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# Generalized quasilinearization method for a second order three point boundary-value problem with nonlinear boundary conditions \*

Bashir Ahmad, Rahmat Ali Khan, & Paul W. Eloe

## Abstract

The generalized quasilinearization technique is applied to obtain a monotone sequence of iterates converging uniformly and quadratically to a solution of three point boundary value problem for second order differential equations with nonlinear boundary conditions. Also, we improve the convergence of the sequence of iterates by establishing a convergence of order  $k$ .

## 1 Introduction

The method of quasilinearization pioneered by Bellman and Kalaba [1] and generalized by Lakshmikantham [8, 9] has been applied to a variety of problems [2, 10, 11, 12, 13, 16].

Multipoint boundary value problems for second order differential equations have also been receiving considerable attention recently. Kiguradze and Lomtatidze [7] and Lomtatidze [14, 15] have studied closely related problems. Gupta et.al. [4, 5, 6] have studied problems related to three point boundary value problems. More recently, Paul Eloe and Yang Gao [3] discussed the method of quasilinearization for a three point boundary value problem. In this paper, we develop the method of generalized quasilinearization for a three point boundary value problem involving nonlinear boundary conditions and obtain a monotone sequence of approximate solutions converging uniformly and quadratically to a solution of the problem. Also, we have discussed the convergence of order  $k$ .

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## Basic Results

Consider the three point boundary value problem with nonlinear boundary conditions

$$\begin{aligned}x'' &= f(t, x(t)), \quad t \in [0, 1] = J \\x(0) &= a, \quad x(1) = g(x(\frac{1}{2})),\end{aligned}\tag{1.1}$$

where  $f \in C[J \times \mathbb{R}, \mathbb{R}]$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is continuous. Let  $G(t, s)$  denote the Green's function for the conjugate or Dirichlet boundary value problem and is given by

$$G(t, s) = \begin{cases} t(s-1), & 0 \leq t < s \leq 1 \\ s(t-1), & 0 \leq s < t \leq 1. \end{cases}$$

We note that  $G(t, s) < 0$  on  $(0, 1) \times (0, 1)$ . If  $x(t)$  is the solution of (1.1) and (1.2), then

$$x(t) = a(1-t) + g(x(\frac{1}{2}))t + \int_0^1 G(t, s)f(s, x(s))ds.\tag{1.2}$$

Let  $\alpha, \beta \in C^2[0, 1]$ . We say that  $\alpha$  is a lower solution of the BVP (1.1), if

$$\begin{aligned}\alpha'' &\geq f(t, \alpha), \quad t \in [0, 1] \\ \alpha(0) &\leq a, \quad \alpha(1) \leq g(\alpha(\frac{1}{2})),\end{aligned}$$

and  $\beta$  be an upper solution of the BVP (1.1), if

$$\begin{aligned}\beta'' &\leq f(t, \beta), \quad t \in [0, 1] \\ \beta(0) &\geq a, \quad \beta(1) \geq g(\beta(\frac{1}{2})).\end{aligned}$$

Now, we state the following theorems without proof [3].

**Theorem 1.1** *Assume that  $f$  is continuous with  $f_x > 0$  on  $[0, 1] \times \mathbb{R}$  and  $g$  is continuous with  $0 \leq g' < 1$  on  $\mathbb{R}$ . Let  $\beta$  and  $\alpha$  be the upper and lower solutions of the BVP (1.1) respectively. Then  $\alpha(t) \leq \beta(t)$ ,  $t \in [0, 1]$ .*

**Theorem 1.2 (Method of upper and lower solutions)** *Assume that  $f$  is continuous on  $[0, 1] \times \mathbb{R}$  and  $g$  is continuous on  $\mathbb{R}$  satisfying  $0 \leq g' < 1$ . Further, we assume that there exists an upper solution  $\beta$  and a lower solution  $\alpha$  of the BVP (1.1) such that  $\alpha(t) \leq \beta(t)$ ,  $t \in [0, 1]$ . Then there exists a solution  $x$  of the BVP (1.1) such that*

$$\alpha(t) \leq x \leq \beta(t), \quad t \in [0, 1].$$

## 2 Main Result

### Theorem 2.1 (Generalized quasilinearization method)

(A<sub>1</sub>)  $f, f_x$  are continuous on  $[0, 1] \times \mathbb{R}$  and  $f_{xx}$  exists on  $[0, 1] \times \mathbb{R}$ . Further,  $f_x > 0$  and  $f_{xx} + \phi_{xx} \leq 0$ , where  $\phi, \phi_x$  are continuous on  $[0, 1] \times \mathbb{R}$  and  $\phi_{xx} \leq 0$ .

(A<sub>2</sub>)  $g, g'$  are continuous on  $\mathbb{R}$  and  $g''$  exists and  $0 \leq g' < 1, g''(x) \geq 0, x \in \mathbb{R}$ .

(A<sub>3</sub>)  $\alpha$  and  $\beta$  are lower and upper solutions of the BVP (1.1) respectively.

Then there exists a monotone sequence  $\{w_n\}$  of solutions converging quadratically to the unique solution  $x$  of the BVP (1.1).

**Proof.** Define  $F : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  as

$$F(t, x) = f(t, x) + \phi(t, x).$$

Then, in view of (A<sub>1</sub>), we note that  $F, F_x$  are continuous on  $[0, 1] \times \mathbb{R}$ , and  $F_{xx}$  exists such that

$$F_{xx}(t, x) \leq 0. \tag{2.1}$$

Using the mean value theorem and the assumptions (A<sub>1</sub>) and (A<sub>2</sub>), we obtain

$$f(t, x) \leq F(t, y) + F_x(t, y)(x - y) - \phi(t, x), \tag{2.2}$$

$$g(x) \geq g(y) + g'(y)(x - y), \tag{2.3}$$

where  $x, y \in \mathbb{R}$  such that  $x \geq y$  and  $t \in [0, 1]$ . Here, we remark that (2.2) and (2.3) are also valid independent of the requirement  $x \geq y$ . Define the functions  $F^*(t, x, y)$  and  $h(x, y)$  as

$$\begin{aligned} F^*(t, x, y) &= F(t, y) + F_x(t, y)(x - y) - \phi(t, x), \\ h(x, y) &= g(y) + g'(y)(x - y). \end{aligned}$$

We observe that

$$f(t, x) = \min_y F^*(t, x, y). \tag{2.4}$$

Further

$$\begin{aligned} F_x^*(t, x, y) &= F_x(t, y) - \phi_x(t, x) \geq F_x(t, x) - \phi_x(t, x) \\ &= f_x(t, x) > 0, \end{aligned} \tag{2.5}$$

implies that  $F^*(t, x, y)$  is increasing in  $x$  for each fixed  $(t, y) \in [0, 1] \times \mathbb{R}$ . Similarly

$$g(x) = \max_y h(x, y), \tag{2.6}$$

$$0 \leq h'(x, y) < 1. \tag{2.7}$$

Now, set  $\alpha = w_0$ , and consider the three point BVP

$$\begin{aligned} x'' &= \overset{*}{F}(t, x(t), w_0(t)), \quad t \in [0, 1] = J \\ x(0) &= a, \quad x(1) = h(x(\frac{1}{2}), w_0(\frac{1}{2})). \end{aligned} \quad (2.8)$$

Using  $(A_3)$  together with (2.4) and (2.6), we have

$$\begin{aligned} w_0'' &\geq f(t, w_0) = \overset{*}{F}(t, w_0, w_0), \quad t \in [0, 1] \\ w_0(0) &\leq a, \quad w_0(1) \leq g(w_0(\frac{1}{2})) = h(w_0(\frac{1}{2}), w_0(\frac{1}{2})), \end{aligned}$$

and

$$\begin{aligned} \beta'' &\leq f(t, \beta) \leq \overset{*}{F}(t, \beta, w_0), \quad t \in [0, 1] \\ \beta(0) &\geq a, \quad \beta(1) \geq g(\beta(\frac{1}{2})) \geq h(\beta(\frac{1}{2}), w_0(\frac{1}{2})), \end{aligned}$$

which imply that  $w_0$  and  $\beta$  are lower and upper solutions of the BVP (2.8) respectively. In view of (2.5) (2.7) and the fact that  $w_0$  and  $\beta$  are lower and upper solutions of the BVP (2.8) respectively, it follows by Theorems 1.1 and 1.2 that there exists a unique solution  $w_1$  of the BVP (2.8) such that

$$w_0(t) \leq w_1(t) \leq \beta(t), \quad t \in [0, 1].$$

Now, consider the BVP

$$\begin{aligned} x'' &= \overset{*}{F}(t, x(t), w_1(t)), \quad t \in [0, 1] = J \\ x(0) &= a, \quad x(1) = h(x(\frac{1}{2}), w_1(\frac{1}{2})). \end{aligned} \quad (2.9)$$

Again, using  $(A_3)$ , (2.4) and (2.6), we find that  $w_1$  and  $\beta$  are lower and upper solutions of (2.9) respectively, that is,

$$\begin{aligned} w_1'' &= \overset{*}{F}(t, w_1, w_0) \geq \overset{*}{F}(t, w_1, w_1), \quad t \in [0, 1] \\ w_1(0) &= a, \quad w_1(1) = h(w_1(\frac{1}{2}), w_0(\frac{1}{2})) \leq h(w_1(\frac{1}{2}), w_1(\frac{1}{2})), \end{aligned}$$

and

$$\begin{aligned} \beta'' &\leq f(t, \beta) \leq \overset{*}{F}(t, \beta, w_1), \quad t \in [0, 1] \\ \beta(0) &\geq a, \quad \beta(1) \geq g(\beta(\frac{1}{2})) \geq h(\beta(\frac{1}{2}), w_1(\frac{1}{2})). \end{aligned}$$

Hence, by Theorems 1.1 and 1.2, there exists a unique solution  $w_2$  of (2.9) such that

$$w_1(t) \leq w_2(t) \leq \beta(t), \quad t \in [0, 1].$$

Continuing this process successively, we obtain a monotone sequence  $\{w_n\}$  of solutions satisfying

$$w_0(t) \leq w_1(t) \leq \dots \leq w_n(t) \leq \beta(t), \quad t \in [0, 1],$$

where each element  $w_n$  of the sequence is a solution of the BVP

$$\begin{aligned} x'' &= F^*(t, x(t), w_{n-1}(t)), \quad t \in [0, 1] = J \\ x(0) &= a, \quad x(1) = h(x(\frac{1}{2}), w_{n-1}(\frac{1}{2})), \end{aligned}$$

and

$$w_n(t) = a(1-t) + h(w_n(\frac{1}{2}), w_{n-1}(\frac{1}{2}))t + \int_0^1 G(t, s) F^*(s, w_n, w_{n-1}) ds. \quad (2.10)$$

Employing the fact that  $[0, 1]$  is compact and the monotone convergence is pointwise, it follows that the convergence of the sequence is uniform. If  $x(t)$  is the limit point of the sequence, then passing onto the limit  $n \rightarrow \infty$ , (2.10) gives

$$\begin{aligned} x(t) &= a(1-t) + h(x(\frac{1}{2}), x(\frac{1}{2}))t + \int_0^1 G(t, s) F^*(s, x(s), x(s)) ds \\ &= a(1-t) + g(x(\frac{1}{2}))t + \int_0^1 G(t, s) f(s, x(s)) ds. \end{aligned}$$

Thus,  $x(t)$  is the solution of the BVP (1.1). Now, we show that the convergence of the sequence is quadratic. For that, set

$$e_n(t) = x(t) - w_n(t), \quad t \in [0, 1].$$

Observe that

$$\begin{aligned} e_n(t) &\geq 0, \quad e_n(0) = 0, \\ e_n(1) &= g(x(\frac{1}{2})) - h(w_n(\frac{1}{2}), w_{n-1}(\frac{1}{2})). \end{aligned}$$

Using the mean value theorem repeatedly,  $(A_1)$  and the nonincreasing property of  $F_x$ , we have

$$\begin{aligned} e''_{n+1}(t) &= x''(t) - w''_{n+1}(t) \\ &= f(t, x) - [F(t, w_n) + F_x(t, w_n)(w_{n+1} - w_n) - \phi(t, w_{n+1})] \\ &= F_x(t, c_1)(x - w_n) - F_x(t, w_n)(x - w_n) + F_x(t, w_n)(x - w_{n+1}) \\ &\quad - \phi_x(t, c_2)(x - w_{n+1}) \\ &= (F_{xx}(t, c_3)(c_1 - w_n)(x - w_n) + (F_x(t, w_n) - \phi_x(t, c_2))(x - w_{n+1})) \\ &\geq F_{xx}(t, c_3)(x - w_n)^2 + (F_x(t, c_2) - \phi_x(t, c_2))(x - w_{n+1}) \\ &= F_{xx}(t, c_3)(e_n)^2 + f_x(t, c_2)e_{n+1} \\ &\geq F_{xx}(t, c_2)(e_n)^2 \geq -M \|e_n\|^2, \end{aligned}$$

where  $M$  is a bound on  $F_{xx}(t, x)$  for  $t \in [0, 1]$ ,  $w_n < c_3 < c_1 < x(t)$ ,  $w_{n+1} < c_2 < x(t)$ , and  $\|\cdot\|$  denotes the supremum norm on  $C[0, 1]$ . Thus, we have

$$\begin{aligned}
 e_{n+1}(t) &= [g(x(\frac{1}{2})) - h(w_{n+1}(\frac{1}{2}), w_n(\frac{1}{2}))]t + \int_0^1 G(t, s)e''_{n+1}(s)ds \\
 &\leq [g(x(\frac{1}{2})) - g(w_n(\frac{1}{2})) - g'(w_n(\frac{1}{2}))(w_{n+1}(\frac{1}{2}) - w_n(\frac{1}{2}))]t \\
 &\quad + \int_0^1 G(t, s)M\|e_n\|^2 ds \\
 &\leq [g'(c_o)(x(\frac{1}{2}) - w_n(\frac{1}{2})) - g'(w_n(\frac{1}{2}))(w_{n+1}(\frac{1}{2}) - w_n(\frac{1}{2}))]t \\
 &\quad + M\|e_n\|^2 \int_0^1 |G(t, s)| ds \\
 &= [g''(c_1)(c_o - w_n(\frac{1}{2}))(x(\frac{1}{2}) - w_n(\frac{1}{2})) + g'(w_n(\frac{1}{2}))e_{n+1}]t \\
 &\quad + M_1\|e_n\|^2 \\
 &\leq [g''(c_1)e_n^2(t) + g'(w_n(\frac{1}{2}))e_{n+1}]t + M_1 \|e_n\|^2,
 \end{aligned}$$

where  $w_n(\frac{1}{2}) < c_1 < c_o < x(\frac{1}{2})$ . Taking the maximum over the interval  $[0, 1]$ , we get

$$\|e_{n+1}\| \leq M_2\|e_n\|^2 + \lambda\|e_{n+1}\| + M_1\|e_n\|^2.$$

Solving algebraically, we get

$$\|e_{n+1}\| \leq \frac{M_3}{1 - \lambda}\|e_n\|^2,$$

where,  $|g'| \leq \lambda < 1$ ,  $M_1$  provides a bound on  $M \int_0^1 |G(t, s)| ds$ ,  $M_2$  provides a bound for  $|g''|$  on  $[w_n(\frac{1}{2}), x(\frac{1}{2})]$  and  $M_3 = M_1 + M_2$ . This establishes the quadratic convergence.

### 3 Rapid Convergence

**Theorem 3.1** *Assume that*

(B<sub>1</sub>)  $\frac{\partial^i}{\partial x^i} f(t, x)$  ( $i = 0, 1, 2, \dots, k$ ) are continuous on  $[0, 1] \times \mathbb{R}$  satisfying

$$\begin{aligned}
 \frac{\partial^i}{\partial x^i} f(t, x) &\geq 0, \quad (i = 0, 1, 2, \dots, k - 1) \\
 \frac{\partial^k}{\partial x^k} (f(t, x) + \phi(t, x)) &\leq 0,
 \end{aligned}$$

where  $\frac{\partial^i}{\partial x^i} \phi(t, x)$  ( $i = 0, 1, 2, \dots, k$ ) are continuous and  $\frac{\partial^k}{\partial x^k} \phi(t, x) < 0$  for some function  $\phi(t, x)$ .

(B<sub>2</sub>)  $\alpha, \beta \in C^2[J, \mathbb{R}]$  are lower and upper solutions of the BVP (1.1).

(B<sub>3</sub>)  $\frac{d^i}{dx^i}g(x)$  ( $i = 0, 1, 2, \dots, k$ ) are continuous on  $\mathbb{R}$  satisfying

$$0 \leq \frac{d^i}{dx^i}g(x) < \frac{M}{(\beta - \alpha)^{i-1}},$$

with  $0 < M < \frac{1}{3}$  and  $\frac{d^k}{dx^k}g(x) \geq 0$ .

Then there exists a monotone sequence of solutions  $\{w_n\}$  that converge to the unique solution,  $x$ , of the BVP (1.1) with the order of convergence  $k \geq 2$ .

**Proof.** Define  $F : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  as

$$F(t, x) = f(t, x) + \phi(t, x).$$

Using (B<sub>1</sub>), (B<sub>3</sub>) and the generalized mean value theorem, we obtain

$$\begin{aligned} f(t, x) &\leq \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} F(t, y) \frac{(x-y)^i}{i!} - \phi(t, x), \\ g(x) &\geq \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^i}{i!}. \end{aligned}$$

Define

$$F^{**}(t, x, y) = \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} F(t, y) \frac{(x-y)^i}{i!} - \phi(t, x), \tag{3.1}$$

and

$$h^*(x, y) = \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^i}{i!}. \tag{3.2}$$

Observe that  $F^{**}(t, x, y)$  and  $h^*(x, y)$  are continuous and further

$$f(t, x) = \min_y F^{**}(t, x, y), \tag{3.3}$$

$$g(x) = \max_y h^*(x, y). \tag{3.4}$$

Using generalized mean value theorem, (3.1) can be written as

$$F^{**}(t, x, y) = \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, y) \frac{(x-y)^i}{i!} - \frac{\partial^k}{\partial x^k} \phi(t, \xi) \frac{(x-y)^k}{k!}. \tag{3.5}$$

Differentiating (3.5) and using (B<sub>1</sub>), we get

$$F_x^{**}(t, x, y) > \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, y) \frac{(x-y)^{i-1}}{(i-1)!} \geq 0, \tag{3.6}$$



which implies that  $F^{**}(t, x, y)$  is increasing in  $x$  for each  $(t, y) \in [0, 1] \times \mathbb{R}$ . Similarly, differentiation of (3.2), in view of  $(B_3)$ , yields

$$h^*(x, y) = \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^{i-1}}{(i-1)!}.$$

Clearly  $h^*(x, y) \geq 0$  and

$$\begin{aligned} h^*(x, y) &= \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^{i-1}}{(i-1)!} \leq \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(\beta-\alpha)^{i-1}}{(i-1)!} \\ &\leq \sum_{i=0}^{k-1} \frac{M}{(i-1)!} < M(1 + \sum_{i=0}^{k-2} \frac{1}{2^{k-1}}) = M(3 - \frac{1}{2^{k-3}}) \\ &< 3M < 1. \end{aligned}$$

Now, set  $\alpha = w_0$ , and consider the linear BVP

$$\begin{aligned} x'' &= F^{**}(t, x(t), w_0(t)), \quad t \in [0, 1] = J \\ x(0) &= a, \quad x(1) = h^*(x(\frac{1}{2}), w_0(\frac{1}{2})). \end{aligned} \tag{3.7}$$

Using  $(B_2)$ , (3.3) and (3.4), we find that

$$\begin{aligned} w_0'' &\geq f(t, w_0) = F^{**}(t, w_0, w_0), \quad t \in [0, 1] \\ w_0(0) &\leq a, \quad w_0(1) \leq g(w_0(\frac{1}{2})) = h^*(w_0(\frac{1}{2}), w_0(\frac{1}{2})), \end{aligned}$$

and

$$\begin{aligned} \beta'' &\leq f(t, \beta) \leq F^{**}(t, \beta, w_0), \quad t \in [0, 1] \\ \beta(0) &\geq a, \quad \beta(1) \geq g(\beta(\frac{1}{2})) \geq h^*(\beta(\frac{1}{2}), w_0(\frac{1}{2})), \end{aligned}$$

imply that  $w_0$  and  $\beta$  are lower and upper solutions of the BVP (3.7) respectively. It follows by Theorems 1.1 and 1.2 that there exists a unique solution  $w_1$  of the BVP (3.7) such that

$$w_0(t) \leq w_1(t) \leq \beta(t), \quad t \in [0, 1].$$

Continuing this process successively, we obtain a monotone sequence  $\{w_n\}$  of solutions satisfying

$$w_0(t) \leq w_1(t) \leq w_2(t) \leq \cdots \leq w_n(t) \leq \beta(t), \quad t \in [0, 1],$$

where each element  $w_n$  of the sequence is a solution of the BVP

$$\begin{aligned} x'' &= F^{**}(t, x(t), w_{n-1}(t)), \quad t \in [0, 1] = J \\ x(0) &= a, \quad x(1) = h^*(x(\frac{1}{2}), w_{n-1}(\frac{1}{2})), \end{aligned}$$

and is given by

$$w_n(t) = a(1-t) + {}^*h(w_n(\frac{1}{2}), w_{n-1}(\frac{1}{2}))t + \int_0^1 G(t,s) {}^{**}F(s, w_n, w_{n-1})ds. \quad (3.8)$$

Again, using the standard arguments employed in the last section, it follows that

$$\begin{aligned} x(t) &= a(1-t) + {}^*h(x(\frac{1}{2}), x(\frac{1}{2}))t + \int_0^1 G(t,s) {}^{**}F(s, x(s), x(s))ds \\ &= a(1-t) + g(x(\frac{1}{2}))t + \int_0^1 G(t,s)f(s, x(s))ds. \end{aligned}$$

Hence  $x(t)$  is the solution of the BVP (1.1). Now, we show that the convergence of the sequence of iterates is of order  $k \geq 2$ . For that, we set

$$e_n(t) = x(t) - w_n(t), \quad a_n(t) = w_{n+1}(t) - w_n(t), \quad t \in [0, 1].$$

Note that  $e_n(t) \geq 0, a_n(t) \geq 0, e_n(t) - a_n(t) = e_{n+1}(t)$ , and

$$e_n(0) = 0, \quad e_n(1) = g(x(\frac{1}{2})) - h(w_n(\frac{1}{2}), w_{n-1}(\frac{1}{2})).$$

Also,  $e_n(t) \geq a_n(t)$  and hence by induction  $e_n^k(t) \geq a_n^k(t)$ . Using the generalized mean value theorem, we have

$$\begin{aligned} e''_{n+1}(t) &= x'' - w''_{n+1} \\ &= \left[ \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_n) \frac{(x - w_n)^i}{i!} + \frac{\partial^k}{\partial x^k} f(t, \xi) \frac{(x - w_n)^k}{k!} \right] \\ &\quad - \left[ \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_n) \frac{(w_{n+1} - w_n)^i}{i!} - \frac{\partial^k}{\partial x^k} \phi(t, \xi) \frac{(w_{n+1} - w_n)^k}{k!} \right] \\ &= \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_n) \frac{(e_n^i - a_n^i)}{i!} + \frac{\partial^k}{\partial x^k} f(t, \xi) \frac{(e_n)^k}{k!} + \frac{\partial^k}{\partial x^k} \phi(t, \xi) \frac{(a_n)^k}{k!} \\ &\geq \left( \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_n) \frac{1}{i!} \sum_{j=0}^{k-1} e_n^j a_n^{i-1-j} \right) e_{n+1} + \left( \frac{\partial^k}{\partial x^k} f(t, \xi) + \frac{\partial^k}{\partial x^k} \phi(t, \xi) \right) \frac{(e_n)^k}{k!} \\ &\geq \frac{\partial^k}{\partial x^k} F(t, \xi) \frac{(e_n)^k}{k!} \geq -M \|e_n\|^k, \end{aligned} \quad (3.9)$$

where  $M$  is a bound on  $\frac{1}{k!} \frac{\partial^k}{\partial x^k} F(t, \xi)$  for  $t \in [0, 1]$ . Thus, in view of (3.9), we have

$$\begin{aligned}
& e_{n+1}(t) \\
&= \left( g\left(x\left(\frac{1}{2}\right)\right) - h\left(w_{n+1}\left(\frac{1}{2}\right), w_n\left(\frac{1}{2}\right)\right) \right) t + \int_0^1 G(t, s) e''_{n+1}(t) ds \\
&\leq \left( g\left(x\left(\frac{1}{2}\right)\right) - h\left(w_{n+1}\left(\frac{1}{2}\right), w_n\left(\frac{1}{2}\right)\right) \right) t + M \|e_n\|^k \int_0^1 |G(t, s)| ds \\
&= \left[ \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g\left(w_n\left(\frac{1}{2}\right)\right) \frac{\left(x\left(\frac{1}{2}\right) - w_n\left(\frac{1}{2}\right)\right)^i}{i!} + \frac{d^k}{dx^k} g\left(\xi\left(\frac{1}{2}\right)\right) \frac{\left(x\left(\frac{1}{2}\right) - w_n\left(\frac{1}{2}\right)\right)^k}{k!} \right. \\
&\quad \left. - \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g\left(w_n\left(\frac{1}{2}\right)\right) \frac{\left(w_{n+1}\left(\frac{1}{2}\right) - w_n\left(\frac{1}{2}\right)\right)^i}{i!} \right] t + M_1 \|e_n\|^k \\
&= \left[ \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g\left(w_n\left(\frac{1}{2}\right)\right) \frac{\left(e_n^i\left(\frac{1}{2}\right) - a_n\left(\frac{1}{2}\right)\right)^i}{i!} + \frac{d^k}{dx^k} g\left(\xi\left(\frac{1}{2}\right)\right) \frac{\left(e_n\left(\frac{1}{2}\right)\right)^k}{k!} \right] t + M_1 \|e_n\|^k \\
&= \left[ \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g\left(w_n\left(\frac{1}{2}\right)\right) \frac{1}{i!} \sum_{j=0}^{k-1} e_n^j\left(\frac{1}{2}\right) a_n^{i-1-j}\left(\frac{1}{2}\right) e_{n+1}\left(\frac{1}{2}\right) \right. \\
&\quad \left. + \frac{d^k}{dx^k} g\left(\xi\left(\frac{1}{2}\right)\right) \frac{\left(e_n\left(\frac{1}{2}\right)\right)^k}{k!} \right] t + M_1 \|e_n\|^k \tag{3.10} \\
&\leq \left[ \sum_{i=1}^{k-1} \frac{M}{(\beta - \alpha)^{i-1}} \frac{1}{i!} \sum_{j=0}^{i-1} e_n^{i-1-j}\left(\frac{1}{2}\right) a_n^j\left(\frac{1}{2}\right) \right] e_{n+1}\left(\frac{1}{2}\right) + M_2 \|e_n\|^k + M_1 \|e_n\|^k.
\end{aligned}$$

Letting

$$P_n(t) = \sum_{i=1}^{k-1} \frac{M}{(\beta - \alpha)^{i-1}} \frac{1}{i!} \sum_{j=0}^{i-1} e_n^{i-1-j}\left(\frac{1}{2}\right) a_n^j\left(\frac{1}{2}\right),$$

we observe that

$$\lim_{n \rightarrow \infty} P_n(t) = \lim_{n \rightarrow \infty} \sum_{i=1}^{k-1} \frac{M}{(\beta - \alpha)^{i-1}} \frac{1}{i!} \sum_{j=0}^{i-1} e_n^{i-1-j}\left(\frac{1}{2}\right) a_n^j\left(\frac{1}{2}\right) = M < \frac{1}{3}.$$

Therefore, we can choose  $\lambda < 1/3$  and  $n_0 \in \mathbb{N}$  such that for  $n \geq n_0$ , we have  $P_n(t) < \lambda$  and consequently (3.10) becomes

$$\|e_{n+1}\| < \lambda \|e_{n+1}\| + M_3 \|e_n\|^k.$$

Solving algebraically, we obtain

$$\|e_{n+1}\| \leq \frac{M_3}{1 - \lambda} \|e_n\|^k,$$

where  $M_3 = M_1 + M_2$ ,  $M_1$  provides bound for  $M \int_0^1 |G(t, s)| ds$ , and  $M_2$  provides bound for

$$\frac{d^k}{dx^k} g\left(\xi\left(\frac{1}{2}\right)\right) \frac{1}{k!}.$$

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