

1998

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Doddaballapur, Vidya; Eloë, Paul W.; and Zhang, Yongzhi, "Quadratic convergence of approximate solutions of two-point boundary value problems with impulse" (1998). *Mathematics Faculty Publications*. 95.  
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# Quadratic Convergence of Approximate Solutions of Two-Point Boundary Value Problems with Impulse \*

Vidya Doddaballapur, Paul W. Eloe, & Yongzhi Zhang

## Abstract

The method of quasilinearization, coupled with the method of upper and lower solutions, is applied to a boundary value problem for an ordinary differential equation with impulse that has a unique solution. The method generates sequences of approximate solutions which converge monotonically and quadratically to the unique solution. In this work, we allow nonlinear terms with respect to velocity; in particular, Nagumo conditions are employed.

## 1 Introduction

Let  $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$  be given. In this paper, we shall apply the method of quasilinearization to the two-point conjugate boundary value problem (BVP) with impulse,

$$x''(t) = f(t, x(t), x'(t)), \quad t_k < t < t_{k+1}, \quad k = 0, \dots, m, \quad (1)$$

$$x(0) = a, \quad x(1) = b, \quad (2)$$

and for  $k = 1, \dots, m$ ,

$$\begin{aligned} \Delta x(t_k) &= u_k \\ \Delta x'(t_k) &= v_k(x(t_k), x'(t_k)), \end{aligned} \quad (3)$$

where  $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuous,  $u_k \in \mathbb{R}$ ,  $v_k : \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuous,  $k = 1, \dots, m$ . Define the impulse,  $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ , and by convention, let  $x(t_k) = x(t_k^-)$ ,  $k = 1, \dots, m$ . We shall employ the method of upper and lower solutions and the method of quasilinearization to obtain a bilateral iteration scheme in which the iterates converge quadratically to the unique solution of the BVP with impulse, (1), (2), (3).

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\* 1991 Mathematics Subject Classifications: 34A37, 34B15.

Key words and phrases: Quasilinearization, boundary value problem with impulse, quadratic convergence, Nagumo conditions.

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Published November 12, 1998.

The method of quasilinearization is described by Bellman [4, 5], and has recently been generalized by Lakshmikantham, Leela and various co-authors to apply to a wide variety of problems. See, for example, [14, 15], and references therein. The method generates sequences of approximate solutions which converge monotonically and quadratically to the problem of interest. Recently, Vatsala, et. al., [16], [17, 18], have applied the method of quasilinearization to families of two-point BVPs related to (1), in the case that  $f$  is independent of  $x'$ , and the boundary conditions are more general than (2).

More recently, Eloë and Zhang [9] extended the work of Vatsala, et. al. [16, 17, 18] to the BVP, (1), (2), in the case where  $f$  depends on  $x'$ . As pointed out in [9], Knobloch [13] and Jackson and Schrader [12] have obtained conditions such that there exists a sequence of solutions of (1) converging monotonically and in  $C^1[0, 1]$  to a solution of the BVP, (1), (2). Neither Knobloch [13] nor Jackson and Schrader [12] considered the rate of convergence.

Also, recently, Devi, Chandrakala and Vatsala [6] applied the method of quasilinearization to initial value problems for scalar ordinary differential equations with impulse. Doddaballapur and Eloë [7] have extended the work of Vatsala, et. al. [16, 17], [18] to the BVP with impulse, (1), (2), (3), in the case that  $f$  and each  $v_k$  are independent of  $x'$ . Thus, the primary contribution of this paper then is that we extend the work in [16], [17], [18], [9] and [7] to the BVP with impulse, (1), (2), (3), when  $f$  and each  $v_k$  depend on  $x'$ .

The paper is organized in the following manner. We shall obtain a preliminary result in Theorem 1 concerning the properties of upper and lower solutions of the BVP with impulse, (1), (2), (3). In Theorem 2, we shall obtain a fundamental existence of solutions result for the BVP with impulse, (1), (2), (3). The proof of this result employs the Schauder fixed point theorem. Due to the dependence on  $x'$ , technical difficulties arise which require the assumption of Nagumo type conditions and extensions of the Kamke convergence theorem [10, 11]. In Theorem 3, we shall state a uniqueness of solutions result for the BVP with impulse, (1), (2), (3). We shall state and prove the main result of this paper in Theorem 4. The proof of Theorem 4 employs a clever manipulation of Theorems 1 and 2. The iterative details in the proof of Theorem 4 are completely analogous to those found in [7, 9, 16, 17, 9] once Theorems 1 and 2 are obtained. Hence, we consider these details to be standard and only highlight those details in the proof of Theorem 4 that are particular to the BVP with impulse, (1), (2), (3).

## 2 Results

We begin with the definition of an appropriate Banach space,  $B$ . Let  $PC[0, 1]$  denote the piecewise continuous functions on  $[0, 1]$  and let  $PC^1[0, 1]$  denote the functions,  $x$ , such that  $x \in PC[0, 1]$  and  $x' \in PC[0, 1]$ . Define

$$B = \{x \in PC^1[0, 1] : x^{(i)}|_{[t_k, t_{k+1}]} \in C^i[t_k, t_{k+1}], k = 0, \dots, m, i = 0, 1\},$$

with  $\|x\|_B = \max_{k=0,\dots,m} \|x\|_k$  and  $\|x\|_k = \max_{i=0,1} \sup_{t_k \leq t \leq t_{k+1}} |x^{(i)}(t)|$ . We shall say that  $\alpha \in B$  is a lower solution of the BVP with impulse, (1), (2), (3), if

$$\begin{aligned} \alpha''(t) &\geq f(t, \alpha(t), \alpha'(t)), \quad t_k < t < t_{k+1}, \quad k = 0, \dots, m, \\ \alpha(0) &\leq a, \quad \alpha(1) \leq b, \end{aligned}$$

and for  $k = 1, \dots, m$ ,

$$\begin{aligned} \Delta\alpha(t_k) &= u_k \\ \Delta\alpha'(t_k) &\geq v_k(\alpha(t_k), \alpha'(t_k)). \end{aligned}$$

We shall say that  $\beta \in B$  is an upper solution of the BVP with impulse, (1), (2), (3), if

$$\begin{aligned} \beta''(t) &\leq f(t, \beta(t), \beta'(t)), \quad t_k < t < t_{k+1}, \quad k = 0, \dots, m, \\ \beta(0) &\geq a, \quad \beta(1) \geq b, \end{aligned}$$

and for  $k = 1, \dots, m$ ,

$$\begin{aligned} \Delta\beta(t_k) &= u_k \\ \Delta\beta'(t_k) &\leq v_k(\beta(t_k), \beta'(t_k)). \end{aligned}$$

For the remainder of this paper, we shall assume that

$$f \in C([0, 1] \times \mathbb{R}^2), \quad (\partial f / \partial x) = f_x \in C([0, 1] \times \mathbb{R}^2), \quad (4)$$

$$f_x(t, x, y) > 0, \quad (t, x, y) \in [0, 1] \times \mathbb{R}^2, \quad (5)$$

$$v_k \in C^1(\mathbb{R}^2), \quad (6)$$

and for  $k = 1, \dots, m$ ,

$$v_{kx}(x, y) > 0, \quad (x, y) \in \mathbb{R}^2, \quad v_{ky}(x, y) > 0, \quad (x, y) \in \mathbb{R}^2. \quad (7)$$

In order to obtain Theorem 2, we shall define an appropriate fixed point operator,  $T$ . For  $x \in B$ , define an operator  $T$  on  $x$  by

$$Tx(t) = p(t) + I(t, x) + \int_0^1 G(t, s) f(s, x(s), x'(s)) ds, \quad (8)$$

where  $p(t) = a(1-t) + bt$ ,  $I(t, x) = \sum_{k=1}^m I_k(t, x)$ . For  $k = 1, \dots, m$ , let

$$I_k(t, x) = \begin{cases} t(-u_k - (1-t_k)v_k(x(t_k), x'(t_k))) & , 0 \leq t \leq t_k, \\ (1-t)(u_k - t_k v_k(x(t_k), x'(t_k))) & , t_k \leq t \leq 1. \end{cases}$$

Let

$$G(t, s) = \begin{cases} t(s-1) & , 0 \leq t < s \leq 1, \\ s(t-1) & , 0 \leq s < t \leq 1, \end{cases}$$

denote the Green's function for the BVP,  $x''(t) = 0$ ,  $0 \leq t \leq 1$ ,  $x(0) = 0$ ,  $x(1) = 0$ . Eloe and Henderson [8] have argued that  $x$  is a solution of the BVP with impulse, (1), (2), (3), if, and only if,  $x \in B$  and  $Tx = x$ . Finally, we shall define a partial order on  $B$  as follows: for  $\alpha, \beta \in B$ , we say that  $\alpha \leq \beta$  if, and only if,

$$\alpha|_{[t_k, t_{k+1}]}(t) \leq \beta|_{[t_k, t_{k+1}]}(t), t_k \leq t \leq t_{k+1}, k = 0, \dots, m.$$

**Theorem 1** *Assume (4), (5), (6), and (7) hold. Let  $\alpha, \beta$  be lower and upper solutions of the BVP with impulse, (1), (2), (3), respectively. Then  $\alpha \leq \beta$ .*

**Proof.** Set  $w(t) = \alpha(t) - \beta(t)$  and note that  $w$  is continuous on  $[0, 1]$  by (3). Assume, for the sake of contradiction, that  $w$  is positive on  $[0, 1]$ . Since  $w(0) \leq 0$ ,  $w(1) \leq 0$ ,  $w$  has a positive maximum at some  $\tau \in (0, 1)$ . Assume  $\tau \in \cup_{k=0}^m (t_k, t_{k+1})$ . Then  $w''(\tau) \leq 0$  and  $\alpha'(\tau) = \beta'(\tau)$ . However, employing that  $\alpha$  and  $\beta$  are lower and upper solutions of the BVP with impulse, (1), (2), (3), respectively, and employing (5), it follows that

$$w''(\tau) = \alpha''(\tau) - \beta''(\tau) \geq f(\tau, \alpha(\tau), \alpha'(\tau)) - f(\tau, \beta(\tau), \beta'(\tau)) > 0.$$

This provides a contradiction and so,  $\tau \notin \cup_{k=0}^m (t_k, t_{k+1})$ . Now, assume that  $\tau = t_k$  for some  $k \in \{1, \dots, m\}$ . By Taylor's theorem,  $w'(t_k^-) \geq 0$  and  $w'(t_k^+) \leq 0$ , or  $\Delta w'(t_k) \leq 0$  and

$$\alpha'(t_k^-) = \alpha'(t_k) \geq \beta'(t_k) = \beta'(t_k^-).$$

But

$$\Delta w'(t_k) = \Delta \alpha'(t_k) - \Delta \beta'(t_k) \geq v_k(\alpha(t_k), \alpha'(t_k)) - v_k(\beta(t_k), \beta'(t_k)) > 0$$

by (7). Thus,  $\tau \notin \{t_1, \dots, t_m\}$ , and  $w(t) \leq 0$ ,  $0 \leq t \leq 1$ .

**Theorem 2** *Assume  $g \in C([0, 1] \times \mathbb{R}^2)$ ,  $z_k \in C(\mathbb{R}^2)$ ,  $k = 1, \dots, m$ , and assume that each  $z_k(x, y)$  is monotone increasing in  $y$  for fixed  $x$ . Assume that each solution of  $x''(t) = g(t, x(t), x'(t))$  extends to  $[0, 1]$ , or becomes unbounded on its maximal interval of convergence. Let  $\alpha, \beta$  be lower and upper solutions of the BVP,*

$$x''(t) = g(t, x(t), x'(t)), \quad t_k < t < t_{k+1}, \quad (9)$$

$$\Delta x(t_k) = u_k$$

$$\Delta x'(t_k) = z_k(x(t_k), x'(t_k)), \quad (10)$$

with  $k = 1, \dots, m$  and boundary conditions given by (2), respectively, such that

$$\alpha \leq \beta.$$

Then, there exists a solution,  $x$ , of the BVP with impulse, (9), (2), (10), satisfying

$$\alpha \leq x \leq \beta.$$

**Proof.** Define

$$\hat{f}(t, x, y) = \begin{cases} g(t, \beta(t), y) + (x - \beta(t))/[1 + (x - \beta(t))], & x > \beta(t), \\ g(t, x, y), & \alpha(t) \leq x \leq \beta(t), \\ g(t, \alpha(t), y) + (x - \alpha(t))/[1 + |x - \alpha(t)|], & x < \alpha(t), \end{cases}$$

and for  $k = 1, \dots, m$ , define

$$\hat{v}_k(x, y) = \begin{cases} z_k(\beta(t_k), y) + (x - \beta(t_k))/[1 + (x - \beta(t_k))], & x > \beta(t_k), \\ z_k(x, y), & \alpha(t_k) \leq x \leq \beta(t_k), \\ z_k(\alpha(t_k), y) + (x - \alpha(t_k))/[1 + |x - \alpha(t_k)|], & x < \alpha(t_k). \end{cases}$$

Let  $N > 0$  be such that  $|\alpha'(t)| \leq N$ ,  $|\beta'(t)| \leq N$ ,  $t \in [t_k, t_{k+1}]$ ,  $k = 0, \dots, m$ . For each positive integer,  $l$ , define

$$f_l(t, x, y) = \begin{cases} \hat{f}(t, x, N + l), & y > N + l, \\ \hat{f}(t, x, y), & |y| \leq N + l, \\ \hat{f}(t, x, -(N + l)), & y < -(N + l), \end{cases}$$

and

$$v_{kl}(t, x, y) = \begin{cases} \hat{v}_k(x, N + l), & y > N + l, \\ \hat{v}_k(x, y), & |y| \leq N + l, \\ \hat{v}_k(x, -(N + l)), & y < -(N + l). \end{cases}$$

Notice that  $f_l$  and each  $v_{kl}$  are bounded and continuous. With a standard application of the Schauder fixed point theorem to the operator  $T$ , defined by (8), one obtains a solution,  $x_l \in B$ , to the BVP with impulse, (1), (2), (3), with  $f = f_l$  and each  $v_k = v_{kl}$  bounded and continuous.

We now argue that each solution,  $x_l$ , satisfies  $\alpha \leq x_l \leq \beta$ . We shall show that  $x_l \leq \beta$ . As in the proof of Theorem 1, assume for the sake of contradiction that  $x_l - \beta$  has a positive maximum at  $\tau$ . As in the proof of Theorem 1,  $\tau \in (0, 1)$ . If  $\tau \in \cup_{k=0}^m (t_k, t_{k+1})$ , then  $x_l''(\tau) \leq \beta''(\tau)$ , and  $|x_l'(\tau)| = |\beta'(\tau)| \leq N < N + l$ . Thus,

$$(x_l - \beta)''(\tau) \geq (x_l - \beta)(\tau)/[1 + (x_l - \beta)(\tau)] > 0,$$

which is a contradiction. If  $\tau = t_k$ , for some  $k \in \{1, \dots, m\}$ , then  $x_l'(t_k) \geq \beta'(t_k)$ . Since each  $z_k(x, y)$  is monotone increasing in  $y$  for fixed  $x$ , it follows that each  $v_{kl}(x, y)$  is monotone increasing in  $y$  for fixed  $x$ . Moreover, note that  $v_{kl}(\beta(t_k), \beta'(t_k)) = z_k(\beta(t_k), \beta'(t_k))$ . Thus,

$$\begin{aligned} \Delta(x_l - \beta)'(t_k) &\geq v_{kl}(\beta(t_k), x_l'(t_k)) - v_{kl}(\beta(t_k), \beta'(t_k)) \\ &\quad + (x_l - \beta)(t_k)/[1 + (x_l - \beta)(t_k)] \\ &\geq (x_l - \beta)(t_k)/[1 + (x_l - \beta)(t_k)] > 0 \end{aligned}$$

which is also a contradiction. Therefore,  $x_l \leq \beta$ . To show that  $\alpha \leq x_l$  we follow a similar procedure.

For each  $l$  there exists  $t_l \in [0, t_1]$  such that

$$t_1 |x_{kl}'(t_l)| = |x_{kl}(t_l) - a| \leq \max\{|\beta(0) - \alpha(t_1)|, |\beta(t_1) - \alpha(0)|\}.$$

Thus, each of the sequences  $\{x_{kl}(t_l)\}$  and  $\{x'_{kl}(t_l)\}$  are bounded. One can now apply the Kamke convergence theorem (see [11]) for solutions of initial value problems and obtain a subsequence of  $\{x_{kl}\}$  which converges to a solution of  $x''(t) = \hat{f}(t, x(t), x'(t))$  on a maximal subinterval of  $[0, t_1]$ . Clearly,  $\alpha(t) \leq x(t) \leq \beta(t)$  and solutions of  $x''(t) = g(t, x(t), x'(t))$  extend to all of  $[0, 1]$  or become unbounded; thus,  $x''(t) = \hat{f}(t, x(t), x'(t))$  on  $[0, t_1]$ .

Now, apply the impulse defined by (10) at  $t_1$ . Apply the Kamke theorem to the subsequence that was extracted in the preceding paragraph. Because of (10) one can employ  $t_1 = t_l$  for each  $l$ . Thus, one obtains a further subsequence which converges to a solution,  $x$ , of  $x''(t) = \hat{f}(t, x(t), x'(t))$  on  $(0, t_1) \cup (t_1, t_2)$  such that  $x$  satisfies (10) at  $t_1$ .

Continue inductively, first applying (10) at each  $t_j$  and then applying the Kamke convergence theorem on that subinterval  $(t_j, t_{j+1})$ . Finally, since  $\alpha \leq x \leq \beta$ ,  $\hat{f}(t, x(t), x'(t)) = f(t, x(t), x'(t))$  and the proof of Theorem 2 is complete.

**Remark.** For simplicity, we can assume that  $g$  satisfies a Nagumo condition in  $x'$  ([10], [11]). That is, assume that for each  $M > 0$  there exists a positive continuous function,  $h_M(s)$ , defined on  $[0, \infty)$  such that

$$|g(t, x, x')| \leq h_M(|x'|)$$

for all  $(t, x, x') \in [0, 1] \times [-M, M] \times \mathbb{R}$  and such that

$$\int_0^\infty (s/h_M(s))ds = +\infty.$$

The assumption that  $g$  satisfies a Nagumo condition implies that each solution of the differential equation,  $x''(t) = g(t, x(t), x'(t))$ , either extends to  $[0, 1]$  or becomes unbounded on its maximal interval of existence ([10], [11]). In our main result, Theorem 4,  $g$  will represent a modification of  $f$ . Thus, we shall assume in Theorem 4 that  $f$  satisfies a Nagumo condition in  $x'$ .

**Theorem 3** *Assume that (4), (5), (6), and (7) hold. Then, solutions of the BVP with impulse, (1), (2), (3), are unique.*

**Proof.** The uniqueness of solutions result follows immediately from Theorem 1 and the observation that solutions are respectively upper and lower solutions.

**Theorem 4** *Assume that (4), (5), (6), and (7) hold, and assume that*

$$(\partial^2/\partial x^2)f \in C([0, 1] \times \mathbb{R}^2), v''_k \in C(\mathbb{R}^2), k = 1, \dots, m.$$

*Assume that  $f$  satisfies a Nagumo condition in  $x'$ . Assume that  $\alpha_0$  and  $\beta_0$  are lower and upper solutions of the BVP with impulse, (1), (2), (3), respectively. Then there exist monotone sequences,  $\{\alpha_n(t)\}$  and  $\{\beta_n(t)\}$ , which converge in  $B$  to the unique solution,  $x(t)$ , of the BVP with impulse, (1), (2), (3), and the convergence is quadratic.*

**Proof.** Let  $F(t, x) : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  be such that  $F, F_x, F_{xx}$  are continuous on  $[0, 1] \times \mathbb{R}$  and

$$F_{xx}(t, x) \geq 0, (t, x) \in [0, 1] \times \mathbb{R}. \quad (11)$$

Set  $\phi_1(t, x_1, x_2) = F(t, x_1) - f(t, x_1, x_2)$  on  $[0, 1] \times \mathbb{R}^2$ . From (11) it follows that, if  $x_1, y_1 \in \mathbb{R}$ , then  $F(t, x_1) \geq F(t, y_1) + F_x(t, y_1)(x_1 - y_1)$ . In particular, for  $x_1, y_1, x_2, y_2 \in \mathbb{R}$ ,

$$f(t, x_1, x_2) \geq f(t, y_1, y_2) + F_x(t, y_1)(x_1 - y_1) - \phi_1(t, x_1, x_2) + \phi_1(t, y_1, y_2). \quad (12)$$

For each  $k = 1, \dots, m$ , let  $V_k(x) : \mathbb{R} \rightarrow \mathbb{R}$  be such that  $V_k, V'_k, V''_k$  are continuous on  $\mathbb{R}$  and

$$V''_k(x) \geq 0, \quad x \in \mathbb{R}. \quad (13)$$

Set  $\phi_{2k}(x_1, x_2) = V_k(x_1) - v_k(x_1, x_2)$  on  $\mathbb{R}^2$ . From (13) it follows that, if  $x_1, y_1 \in \mathbb{R}$ , then  $V_k(x_1) \geq V_k(y_1) + V'_k(y_1)(x_1 - y_1)$ . In particular, for  $x_1, y_1, x_2, y_2 \in \mathbb{R}$ ,

$$v_k(x_1, x_2) \geq v_k(y_1, y_2) + V'_k(y_1)(x_1 - y_1) - (\phi_{2k}(x_1, x_2) - \phi_{2k}(y_1, y_2)). \quad (14)$$

Define

$$\begin{aligned} g(t, x_1, x_2; \alpha_0, \beta_0, \alpha'_0) &= f(t, \alpha_0(t), \alpha'_0(t)) + F_x(t, \beta_0(t))(x_1 - \alpha_0(t)) \\ &\quad - \phi_1(t, x_1, x_2) + \phi_1(t, \alpha_0(t), \alpha'_0(t)), \\ G(t, x_1, x_2; \beta_0, \beta'_0) &= f(t, \beta_0(t), \beta'_0(t)) + F_x(t, \beta_0(t))(x_1 - \beta_0(t)) \\ &\quad - \phi_1(t, x_1, x_2) + \phi_1(t, \beta_0(t), \beta'_0(t)), \\ h_k(x_1, x_2; \alpha_0, \beta_0, \alpha'_0) &= v_k(\alpha_0(t_k), \alpha'_0(t_k)) + V'_k(\beta_0(t_k))(x_1 - \alpha_0(t_k)) \\ &\quad - (\phi_{2k}(x_1, x_2) - \phi_{2k}(\alpha_0(t_k), \alpha'_0(t_k))), \\ H_k(x_1, x_2; \beta_0, \beta'_0) &= v_k(\beta_0(t_k), \beta'_0(t_k)) + V'_k(\beta_0(t_k))(x_1 - \beta_0(t_k)) \\ &\quad - (\phi_{2k}(x_1, x_2) - \phi_{2k}(\beta_0(t_k), \beta'_0(t_k))). \end{aligned}$$

First consider the BVP with impulse,

$$x''(t) = g(t, x(t), x'(t); \alpha_0, \beta_0, \alpha'_0), \quad t_k < t < t_{k+1}, \quad k = 0, \dots, m, \quad (15)$$

and for  $k = 1, \dots, m$ ,

$$\begin{aligned} \Delta x(t_k) &= u_k \\ \Delta x'(t_k) &= h_k(x(t_k), x'(t_k); \alpha_0, \beta_0, \alpha'_0), \end{aligned} \quad (16)$$

with boundary conditions given by (2). Each  $h_k$  readily satisfies the hypotheses of Theorem 2. A limit comparison implies that  $g$  satisfies a Nagumo condition in  $x'$ .

We now show that  $\alpha_0$  and  $\beta_0$  are lower and upper solutions, respectively, of the BVP with impulse, (15), (2), (16); thus, by Theorem 2, there exists a solution  $\alpha_1(t)$  of the BVP with impulse, (15), (2), (16), satisfying

$$\alpha_0 \leq \alpha_1 \leq \beta_0.$$



To this end, note that for  $t_k < t < t_{k+1}$ ,  $k = 0, \dots, m$ ,

$$\alpha_0''(t) \geq f(t, \alpha_0(t), \alpha_0'(t)) = g(t, \alpha_0(t), \alpha_0'(t); \alpha_0, \beta_0, \alpha_0'),$$

and, for  $k = 1, \dots, m$ ,

$$\Delta \alpha_0'(t_k) \geq v_k(\alpha_0(t_k), \alpha_0'(t_k)) = h_k(\alpha_0(t_k), \alpha_0'(t_k); \alpha_0, \beta_0, \alpha_0').$$

Moreover, from (12) and (14), it follows that for  $t_k < t < t_{k+1}$ ,  $k = 0, \dots, m$ ,

$$\begin{aligned} \beta_0''(t) \leq f(t, \beta_0(t), \beta_0'(t)) &\leq f(t, \alpha_0(t), \alpha_0'(t)) - F_x(t, \beta_0(t))(\alpha_0(t) - \beta_0(t)) \\ &\quad + \phi_1(t, \alpha_0(t), \alpha_0'(t)) - \phi_1(t, \beta_0(t), \beta_0'(t)) \\ &= g(t, \beta_0(t), \beta_0'(t); \alpha_0, \beta_0, \alpha_0'), \end{aligned}$$

and for  $k = 1, \dots, m$ ,

$$\begin{aligned} \Delta \beta_0'(t_k) &\leq v_k(\beta_0(t_k), \beta_0'(t_k)) \\ &\leq v_k(\alpha_0(t), \alpha_0'(t)) - V_k'(\beta_0(t_k))(\alpha_0(t_k) - \beta_0(t_k)) \\ &\quad + (\phi_{2k}(\alpha_0(t_k), \alpha_0'(t_k)) - \phi_{2k}(\beta_0(t_k), \beta_0'(t_k))) \\ &= h_k(\beta_0(t_k), \beta_0'(t_k); \alpha_0, \beta_0, \beta_0'). \end{aligned}$$

Since  $\alpha_0$  and  $\beta_0$  satisfy (2),  $\alpha_0$  and  $\beta_0$  are lower and upper solutions, respectively, of the BVP with impulse, (15), (2), (16), and thus, by Theorem 2, there exists a solution  $\alpha_1(t)$  of the BVP with impulse, (15), (2), (16), such that

$$\alpha_0 \leq \alpha_1 \leq \beta_0.$$

Now, consider the BVP with impulse,

$$\begin{aligned} x''(t) &= G(t, x(t), x'(t); \beta_0, \beta_0'), \quad t_k < t < t_{k+1}, \quad k = 0, \dots, m, \\ x(0) &= a, \quad x(1) = b, \end{aligned} \tag{17}$$

and for  $k = 1, \dots, m$ ,

$$\begin{aligned} \Delta x(t_k) &= u_k \\ \Delta x'(t_k) &= H_k(x(t_k), x'(t_k); \alpha_0, \beta_0, \beta_0'). \end{aligned} \tag{18}$$

Again,  $G$  and each  $H_k$  satisfy the hypotheses of Theorem 2, and, again, (12) and (14) are employed to show that  $\alpha_0$  and  $\beta_0$  are lower and upper solutions, respectively, of the BVP with impulse, (17), (2), (18); thus, there exists a solution  $\beta_1(t)$  of the BVP with impulse, (17), (2), (18), such that

$$\alpha_0 \leq \beta_1 \leq \beta_0.$$

We now show that  $\alpha_1$  and  $\beta_1$  are lower and upper solutions, respectively, of the BVP with impulse, (1), (2), (3). Thus, it will follow by Theorem 1 that

$$\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0.$$

Employ (12) and (11) to see that for  $t \in \cup_{k=0}^m(t_k, t_{k+1})$ ,

$$\begin{aligned}
\alpha_1''(t) &= g(t, \alpha_1(t), \alpha_1'(t); \alpha_0, \beta_0, \alpha_0') \\
&= f(t, \alpha_0(t), \alpha_0'(t)) + F_x(t, \beta_0(t))(\alpha_1(t) - \alpha_0(t)) \\
&\quad - (\phi(t, \alpha_1(t), \alpha_1'(t)) - \phi(t, \alpha_0(t), \alpha_0'(t))) \\
&\geq f(t, \alpha_1(t), \alpha_1'(t)) + F_x(t, \alpha_1(t))(\alpha_0(t) - \alpha_1(t)) + \phi(t, \alpha_1(t), \alpha_1'(t)) \\
&\quad - \phi(t, \alpha_0(t), \alpha_0'(t)) + F_x(t, \beta_0(t))(\alpha_1(t) - \alpha_0(t)) \\
&\quad - (\phi(t, \alpha_1(t), \alpha_1'(t)) - \phi(t, \alpha_0(t), \alpha_0'(t))) \\
&= f(t, \alpha_1(t), \alpha_1'(t)) + (F_x(t, \beta_0(t)) - F_x(t, \alpha_1(t)))(\alpha_1(t) - \alpha_0(t)) \\
&\geq f(t, \alpha_1(t), \alpha_1'(t)).
\end{aligned}$$

Similarly, for  $k = 1, \dots, m$ , employ (14) and (13) to see that

$$\begin{aligned}
\Delta \alpha_1'(t_k) &= h_k(\alpha_1(t_k), \alpha_1'(t_k); \alpha_0, \beta_0, \alpha_0') \\
&= v_k(\alpha_0(t_k), \alpha_0'(t_k)) + V_k'(\beta_0(t_k))(\alpha_1(t_k) - \alpha_0(t_k)) \\
&\quad - (\phi_{2k}(\alpha_1(t_k), \alpha_1'(t_k)) - \phi_{2k}(\alpha_0(t_k), \alpha_0'(t_k))) \\
&\geq v_k(\alpha_1(t_k), \alpha_1'(t_k)) + V_k'(\alpha_1(t_k))(\alpha_0(t_k) - \alpha_1(t_k)) \\
&\quad + \phi_{2k}(\alpha_1(t_k), \alpha_1'(t_k)) - \phi_{2k}(\alpha_0(t_k), \alpha_0'(t_k)) \\
&\quad + V_k'(\beta_0(t_k))(\alpha_1(t_k) - \alpha_0(t_k)) - (\phi_{2k}(\alpha_1(t_k), \alpha_1'(t_k)) \\
&\quad - \phi_{2k}(\alpha_0(t_k), \alpha_0'(t_k))) \\
&= v_k(\alpha_1(t_k)) + (V_k'(\beta_0(t_k)) - V_k'(\alpha_1(t_k)))(\alpha_1(t_k) - \alpha_0(t_k)) \\
&\geq v_k(\alpha_1(t_k)).
\end{aligned}$$

Similarly, it follows by (11)-(14) that for  $t \in \cup_{k=0}^m(t_k, t_{k+1})$ ,

$$\beta_1''(t) \leq f(t, \beta_1(t), \beta_1'(t)),$$

and for  $k \in \{1, \dots, m\}$ ,

$$\Delta \beta_1'(t_k) \leq v_k(\beta_1(t_k), \beta_1'(t_k)).$$

In particular,  $\alpha_1$  and  $\beta_1$  are lower and upper solutions, respectively, of the BVP with impulse, (1), (2), (3), and by Theorem 1,

$$\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0.$$

Inductively, define sequences of functions  $\{g_l\}$ ,  $\{G_l\}$ ,  $\{h_{kl}\}$ , and  $\{H_{kl}\}$  by

$$\begin{aligned}
g_l(t, x_1, x_2) &= g(t, x_1, x_2; \alpha_l, \beta_l, \alpha_l') \\
&= f(t, \alpha_l(t), \alpha_l'(t)) + F_x(t, \beta_l(t))(x_1 - \alpha_l(t)) \\
&\quad - \phi_1(t, x_1, x_2) + \phi_1(t, \alpha_l(t), \alpha_l'(t)) \\
G_l(t, x_1, x_2) &= G(t, x_1, x_2; \beta_l, \beta_l') \\
&= f(t, \beta_l(t), \beta_l'(t)) + F_x(t, \beta_l(t))(x_1 - \beta_l(t))
\end{aligned}$$

$$\begin{aligned}
& -\phi_1(t, x_1, x_2) + \phi_1(t, \beta_l(t), \beta'_l(t)), \\
h_{kl} &= h_k(x_1, x_2; \alpha_l, \beta_l, \alpha'_l) \\
&= v_k(\alpha_l(t_k), \alpha'_l(t_k)) + V'_k(\beta_l(t_k))(x_1 - \alpha_l(t_k)) \\
&\quad - (\phi_{2k}(x_1, x_2) - \phi_{2k}(\alpha_l(t_k))), \\
H_{kl} &= H_k(x_1, x_2; \beta_l, \beta'_l) \\
&= v_k(\beta_l(t_k), \beta'_l(t_k)) + V'_k(\beta_l(t_k))(x_1 - \beta_l(t_k)) \\
&\quad - (\phi_{2k}(x_1, x_2) - \phi_{2k}(\beta_l(t_k), \beta'_l(t_k))).
\end{aligned}$$

Inductively, Theorem 2 implies there exists a solution  $\alpha_{l+1}(t)$  of the BVP with impulse, (15), (2), (16), with  $g = g_l$  and each  $h_k = h_{kl}$  satisfying

$$\alpha_0 \leq \dots \leq \alpha_l \leq \alpha_{l+1} \leq \beta_l \leq \dots \leq \beta_0.$$

Similarly, there exists a solution  $\beta_{l+1}(t)$  of the BVP with impulse, (17), (2), (18), with  $G = G_l$  and each  $H_k = H_{kl}$  satisfying

$$\alpha_0 \leq \dots \leq \alpha_l \leq \beta_{l+1} \leq \beta_l \leq \dots \leq \beta_0.$$

Finally, inductively,  $\alpha_{l+1}$  and  $\beta_{l+1}$  are lower and upper solutions, respectively, of the BVP with impulse, (1), (2), (3), and by Theorem 1,

$$\alpha_0 \leq \dots \leq \alpha_l \leq \alpha_{l+1} \leq \beta_{l+1} \leq \beta_l \leq \dots \leq \beta_0.$$

We now show that each sequence  $\{\alpha_l\}$  and  $\{\beta_l\}$  converge in  $B$  to  $x$ , the unique solution of the BVP with impulse, (1), (2), (3). Recall

$$B = \{x \in PC^1[0, 1] : x^{(i)}|_{[t_k, t_{k+1}]} \in C^i[t_k, t_{k+1}], k = 0, \dots, m, i = 0, 1\},$$

with  $\|x\|_B = \max_{k=0, \dots, m} \|x\|_k$  and  $\|x\|_k = \max_{i=0, 1} \sup_{t_k \leq t \leq t_{k+1}} |x^{(i)}(t)|$ . The Kamke convergence theorem does not apply directly to either sequence,  $\{\alpha_l\}$  or  $\{\beta_l\}$  since neither  $g_l$  nor  $G_l$  converge uniformly on compact sets to  $f$ . To see this, note that

$$g_l(t, x_1, x_2) = f(t, x_1, x_2) + F_x(t, \beta_l(t))(x_1 - \alpha_l(t)) + F(t, \alpha_l(t)) - F(t, x_1)$$

and

$$G_l(t, x_1, x_2) = f(t, x_1, x_2) + F_x(t, \beta_l(t))(x_1 - \beta_l(t)) + F(t, \beta_l(t)) - F(t, x_1).$$

Define

$$\hat{g}_l(t, x_1, x_2) = f(t, x_1, x_2) + F_x(t, \beta_l(t))(\alpha_{l+1} - \alpha_l)(t) + F(t, \alpha_l(t)) - F(t, \alpha_{l+1}(t))$$

and

$$\hat{G}_l(t, x_1, x_2) = f(t, x_1, x_2) + F_x(t, \beta_l(t))(\beta_{l+1} - \beta_l)(t) + F(t, \beta_l(t)) - F(t, \beta_{l+1}(t)).$$

Theorem 3 applies to the BVP with impulse, (1), (2), (3), with  $f = \hat{g}_l$  and each  $v_k = h_{kl}$  and note that  $\alpha_{k+1}$  is the unique solution. The Kamke convergence

theorem now does apply and, with omitted details that are similar to those given in the proof of Theorem 2,  $\{\alpha_l\}$  converges in  $B$  to  $x$ , the unique solution of the BVP with impulse, (1), (2), (3). Similarly,  $\{\beta_l\}$  converges in  $B$  to  $x$ , the unique solution of the BVP with impulse, (1), (2), (3).

We now argue that the convergence is quadratic. Let  $q_n(t) = \beta_n(t) - x(t)$  and  $p_n(t) = x(t) - \alpha_n(t)$ , where  $x(t)$  denotes the unique solution of the BVP with impulse, (1), (2), (3). Set

$$e_n = \max\{\|q_n\|_B, \|p_n\|_B\}.$$

First, consider  $q_{n+1}(t)$  and note that  $q_{n+1} \geq 0$ . For  $t \in \cup_{k=0}^m (t_k, t_{k+1})$ ,

$$\begin{aligned} q''_{n+1}(t) &= F(t, \beta_n(t)) + F_x(t, \beta_n(t))(\beta_{n+1} - \beta_n)(t) \\ &\quad - \phi_1(t, \beta_{n+1}(t), \beta'_{n+1}(t)) - F(t, x(t)) + \phi_1(t, x(t), x'(t)) \\ &= F_x(t, c_1(t))q_n(t) - F_x(t, \beta_n(t))q_n(t) + F_x(t, \beta_n(t))q_{n+1}(t) \\ &\quad - \phi_{1x}(t, c_2(t), c_3(t))q_{n+1}(t) - \phi_{1x'}(t, c_2(t), c_3(t))q'_{n+1}(t), \end{aligned}$$

where  $x(t) \leq c_1(t) \leq \beta_n(t)$ ,  $x(t) \leq c_2(t) \leq \beta_{n+1}(t)$ , and  $c_3(t)$  is between  $x'(t)$  and  $\beta'_{n+1}(t)$ . Thus, there exists  $c_1(t) \leq c_4(t) \leq \beta_n(t)$  such that

$$\begin{aligned} q''_{n+1}(t) &= F_{xx}(t, c_4(t))q_n(t)(c_1(t) - \beta_n(t)) \\ &\quad + (F_x(t, \beta_n(t)) - \phi_{1x}(t, c_2(t), c_3(t)))q_{n+1}(t) - \phi_{1x'}(t, c_2(t), c_3(t))q'_{n+1}(t) \\ &\geq -F_{xx}(t, c_4(t))q_n^2(t) + f_{x'}(t, c_2(t), c_3(t))q'_{n+1}(t). \end{aligned}$$

Note that to obtain this inequality, we have employed the monotonicity of  $F_x$  in the second component. In particular, there exists  $M > 0$ , such that

$$q''_{n+1}(t) - f_{x'}(t, c_2(t), c_3(t))q'_{n+1}(t) \geq -Me_n^2, \quad (19)$$

where  $M > \max_i \max_{(t,x) \in D_i} F_{xx}(t, x)$ , and for  $i = 0, \dots, m$ ,

$$D_i = \{(t, x) : t_i \leq t \leq t_{i+1}, \alpha_0(t) \leq x \leq \beta_0(t)\}.$$

Similarly, there exist appropriate  $c_4$  and  $c_5$  such that for  $k = 1, \dots, m$ ,

$$\Delta q'_{n+1}(t_k) - v_{ky}(c_4, c_5)q'_{n+1}(t_k) \geq -Me_n^2. \quad (20)$$

Let  $m(t) = \exp\left(-\int_0^t f_{x'}(s, c_2(s), c_3(s))ds\right)$  denote the integrating factor associated with (19). Then

$$(q'_{n+1}(t)m(t))' \geq -Mm(t)e_n^2. \quad (21)$$

Thus, for  $t_m \leq t \leq 1$ ,

$$q'_{n+1}(1)m(1) - q'_{n+1}(t)m(t) \geq -Me_n^2 \int_t^1 m(s)ds.$$

Since,  $q'_{n+1}(1) \leq 0$ , it follows that

$$q'_{n+1}(t) \leq Me_n^2 \int_t^1 m(s) ds / m(t).$$

Since  $q_{n+1}$  converges to 0 in  $B$ , eventually  $(s, c_2(s), c_3(s))$  belongs to

$$\hat{D} = \{(s, x_1, x_2) : t_m \leq s \leq 1, \alpha_0(s) \leq x_1 \leq \beta_0(s), x'(s) - 1 \leq x_2 \leq x'(s) + 1\}.$$

Thus, we can bound  $m(t)$  away from both 0 and  $\infty$  for  $n$  sufficiently large; in particular, there exists  $N_1 > 0$  such that for  $t_m \leq t \leq 1$  and  $n$  sufficiently large,

$$q'_{n+1}(t) \leq N_1 e_n^2. \quad (22)$$

Apply (20) at  $t_m$ . Then

$$q'_{n+1}(t_m^+) - q'_{n+1}(t_m) - v_{my}(c_4, c_5)q'_{n+1}(t_m) \geq -Me_n^2.$$

Employ (7) and also bound  $v_{my}$  away from both 0 and  $\infty$  to obtain some  $\hat{M} > 0$  such that

$$q'_{n+1}(t_m^-) \geq -\hat{M}e_n^2. \quad (23)$$

Now, employ (21) and (23) to obtain (22) for  $t_{m-1} \leq t \leq t_m$  for some  $N_2 > 0$ . Again, apply (20) to obtain a suitable (23) at  $t_{m-1}$ . Proceed inductively and obtain that there exists  $N > 0$  such that for  $t \in \cup_{k=0}^m [t_k, t_{k+1}]$  and  $n$  sufficiently large,

$$q'_{n+1}(t) \leq Ne_n^2. \quad (24)$$

Recall that  $q_{n+1}(t) \geq 0$ , and that  $q_{n+1} \in C[0, 1]$ . Integrate (24) from 0 to  $t$ ; then for  $n$  sufficiently large,

$$0 \leq q_{n+1} \leq Ne_n^2. \quad (25)$$

Beginning again at (21), integrate from 0 to  $t \leq t_1$  to obtain

$$q'_{n+1}(t)m(t) - q'_{n+1}(0) \geq -Me_n^2 \int_0^t m(s) ds.$$

Since,  $q'_{n+1}(0) \geq 0$ , it follows that for  $0 \leq t \leq t_1$ , there exists  $N_1 > 0$ , such that

$$q'_{n+1}(t) \geq -Me_n^2 \int_0^t m(s) ds / m(t) \geq -N_1 e_n^2,$$

for  $n$  sufficiently large. This is analogous to (22). Proceed analogously, then, and choose  $N$  large enough such that for  $t \in \cup_{k=0}^m [t_k, t_{k+1}]$  for  $n$  sufficiently large,

$$q'_{n+1}(t) \geq -Ne_n^2. \quad (26)$$

It now follows from (24), (25), and (26) that  $\beta_n$  converges to  $x$  quadratically in  $B$ .

The argument that  $\{\alpha_n\}$  converges quadratically to  $x$  in  $B$  is similar and we provide some details.

$$\begin{aligned}
p''_{n+1}(t) &= F(t, x(t)) - \phi_1(t, x(t), x'(t)) \\
&\quad - (F(t, \alpha_n(t)) + F_x(t, \beta_n(t))(\alpha_{n+1} - \alpha_n)(t) - \phi_1(t, \alpha_{n+1}(t), \alpha'_{n+1}(t))) \\
&= F_x(t, c_1(t))p_n(t) - F_x(t, \beta_n(t))p_n(t) + F_x(t, \beta_n(t))p_{n+1}(t) \\
&\quad - \phi_{1x}(t, c_2(t), c_3(t))p_{n+1}(t) - \phi_{1x'}(t, c_2(t), c_3(t))p'_{n+1}(t) \\
&= F_{xx}(t, c_4(t))p_n(t)(c_1(t) - \beta_n(t)) \\
&\quad + (F_x(t, \beta_n(t)) - \phi_{1x}(t, c_2(t), c_3(t)))p_{n+1}(t) - \phi_{1x'}(t, c_2(t), c_3(t))p'_{n+1}(t) \\
&\geq -F_{xx}(t, c_4(t))p_n(t)(p_n(t) + q_n(t)) + f_{x'}(t, c_2(t), c_3(t))p'_{n+1}(t).
\end{aligned}$$

In particular,

$$p''_{n+1}(t) - f_{x'}(t, c_2(t), c_3(t))p'_{n+1}(t) \geq -2Me_n^2$$

on an appropriate set and for sufficiently large  $n$ . A similar inequality is obtained with respect to the impulse and the details for quadratic convergence follow as above.

**Corollary 5** *The sequence  $\{\beta''_n(t) - f(t, \beta_n(t), \beta'_n(t))\}$  converges quadratically to 0 in  $B$ .*

**Proof:** There exist  $\beta_n \geq c_2 \geq c_1 \geq \beta_{n+1}$  such that

$$\begin{aligned}
f(t, \beta_{n+1}(t), \beta'_{n+1}(t)) &\geq \beta''_{n+1}(t) \\
&= f(t, \beta_n(t), \beta'_n(t)) + F_x(t, \beta_n(t))(\beta_{n+1}(t) - \beta_n(t)) \\
&\quad - (\phi_1(t, \beta_{n+1}(t), \beta'_{n+1}(t)) - \phi_1(t, \beta_n(t), \beta'_n(t))) \\
&= f(t, \beta_{n+1}(t), \beta'_{n+1}(t)) \\
&\quad + F_{xx}(t, c_2(t))(\beta_{n+1}(t) - \beta_n(t))(\beta_n(t) - c_1(t)).
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &\leq f(t, \beta_{n+1}(t), \beta'_{n+1}(t)) - \beta''_{n+1}(t) \\
&\leq F_{xx}(t, c_2(t))(\beta_{n+1}(t) - \beta_n(t))^2 \\
&\leq F_{xx}(t, c_2(t))e_n^2.
\end{aligned}$$

Similar inequalities are obtained for the impulse. Quadratic convergence can also be obtained for the sequence

$$\{f(t, \alpha_n(t), \alpha'_n(t)) - \alpha''_n(t)\}.$$

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