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OPTIMAL INTERVALS FOR THIRD ORDER LIPSCHITZ EQUATIONS

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Abstract. For the third order differential equation $y''' = f(t, y, y', y'')$, where

$$|f(t, y_1, y_2, y_3) - f(t, z_1, z_2, z_3)| \leq \sum_{i=1}^{3} k_i |y_i - z_i|$$

on $(a, b) \times \mathbf{R}^3$, subintervals of (a, b) of maximal length are characterized, in terms of the Lipschitz coefficients k_i, $i = 1, 2, 3$, on which certain boundary value problems possess unique solutions. The techniques for determining best interval length involve applications of the Pontryagin Maximum Principle along with uniqueness implies existence arguments.

1. Introduction. In this paper, we are concerned with solutions of boundary value problems for the third order differential equation

$$y''' = f(t, y, y', y''), \tag{1}$$

satisfying

$$y'(t_1) = y_1, \quad y(t_2) = y_2, \quad y'(t_3) = y_3, \quad a < t_1 \leq t_2 \leq t_3 < b, \tag{2}$$

where we assume that f is continuous on the slab $(a, b) \times \mathbf{R}^3$ and satisfies the Lipschitz condition

$$|f(t, y_1, y_2, y_3) - f(t, z_1, z_2, z_3)| \leq \sum_{i=1}^{3} k_i |y_i - z_i| \tag{3}$$

on the slab.

Aftabizadeh, Gupta and Xu [1] recently studied the existence and uniqueness of solutions of (1), (2) under conditions for which Leray-Schauder continuation theory was applicable. Partial motivation for their paper involved a boundary value problem of the form (1), (2) describing the deflection of an equally-loaded beam composed of three parallel layers of different materials.

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The focus of this paper is on characterizing maximal length subintervals of (a,b) in terms of the Lipschitz coefficients k_i, $i = 1,2,3$, on which there exist unique solutions of (1), (2). The techniques we employ here involve an application of the Pontryagin Maximum Principle [2, 12] in conjunction with uniqueness implies existence results for solutions of boundary value problems. This approach follows a pattern set in the papers by Melentsova [13], Melentsova and Mil'shtein [14, 15], Jackson [9, 10], Henderson [6, 7], Henderson and McGwier [8], and Hankerson and Henderson [3]. In order to adapt the arguments of Jackson and Henderson, *et al.* for our boundary value problems (1), (2), we distinguish between the classes of two-point problems,

$$y'(t_1) = y_1, \quad y(t_1) = y_2, \quad y'(t_2) = y_3, \quad a < t_1 < t_2 < b, \tag{4}$$

$$y'(t_1) = y_1, \quad y(t_2) = y_2, \quad y'(t_2) = y_3, \quad a < t_1 < t_2 < b, \tag{5}$$

and the three-point problems,

$$y'(t_1) = y_1, \quad y(t_2) = y_2, \quad y'(t_3) = y_3, \quad a < t_1 < t_2 < t_3 < b. \tag{6}$$

In following along the lines taken in [3, 6–10], we show in Section 2, for third order linear equations, that uniqueness of solutions (hence existence) of problems satisfying two-point conditions, ((4) and (5)), implies the uniqueness of solutions of problems satisfying three-point conditions (6). This then contributes significantly to our discussion in Section 3, in which we apply optimal control theory arguments to the two-point boundary value problems for a class of linear equations, for which a linearization of (1) belongs. These optimal control arguments yield optimal length intervals in terms of the k_i, $i = 1,2,3$, on which solutions of (1), (2) are unique. Then in Section 4, we show that in fact uniqueness of solutions of (1), (2) implies their existence, and hence obtain our desired result.

2. Uniqueness and linear equations. In this section, we consider uniqueness results concerning solutions of boundary value problems satisfying (4), (5), or (6) for the linear equation

$$y''' = a_1(t)y + a_2(t)y' + a_3(t)y'', \tag{7}$$

where the $a_i(t)$ are bounded Lebesgue measurable functions on (a,b). By a solution $y(t)$ of (7), we mean in the usual sense that $y(t) \in C^{(2)}(a,b)$, $y''(t)$ is absolutely continuous on (a,b), and $y(t)$ satisfies (7) for almost all $t \in (a,b)$.

For a couple of other studies devoted to uniqueness relationships between two-point and multipoint problems for linear equations, see, for example [16, 18]. For our uniqueness discussion, we will be concerned with relations between solutions of (7) satisfying the two-point conditions,

$$y'(\tau_1) = y(\tau_1) = y'(\tau_2) = 0, \quad a < \tau_1 < \tau_2 < b, \tag{8}$$

$$y'(\tau_1) = y(\tau_2) = y'(\tau_2) = 0, \quad a < \tau_1 < \tau_2 < b, \tag{9}$$

and the three-point conditions,

$$y'(\tau_1) = y(\tau_2) = y'(\tau_3) = 0, \quad a < \tau_1 < \tau_2 < \tau_3 < b. \tag{10}$$

Theorem 1. *Assume that each of the boundary value problems (7), (8), and (7), (9) has only the trivial solution on (a,b). Then the only solution of (7), (10) on (a,b) is the trivial solution.*

Proof: Let $u_1(t)$, $u_2(t)$, $u_3(t)$ be a fundamental set of solutions of (7). In order to obtain a contradiction, assume the conclusion of the theorem is false. Then there exists a nontrivial solution $y(t)$ of (7) satisfying condition (10) for some points $a < t_1 < t_2 < t_3 < b$. Moreover, $y(t)$ is essentially unique because of the hypotheses concerning solutions of (7), (8) and (7), (9). Now define $\tau = \inf\{t > t_1 \mid$ there exists a nontrivial solution $z(t)$ of (7) satisfying $z'(t_1) = z(t_2) = z'(t) = 0,\ t_1 < t_2 < t\}$. It follows that there exists a nontrivial solution $z(t)$ of (7) satisfying

$$z'(t_1) = z(t_2) = z'(\tau) = 0,$$

$t_1 < t_2 < \tau$. As a consequence,

$$\det\, X(u_1, u_2, u_3)(t_1, t_2, \tau) = 0,$$

where X is the 3×3 matrix whose ith column, $i = 1, 2, 3$, is given by

$$[u_i'(t_1), u_i(t_2), u_i'(\tau)]^T.$$

Our argument now is that the minor of each entry in the second row of X,

$$[u_1(t_2), u_2(t_2), u_3(t_2)] \qquad (+)$$

is zero. If at least one of the minors is not zero, then by the essential uniqueness of the solution $z(t)$, it follows that for some $c \neq 0$,

$$z(t) = c\, \det\, Z(u_1, u_2, u_3)(t_1, t, \tau),$$

where Z is the 3×3 matrix whose ith column, $i = 1, 2, 3$, is given by

$$[u_i'(t_1), u_i(t), u_i'(\tau)]^T.$$

From the hypothesis concerning solutions of (7), (8), we have $z'(t_2) \neq 0$.

If we define

$$Y(t, s) = c\, \det\, W(u_1, u_2, u_3)(t_1, t, s),$$

where W is the 3×3 matrix whose ith column, $i = 1, 2, 3$, is given by

$$[u_i'(t_1), u_i(t), u_i'(s)]^T,$$

then $Y(t_2, \tau) = z(t_2) = 0$ and $\partial Y/\partial t(t_2, \tau) = z'(t_2) \neq 0$. Applying the Implicit Function Theorem, there exist neighborhoods $U(\tau)$, $V(t_2)$, and a continuous mapping $T : U(\tau) \to V(t_2)$, such that $T(\tau) = t_2$ and $Y(T(s), s) = 0$, for all $s \in U(\tau)$. Thus, for $s < \tau$, but sufficiently near, there exists a nontrivial solution $v(t)$ of (7) satisfying

$$v'(t_1) = v(T(s)) = v'(s) = 0,$$

with $t_1 < T(s) < s < \tau$. This contradicts the definition of τ. Hence, the minor of each entry in row $(+)$ is zero.

Now, if we replace row (+) in the matrix X by the row

$$[u_1(t_1), u_2(t_1), u_3(t_1)],$$

then

$$\det D(u_1, u_2, u_3)(t_1, \tau) = 0,$$

where D is the 3×3 matrix whose ith column, $i = 1, 2, 3$, is given by

$$[u_i'(t_1), u_i(t_1), u_i'(\tau)]^T.$$

More specifically, there exists a nontrivial solution $w(t)$ of (7) satisfying

$$w'(t_1) = w(t_1) = w'(\tau) = 0,$$

which contradicts the assumption of the theorem concerning (7), (8).

Therefore the only solution of (7), (10) is the trivial solution, and the proof is complete.

3. Optimality and uniqueness. In this section, we determine optimal length subintervals of (a, b) in terms of the Lipschitz coefficients k_i, $i = 1, 2, 3$, on which solutions of (1), (2) are unique. Our method for doing this includes applying the Pontryagin Maximum Principle to determine the optimal length interval on which solutions are unique for a related family of linear equations satisfying the two-point conditions (4) and (5). Then from Theorem 1, it will follow that solutions are unique for this family of linear equations satisfying the three-point condition (6). Ultimately, it will also be the case that solutions of (1), (2) are unique on such a subinterval.

In our control theory arguments, a linearization of (1) is utilized. First, we define a control region

$$U = \{(u_1(t), u_2(t), u_3(t)) \mid u_i(t) \text{ is Lebesgue measurable and } |u_i(t)| \le k_i,\ i = 1, 2, 3, \text{ on } (a, b)\}.$$

We will be concerned with solutions of boundary value problems associated with the linear equations

$$x''' = u_1(t)x + u_2(t)x' + u_3(t)x'' \tag{11}$$

satisfying (8), (9), or (10), where $(u_1(t), u_2(t), u_3(t)) \in U$.

In much of the discussion that follows, we assume that solutions of (1), (2) are not unique. Then there exist distinct solutions $v(t)$ and $w(t)$ of (1) satisfying conditions (2) for some points $a < t_1 \le t_2 \le t_3 < b$. Define the three functions,

$$\begin{aligned} h_1(t) &= f(t, v(t), v'(t), v''(t)) - f(t, w(t), v'(t), v''(t)), \\ h_2(t) &= f(t, w(t), v'(t), v''(t)) - f(t, w(t), w'(t), v''(t)), \\ h_3(t) &= f(t, w(t), w'(t), v''(t)) - f(t, w(t), w'(t), w''(t)), \end{aligned}$$

and, for $i = 1, 2, 3$, define

$$\tilde{u}_i(t) = \begin{cases} \frac{h_i(t)}{v^{(i-1)}(t) - w^{(i-1)}(t)}, & \text{for } v^{(i-1)}(t) \ne w^{(i-1)}(t), \\ 0, & \text{for } v^{(i-1)}(t) = w^{(i-1)}(t). \end{cases}$$

Then $\tilde{u}_i(t)$ is Lebesgue measurable and $|\tilde{u}_i(t)| \le k_i$ on (a,b), $i = 1,2,3$, and moreover, the difference $y(t) \equiv v(t) - w(t)$ is a nontrivial solution of the linear equation (11), for $\tilde{u} = (\tilde{u}_1(t), \tilde{u}_2(t), \tilde{u}_3(t))$, and satisfies (8), (9), or (10). As a consequence of Theorem 1, there exists a nontrivial solution of (11), for $\tilde{u}$, satisfying either (8) or (9), for some points $a < \tau_1 < \tau_2 < b$.

It follows from arguments in optimal control theory that there is a boundary value problem in the collection (11), (8) (or (11), (9)), which has a nontrivial time optimal solution (see Gamkrelidze [2, p. 147] or Lee and Markus [12, p. 259]); that is, there is at least one nontrivial $u^* \in U$ and $\tau_1 \le c < d \le \tau_2$ such that

$$x''' = u_1^*(t)x + u_2^*(t)x' + u_3^*(t)x'',$$

$$x'(c) = x(c) = x'(d) = 0, \quad (\text{or } x'(c) = x(d) = x'(d) = 0),$$

has a nontrivial solution $x(t)$, and $d - c$ is a minimum over all such solutions. For this time optimal solution $x(t)$, let $z(t) = (x(t), x'(t), x''(t))^T$. Then $z(t)$ is a time optimal solution of the first order system

$$z' = A[u^*(t)]z$$

satisfying

$$z_2(c) = z_1(c) = z_2(d) = 0, \quad (\text{or } z_2(c) = z_1(d) = z_2(d) = 0).$$

By the Pontryagin Maximum Principle, associated with this time optimal solution is a nontrivial time optimal solution $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))^T$ of the adjoint equation

$$\gamma' = -A^T[u^*(t)]\gamma$$

satisfying the complementary conditions

$$\gamma_3(c) = \gamma_1(d) = \gamma_3(d) = 0, \quad (\text{or } \gamma_3(c) = \gamma_1(c) = \gamma_3(d) = 0).$$

Furthermore, from the Pontryagin Maximum Principle, the inner product $\langle z'(t), \gamma(t)\rangle$ is a nonnegative constant for almost all $t \in [c,d]$, and

$$\langle z'(t), \gamma(t)\rangle = \max_{u \in U} \langle A[u(t)]z(t), \gamma(t)\rangle$$

for almost all $t \in [c,d]$, which from the nature of $z(t)$ can be written as

$$\gamma_3(t)\sum_{i=1}^{3} u_i^*(t)x^{(i-1)}(t) = \max_{u \in U}\{\gamma_3(t)\sum_{i=1}^{3} u_i(t)x^{(i-1)}(t)\} \tag{12}$$

for almost all $t \in [c,d]$.

Now, from the time optimality of $x(t)$, (satisfying either conditions (8) or (9) at $c < d$), we have $x'(t) \ne 0$ on (c,d). Therefore $x(t) \ne 0$ on (c,d) (we may assume $x(t) > 0$ on (c,d)). Moreover, from the time optimality of $\gamma(t)$, we also conclude $\gamma_3(t) \ne 0$ on (c,d). As a consequence, (12) can be used to determine an optimal control $u^*(t)$, for almost all $t \in [c,d]$. In particular, if $x(t) > 0$ and $\gamma_3(t) < 0$ on (c,d), then the time optimal solution $x(t)$ is a solution of

$$x''' = -[k_1 x + k_2|x'| + k_3|x''|] \tag{13}$$

on $[c,d]$, while if $x(t) > 0$ and $\gamma_3(t) > 0$ on (c,d), then the time optimal solution $x(t)$ is a solution of

$$x''' = k_1 x + k_2|x'| + k_3|x''| \tag{14}$$

on $[c,d]$.

From this discussion, we have the following lemma.

Lemma 1. *If there is a vector $u \in U$ such that the corresponding linear equation (11) has a nontrivial solution satisfying (8), for some $a < \tau_1 < \tau_2 < b$, and if $x(t)$ is a time optimal solution with*

$$x'(c) = x(c) = x'(d) = 0,$$

and with a $d - c$ minimum, then $x(t)$ is a solution of (13) on $[c, d]$.

Proof: By the time optimality, $x'(t) \neq 0$ on (c, d); thus $x(t) \neq 0$ on $(c, d]$. Without loss of generality, we may also assume that $x''(c) > 0$ so that $x(t) > 0$ on $(c, d]$.

Now, if $\gamma(t)$ is a nontrivial optimal solution of the adjoint system associated with $x(t)$ by the Pontryagin Maximum Principle, then $\gamma_3(c) = \gamma_1(d) = \gamma_3(d) = 0$, and as observed above, by its own time optimality $\gamma_3(t) \neq 0$ on (c, d). Thus, $x(t)$ is a solution of (13) or (14) on $[c, d]$, so that $x'''(t)$ is of constant sign on (c, d). Consequently, $x''(t)$ is strictly monotone on $[c, d]$. This, in conjunction with the positivity of $x(t)$ and the boundary conditions, implies that $x''(t)$ is strictly decreasing, which in turn implies $x'''(t) < 0$ on (c, d). Therefore $x(t)$ is a solution of (13) on $[c, d]$. The proof is complete. ∎

We recall at this point that most of our discussion preceding Lemma 1 was based on the assumption that (1) had distinct solutions satisfying the same boundary conditions (2). That discussion and Lemma 1 are now applied to formulate optimal length intervals in terms of the Lipschitz coefficients k_1, k_2, k_3 on which solutions of (1), (2) are unique.

Theorem 2. *Let $h = h(k_1, k_2, k_3) > 0$ be the smallest positive number such that there is a solution $x(t)$ of the boundary value problem for (13) satisfying*

$$x'(0) = x(0) = x'(h) = 0,$$

with $x(t) > 0$ on $(0, h]$, or $h = +\infty$ if no such solution exists. Then each of the boundary value problems (1), (2) has at most one solution, provided $t_3 - t_1 < h$. Furthermore, this result is best possible for the class of all third order ordinary differential equations satisfying the Lipschitz condition (3).

Proof: First, since equation (13) is autonomous, it suffices to apply Lemma 1 by specifying boundary conditions at 0 and h (i.e., h corresponds to $d - c$ in Lemma 1).

Suppose to the contrary that there exist distinct solutions $v(t)$ and $w(t)$ of (1), (2) for some points $a < t_1 \le t_2 \le t_3 < b$ with $t_3 - t_1 < h$. Then $y(t) \equiv v(t) - w(t)$ is a nontrivial solution of the linear equation (11), for suitably defined $u = (u_1(t), u_2(t), u_3(t)) \in U$, and satisfies (8), (9), or (10). By Theorem 1, there is a nontrivial solution $z(t)$ of (11), for this u, satisfying either

(i) $z'(t_1) = z(t_1) = z'(t_3) = 0$, or

(ii) $z'(t_1) = z(t_3) = z'(t_3) = 0$.

Case (i) leads to a contradiction of Lemma 1, since $t_3 - t_1 < h$. For case (ii), it can be argued as in earlier parts of this section that there is a nontrivial solution $r(t)$ of (14) satisfying $r'(0) = r(k) = r'(k) = 0$, for some $0 < k < h$. Setting $s(t) = r(-t + k)$, then $s(t)$ is a nontrivial solution of (13) and satisfies $s'(0) = s(0) = s'(k) = 0$, where $0 < k < h$; again, a contradiction of Lemma 1.

Therefore, solutions of (1), (2) are unique on subintervals of length less than h. That this result is best possible follows from the fact that (13) satisfies the Lipschitz condition (3), and $x(t)$ in the statement of the theorem is a nontrivial solution satisfying (8) on $[0, h]$. However, it is the case that (13), (8) also has the zero solution.

4. Uniqueness, existence and optimality. In this section, we first show that if solutions to the boundary value problems (1), (2) are unique on an interval (a, b), then solutions of (1), (2) exist on (a,b). The proof of this utilizes standard shooting methods such as those used by Peterson [17] or Henderson [5], and we will give only a sketch of the proof.

Theorem 3. *Assume that solutions of the boundary value problems (1), (2) are unique on* (a,b). *Then there exist unique solutions of (1), (2) on* (a,b).

Proof: We remark first, from the uniqueness assumption concerning solutions (1), (2), that (1) is disconjugate on (a,b). That being the case, it follows that conjugate boundary value problems for (1) have unique solutions on (a,b); see [4, 11].

It suffices to show the existence of solutions of boundary value problems satisfying (1), (4) (then (1), (5) by analogy), and then (1), (6). We apply a shooting method successively. Let $y_1, y_2, y_3 \in \mathbf{R}$ be given throughout.

(1), (4): For this problem, let $a < t_1 < t_2 < b$ and let $y(t)$ be the solution of the conjugate boundary value problem for (1) satisfying

$$y'(t_1) = y_1, \quad y(t_1) = y_2, \quad y(t_2) = 0.$$

Define $S_1 \equiv \{z'(t_2) \mid z(t) \text{ is a solution of (1) and } z'(t_1) = y'(t_1),\ z(t_1) = y(t_1)\}$. Using the uniqueness of solutions of (1), (2) and the continuous dependence of solutions of (1), (4) on boundary conditions, it can be shown that $S_1 = \mathbf{R}$; see [5]. Thus choosing $y_3 \in S_1$, the corresponding solution $z(t)$ satisfies (1), (4).

(1), (5): The argument for this problem is completely analogous to the one given above.

(1), (6): For this problem, let $a < t_1 < t_2 < t_3 < b$ and let $y(t)$ be the solution of (1) satisfying conditions of type (4) given by

$$y'(t_1) = y_1, \quad y(t_1) = 0, \quad y'(t_3) = y_3.$$

This time, define $S_2 \equiv \{z(t_2) \mid z(t) \text{ is a solution of (1) and } z'(t_1) = y'(t_1),\ z'(t_3) = y'(t_3)\}$. Again, it can be shown that $S_2 = \mathbf{R}$. Thus, for $y_2 \in S_2$, the corresponding solution $z(t)$ satisfies (1), (6), and the proof is complete. ∎

As a consequence of Theorems 2 and 3, we can now state the theorem establishing optimal length subintervals of (a,b), in terms of the Lipschitz coefficients k_i, $i = 1,2,3$, on which there exist unique solutions of (1), (2).

Theorem 4. *Let* $h = h(k_1, k_2, k_3)$ *be the positive number defined in Theorem 2. Then each of the boundary value problems (1), (2) has a unique solution, provided* $t_3 - t_1 < h$. *Furthermore, this result is best possible for the class of all third order ordinary differential equations satisfying the Lipschitz condition (3).*

Example. In this example, suppose the Lipschitz coefficients satisfy $k_1 = k_2 = k_3 = 1$. From Theorem 4, boundary value problems (1), (2) have unique solutions on any open subinterval of (a,b) of length less than h, where $x(t)$ is the solution of the initial value problem

$$\begin{aligned} x''' &= -[x + |x'| + |x''|], \\ x(0) &= x'(0) = 0, \quad x''(0) = 1, \end{aligned}$$

and $h > 0$ is the first positive number such that $x'(h) = 0$. We obtain the best possible result to be $h = 1.923239$.

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