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A LEGGETT-WILLIAMS TYPE THEOREM APPLIED TO A FOURTH ORDER PROBLEM

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In Memory of Professor Drumi D. Bainov.

ABSTRACT. In this article we apply an extension of a Leggett-Williams type fixed point theorem to a two-point boundary value problem for a fourth order ordinary differential equation. The fixed point theorem employs concave and convex functionals defined on a cone in a Banach space. Inequalities that extend the notion of concavity to fourth order differential inequalities are derived and employed to provide the necessary estimates. Symmetry is employed in the construction of the appropriate Banach space.

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1. INTRODUCTION

Richard Avery and co-authors [3, 4, 5, 6] have extended the Leggett-Williams fixed point theorem [9] in various ways; a recent extension [4] employs topological methods rather than index theory and as a result the recent extension does not require the functional boundaries to be invariant with respect to a functional wedge. It is shown [5] that this extension applies in a natural way to second order right focal boundary value problems. The concept of concavity provides estimates that are useful in multiple technical arguments with respect to the concave and convex functionals; the increasing nature of functions give rise to natural constructions of convex or concave functions.

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Recently, Al Twaty and Eloe [2] applied these types of theorems to a two point conjugate type boundary value problem for a second order ordinary differential equation. Concavity was employed as in [5] and symmetry of functions was employed to construct appropriate concave or convex functionals.

In this article we shall apply the fixed point theorem to a two-point conjugate type boundary value problem for a fourth order ordinary differential equation. Symmetry will be employed as in [2]. A new inequality representing concavity will be obtained for functions satisfying a fourth order differential inequality (and more importantly, a new inequality will be obtained for an associated Green's function). Hence, we shall exhibit sufficient conditions for the existence of solutions for a family of fourth order two-point conjugate boundary value problems.

There has been particular interest in the application of fixed point theory to two point boundary value problems for a fourth order equation as these boundary value problems serve as models for cantilever beam problems. Fixed point applications have been of interest for many years [1, 10] and interest has recently been renewed, [7, 8, 11, 12, 13], for example.

In Section 2 we shall introduce the appropriate definitions and state the fixed point theorem. In Section 3, we shall apply the fixed point theorem to a conjugate boundary value problem for the fourth order problem. To do so, we first obtain Lemma 3.1 which gives a new estimate for an associated Green's function and represents the primary contribution of this work.

2. PRELIMINARIES

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* if it satisfies the following two conditions:

(i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P, -x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y \text{ if and only if } y - x \in P.$$

Definition 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Let α and ψ be non-negative continuous concave functionals on P and δ and β be non-negative continuous convex functionals on P; then, for non-negative real numbers a, b, c and d, we define the following sets:

$$A := A(\alpha, \beta, a, d) = \{x \in P : a \leq \alpha(x) \text{ and } \beta(x) \leq d\}, \tag{2.1}$$

$$B := B(\alpha, \delta, \beta, a, b, d) = \{x \in A : \delta(x) \leq b\}, \tag{2.2}$$

and

$$C := C(\alpha, \psi, \beta, a, c, d) = \{x \in A : c \leq \psi(x)\}. \tag{2.3}$$

We say that A is a *functional wedge with concave functional boundary* defined by the concave functional α and convex functional boundary defined by the convex functional β. We say that an operator $T : A \to P$ is *invariant with respect to the concave functional boundary*, if $a \leq \alpha(Tx)$ for all $x \in A$, and that T is *invariant with respect to the convex functional boundary*, if $\beta(Tx) \leq d$ for all $x \in A$. Note that A is a convex set. The following theorem, proved in [4], is an extension of the original Leggett-Williams fixed point theorem [9].

Theorem 2.4. *Suppose P is a cone in a real Banach space E, α and ψ are non-negative continuous concave functionals on P, δ and β are non-negative continuous convex functionals on P, and for non-negative real numbers a, b, c and d the sets A, B and C are as defined in* (2.1), (2.2) *and* (2.3). *Furthermore, suppose that A is a bounded subset of P, that $T : A \to P$ is completely continuous and that the following conditions hold:*

(A1) $\{x \in A : \ c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset$, $\{x \in P : \ \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset$;
(A2) $\alpha(Tx) \geq a$ *for all* $x \in B$;
(A3) $\alpha(Tx) \geq a$ *for all* $x \in A$ *with* $\delta(Tx) > b$;
(A4) $\beta(Tx) \leq d$ *for all* $x \in C$; *and,*
(A5) $\beta(Tx) \leq d$ *for all* $x \in A$ *with* $\psi(Tx) < c$.

Then T has a fixed point $x^* \in A$.

A fixed point of T will also be called a solution of T.

3. THE APPLICATION

Let $f : \mathbb{R} \to \mathbb{R}^+$ is a continuous map and let $n > 0$ be fixed. We consider the two point conjugate boundary value problem,

$$x^{(iv)}(t) = f(x(t)), \quad t \in [0,1], \tag{3.1}$$

$$x^{(i)}(0) = 0, \quad x^{(i)}(1) = 0, \quad i = 0, 1. \tag{3.2}$$

The Green's function for this problem has the form,

$$G(t,s) = \begin{cases} \frac{t^2}{6}(1-s)^2(3(s-t)+2(1-s)t) & : \ 0 \le t < s \le 1, \\ \frac{s^2}{6}(1-t)^2(3(t-s)+2(1-t)s) & : \ 0 \le s < t \le 1. \end{cases}$$

Note that G satisfies the symmetry property

$$G(1-t, 1-s) = G(t,s), \quad (t,s) \in [0,1] \times [0,1].$$

We shall state and prove a lemma which characterizes the generalized notion of concavity and motivates the construction of the appropriate cone in which to apply Theorem 2.4. This lemma provides the primary contribution of this article.

Lemma 3.1. *If $y, w \in [0,1]$ with $y < w$ and $t \in (0,1)$ then*

$$\frac{G(t,y)}{G(t,w)} \ge \frac{y^2}{w^2}. \tag{3.3}$$

Proof. Suppose $y, w \in [0,1]$ with $y < w$ and $t \in (0,1)$.

Case 1: $y < w < t$.

$$\begin{aligned} \frac{G(t,y)}{G(t,w)} &= \frac{\frac{y^2}{6}(1-t)^2[3(t-y)+2(1-t)y]}{\frac{w^2}{6}(1-t)^2[3(t-w)+2(1-t)w]} \\ &= \frac{y^2(3t-y-2ty)}{w^2(3t-w-2tw)} \ge \frac{y^2}{w^2} \end{aligned}$$

since $3t - y - 2ty \ge 3t - w - 2tw$.

Case 2: $y < t < w$.

For $w \in [t,1]$ the function

$$z(w) = \frac{t^2(1-w)^2(3w-t-2wt)}{w^2} = t^2\left(1 - \frac{1}{w}\right)^2 (3w-t-2wt)$$

is decreasing, thus $z(t) \ge z(w)$, that is,

$$\begin{aligned} (1-t)^2(3t-t-2t^2) &= \frac{t^2(1-t)^2(3t-t-2t^2)}{t^2} \\ &\ge \frac{t^2(1-w)^2(3w-t-2wt)}{w^2} \\ &= t^2\left(1-\frac{1}{w}\right)^2(3w-t-2wt) \end{aligned}$$

and since

$$(1-t)^2[3(t-y)+2(1-t)y] \geq (1-t)^2(3t-t-2t^2)$$

we have

$$\begin{aligned}\left(\frac{y^2}{6}\right)(1-t)^2[3(t-y)+2(1-t)y] &\geq \left(\frac{y^2}{6}\right)\left(\frac{t^2(1-t)^2(3t-t-2t^2)}{t^2}\right)\\ &\geq \left(\frac{y^2}{6}\right)\left(\frac{t^2(1-w)^2[3(t-w)+2(1-t)w]}{w^2}\right)\end{aligned}$$

which implies that

$$\frac{G(t,y)}{G(t,w)} = \frac{\left(\frac{y^2}{6}\right)(1-t)^2[3(t-y)+2(1-t)y]}{\left(\frac{t^2}{6}\right)(1-w)^2[3(w-t)+2(1-w)t]} \geq \frac{y^2}{w^2}.$$

Case 3: $t < y < w$.

For $r \in [t,1]$ the function

$$h(r) = \left(1-\frac{1}{r}\right)^2(3r-t-2rt) = \frac{(1-r)^2[3(r-t)+2(1-r)t]}{r^2}$$

is decreasing, thus, $h(y) \geq h(w)$. Hence,

$$\frac{(1-y)^2[3(y-t)+2(1-y)t]}{y^2} \geq \frac{(1-w)^2[3(w-t)+2(1-w)t]}{w^2}$$

which implies that

$$\frac{G(t,y)}{G(t,w)} = \frac{\left(\frac{t^2}{6}\right)(1-y)^2[3(y-t)+2(1-y)t]}{\left(\frac{t^2}{6}\right)(1-w)^2[3(w-t)+2(1-w)t]} \geq \frac{y^2}{w^2}.$$

□

Remark 3.2. Note that if $x \in C^4[0,1]$, $(x^{(iv)})(t) \geq 0$, $0 < t < 1$, and x satisfies (3.2). Then for any $y, w \in [0,1]$ with $y < w$ we have

$$\frac{x(y)}{x(w)} \geq \frac{y^2}{w^2} \tag{3.4}$$

since

$$\begin{aligned}x(y) &= \int_0^1 G(y,s)\,x^{(iv)}(s)\,ds\\ &\geq \int_0^1 \left(\frac{y^2}{w^2}\right) G(w,s)\,x^{(iv)}(s)\,ds\\ &\geq \left(\frac{y^2}{w^2}\right) x(w).\end{aligned}$$

Let $E = C[0,1]$, equipped with the usual supremum norm denote the Banach space. Define the cone $P \subset E = C[0,1]$ by

$$P := \{x \in E \ : x(1-t) = x(t), 0 < t < 1, \quad x(t) \geq 0, 0 < t < 1,$$
$$x(t) \text{ nondecreasing on } t \in [0,1/2], \text{ and if } 0 \leq y \leq w \leq 1, \text{then} w^2 x(y) \geq y^2 x(w)\}.$$

Define $T : E \to E$ by

$$Tx(t) = \int_0^1 G(t,s) f(x(s)) ds.$$

Lemma 3.3. *Assume $f : \mathbb{R} \to \mathbb{R}^+$ is a continuous map. Then*

$$T : P \to P.$$

Proof. To see that $Tx(1-t) = Tx(t)$,

$$\begin{aligned} Tx(1-t) &= \int_0^1 G(1-t,s) f(x(s)) ds \\ &= -\int_1^0 G(1-t,1-\sigma) f(x(1-\sigma)) d\sigma \\ &= \int_0^1 G(1-t,1-\sigma) f(x(\sigma)) d\sigma = \int_0^1 G(t,\sigma) f(x(\sigma)) d\sigma \\ &= Tx(t). \end{aligned}$$

Clearly, $Tx(t) \geq 0$ on $[0,1]$ since $G(t,s) \geq 0$ on $[0,1] \times [0,n]$ and $f : \mathbb{R} \to \mathbb{R}^+$. Let

$$H(t,s) = \begin{cases} \frac{t}{6}(1-s)^2(6s - 3t - 6st) & : \ 0 \leq t < s \leq 1, \\ \frac{s^2}{6}(1-t)(3 + 6st - 9t) & : \ 0 \leq s < t \leq 1. \end{cases}$$

Note that

$$(Tx)'(t) = \int_0^1 H(t,s) \ f(x(s))$$

Also note that for $t \in [0,1/2], t < s$, $H(t,s) \geq 0$. Then, using that x is symmetric about $1/2$, if $t \in [0,1/2]$, $\int_0^1 H(t,s) \ f(x(s)) ds =$

$$\int_0^t H(t,s) \ f(x(s)) ds + \int_t^{1-t} H(t,s) \ f(x(s)) ds + \int_{1-t}^1 H(t,s) \ f(x(s)) ds$$
$$= \int_0^t \frac{s^2}{2}(\frac{1}{2} - t) \ f(x(s)) ds + \int_t^{1-t} H(t,s) \ f(x(s)) ds \geq 0$$

and we have that Tx is nondecreasing on $[0,1/2]$.

Finally, $(Tx)^{(iv)}(t) = f(x(t)) \geq 0$, $0 < t < 1$ and Tx satisfies (3.2). So by Lemma 3.1, $w^2 Tx(y) \geq y^2 Tx(w)$ and Tx satisfies the concavity condition.

□

For fixed $\nu, \tau, \mu \in [0, \frac{1}{2}]$ and $x \in P$, define the concave functionals α and ψ on P by

$$\alpha(x) := \min_{t\in[\tau,\frac{1}{2}]} x(t) = x(\tau), \quad \psi(x) := \min_{t\in[\mu,\frac{1}{2}]} x(t) = x(\mu),$$

and the convex functionals δ and β on P by

$$\delta(x) := \max_{t\in[0,\nu]} (x(t)) = x(\nu), \quad \beta(x) := \max_{t\in[0,\frac{1}{2}]} x(t) = x(\frac{1}{2}).$$

Theorem 3.4. *Assume $\tau, \nu, \mu \in (0, \frac{1}{2}]$ are fixed with $\tau \le \mu < \nu$, d and L are positive real numbers with $0 < L \le 4\mu^2 d$ and $f : [0, \infty) \to [0, \infty)$ is a continuous function such that*

(a) $f(w) \ge \frac{4!d}{\tau^2(\nu-\tau)(1-2\tau)(1-\tau-\nu)} \equiv M$ *for* $w \in [4\tau^2 d, 4\nu^2 d]$,
(b) $f(w)$ *is decreasing for* $w \in [0, L]$ *with* $f(L) \ge f(w)$ *for* $w \in [L, d]$, *and*
(c) $\frac{1}{8}\int_0^{\mu} s^2 f\left(\frac{Ls^2}{\mu^2}\right) ds \le d - \frac{f(L)}{24}(\frac{1}{8} - \mu^3)$.

Then the operator T has at least one positive solution $x^* \in A(\alpha, \beta, 4\tau^2 d, d)$.

Proof. Let $a = 4\tau^2 d$, $b = 4\nu^2 d = \frac{\nu^2}{\tau^2}a$, and $c = 4\mu^2 d$. Let $x \in A(\alpha, \beta, a, d)$. An immediate corollary of Lemma 3.3 is

$$T : A(\alpha, \beta, a, d) \to P.$$

By the Arzela-Ascoli Theorem it is a standard exercise to show that T is a completely continuous operator using the properties of G and f; by the definition of β, A is a bounded subset of the cone P. Also, if $x \in P$ and $\beta(x) > d$, then by the properties of the cone P (in particular, the concavity of x),

$$\alpha(x) = x(\tau) \ge 4\tau^2 x(\frac{1}{2}) = 4\tau^2 \beta(x) > 4\tau^2 d = a.$$

Thus,

$$\{x \in P \ : \ \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset.$$

For any $r \in \left(\frac{4!(4d)}{(1-\mu)^2}, \frac{4!(4d)}{(1-\nu)^2}\right)$ define x_r by

$$x_r(t) \equiv \int_0^1 rG(t, s)ds = \frac{rt^2(1-t)^2}{4!}.$$

We claim $x_r \in A$.

$$\alpha(x_r) = x_r(\tau) = \frac{r\tau^2(1-\tau)^2}{4!} > \frac{4!(4d)}{(1-\mu)^2}\frac{\tau^2(1-\tau)^2}{4!} \ge 4\tau^2 d = a,$$

$$\beta(x_r) = x_r(\frac{1}{2}) = \frac{r(\frac{1}{2})^2(\frac{1}{2})^2}{4!} < \frac{4!(4d)}{(1-\nu)^2}\frac{(\frac{1}{2})^2(\frac{1}{2})^2}{4!} = \frac{(\frac{1}{2})^2}{(1-\nu)^2}d \le d.$$

Thus, the claim is true. Moreover, x_r has the properties that

$$\psi(x_r) = x_r(\mu) = \frac{r\mu^2(1-\mu)^2}{4!} > \left(\frac{4!(4d)}{(1-\mu)^2}\right)\left(\frac{\mu^2(1-\mu)^2}{4!}\right) = 4\mu^2 d = c$$

and

$$\delta(x_r) = x_r(\nu) = \frac{r\nu^2(1-\nu)^2}{4!} < \left(\frac{4!(4d)}{(1-\nu)^2}\right)\left(\frac{\nu^2(1-\nu)^2}{4!}\right) = 4\nu^2 d = b.$$

In particular,

$$\{x \in A \;:\; c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset.$$

We have shown that condition (A1) of Theorem 2.4 is satisfied.

We now verify that condition (A2) of Theorem 2.4, $\alpha(Tx) \geq a$ for all $x \in B$, is satisfied. Let $x \in B$. Apply condition (a) of Theorem 3.4, and

$$\begin{aligned}
\alpha(Tx) &= \int_0^1 G(\tau,s)\, f(x(s))\, ds \\
&\geq M\left[\int_\tau^\nu G(\tau,s)\, ds + \int_{1-\nu}^{1-\tau} G(\tau,s)\, ds\right] \\
&\geq M\left[\int_\tau^\nu \frac{\tau^3(1-s)^3}{3}\, ds + \int_{1-\nu}^{1-\tau} \frac{\tau^3(1-s)^3}{3}\, ds\right] \\
&= M\left[\int_\tau^\nu \frac{\tau^3(1-s)^3}{3}\, ds + \int_{1-\nu}^{1-\tau} \frac{\tau^3(1-s)^3}{3}\, ds\right] \\
&= M\left(\frac{\tau^3}{12}\right)\left[(1-\tau)^4 - (1-\nu)^4 + \nu^4 - \tau^4\right] \\
&= M\left(\frac{\tau^3}{12}\right)\left[((1-\tau)^2+\tau^2)((1-\tau)^2-\tau^2) + (\nu^2+(1-\nu)^2)(\nu^2-(1-\nu)^2)\right] \\
&= M\left(\frac{\tau^3}{12}\right)\left[((1-\tau)^2+\tau^2)(1-2\tau) + (\nu^2+(1-\nu)^2)(2\nu-1)\right] \\
&\geq M\left(\frac{\tau^3}{12}\right)\left[((1-\tau)^2+\tau^2)(1-2\tau) + (\nu^2+(1-\nu)^2)(2\tau-1)\right] \\
&= M\left(\frac{\tau^3}{12}\right)\left[2(1-2\tau)(\nu-\tau)(1-\tau-\nu)\right] \\
&= M\left(\frac{\tau^3}{6}\right)\left[(1-2\tau)(\nu-\tau)(1-\tau-\nu)\right] = 4\tau d = a.
\end{aligned}$$

We now verify that condition (A3) of Theorem 2.4, $\alpha(Tx) \geq a$, for all $x \in A$ with $\delta(Tx) > b$, is satisfied. Let $x \in A$ with $\delta(Tx) > b$. Apply Lemma 3.1 to obtain

$$\alpha(Tx) = (Tx)(\tau) \geq \left(\frac{\tau}{\nu}\right)^2 (Tx)(\nu) = \left(\frac{\tau}{\nu}\right)^2 \delta(Tx) > \left(\frac{\tau}{\nu}\right)^2 4\nu^2 d = a.$$

Let $x \in C$. Since $c = 4\mu^2 d$ and $0 < L \leq 4\mu^2 d = c$, the concavity of x implies (see the note following Lemma 3.1), for $s \in [0,\mu]$,

$$x(s) \geq \frac{s^2}{\mu^2} x(\mu) \geq \frac{cs^2}{\mu^2} \geq \frac{Ls^2}{\mu^2}.$$

Since x is symmetric about $\frac{1}{2}$ and $G(\frac{1}{2}, s)$ is symmetric about $s = \frac{1}{2}$, it follows that

$$\int_0^1 G(\frac{1}{2}, s)\, f(x(s))\, ds = 2\int_0^{\frac{1}{2}} G(\frac{1}{2}, s)\, f(x(s))\, ds$$

which we will use to abbreviate our calculations. Applying properties (b) and (c) of Theorem 3.4, we have

$$\begin{aligned}\beta(Tx) &= 2\int_0^{\frac{1}{2}} G(\frac{1}{2}, s)\, f(x(s))\, ds \le \frac{1}{8}\int_0^{\frac{1}{2}} s^2\, f(x(s))\, ds \\ &\le \frac{1}{8}\int_0^{\mu} s^2\, f\left(\frac{Ls^2}{\mu^2}\right)\, ds + \frac{f(L)}{8}\int_{\mu}^{\frac{1}{2}} s^2\, ds \\ &\le d - \frac{f(L)}{24}(\frac{1}{8} - \mu^3) + \frac{f(L)}{24}(\frac{1}{8} - \mu^3) = d.\end{aligned}$$

We close the proof by verifying that condition (A5), $\beta(Tx) \le d$, for all $x \in A$ with $\psi(Tx) < c$ is satisfied. Let $x \in A$ with $\psi(Tx) < c$. Apply Lemma 3.1 to obtain

$$\beta(Tx) = (Tx)(\tfrac{1}{2}) \le \left(\tfrac{1}{4\mu^2}\right) Tx(\mu) = \left(\tfrac{1}{4\mu^2}\right)\psi(Tx) \le \left(\tfrac{c}{4\mu^2}\right) = d.$$

Therefore, the hypotheses of Theorem 2.4 have been satisfied; thus the operator T has at least one positive solution $x^* \in A(\alpha, \beta, a, d)$. □

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