. Therefore, for any $\lambda \in [0.061746, 1.213008]$ and $\mu \in (0, 1.885560]$ or $\lambda \in (0, 1.213008]$ and $\mu \in [0.043183, 1.885560]$, by Theorem 3.6 the boundary value problem (3.14) has at least two positive solutions.

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