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Singular Nonlinear $(k, n - k)$ Conjugate Boundary Value Problems

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For $1 \leq k \leq n-1$, solutions are obtained for the boundary value problem, $(-1)^{n-k} y^{(n)} = f(x, y)$, $y^{(i)}(0) = 0$, $0 \leq i \leq k-1$, and $y^{(j)}(1) = 0$, $0 \leq j \leq n-k-1$, where $f(x, y)$ is singular at $y=0$. An application is made of a fixed point theorem for operators that are decreasing with respect to a cone. © 1997 Academic Press

1. INTRODUCTION

Let $1 \leq k \leq n-1$ be fixed. In this paper, we establish the existence of solutions for the $(k, n-k)$ conjugate boundary value problem,

$$(-1)^{n-k} y^{(n)} = f(x, y), \quad 0 < x < 1, \quad (1)$$

$$y^{(i)}(0) = 0, \quad 0 \leq i \leq k-1, \quad (2)$$

$$y^{(j)}(1) = 0, \quad 0 \leq j \leq n-k-1,$$

where $f(x, y)$ has a singularity at $y=0$. Our assumptions throughout are:

- (A) $f(x, y): (0, 1) \times (0, \infty) \rightarrow (0, \infty)$ is continuous,
- (B) $f(x, y)$ is decreasing in y , for each fixed x ,
- (C) $\int_0^1 f(x, y) dx < \infty$, for each fixed y ,

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(D) $\lim_{y \rightarrow 0^+} f(x, y) = \infty$ uniformly on compact subsets of $(0, 1)$, and

(E) $\lim_{y \rightarrow \infty} f(x, y) = 0$ uniformly on compact subsets of $(0, 1)$.

Singular nonlinear two-point boundary value problems appear frequently in applications and usually, only positive solutions are meaningful. This is especially true for the case $n = 2$, with Taliaferro [24] treating the general problem, Callegari and Nachman [4] considering existence questions in boundary layer theory, and Luning and Perry [19] obtaining constructive results for generalized Emden–Fowler problems. Results have also been obtained for singular boundary value problems arising in reaction-diffusion theory and in non-Newtonian fluid theory [5]. Higher order boundary value problems for ordinary differential equations are not documented as well as that for second order problems. Yet, higher order boundary value problems for ordinary differential equations arise naturally in technical applications. Frequently, these occur in the form of a multipoint boundary value problem for an n th order ordinary differential equation (or a system of n first order ordinary differential equations), such as an n -point boundary value problem model of a dynamical system with n degrees of freedom in which n states are observed at n times; see Meyer [21]. Possibly, the best known setting of a boundary value problem for a higher order ordinary differential equation involves two-point problems for the fourth order equation,

$$y^{(4)} - h(x)g(y) = 0,$$

which in certain cases arises in describing deformations of an elastic beam with the boundary conditions often reflecting both ends simply supported, or one end simply supported and the other end clamped by sliding clamps, while vanishing moments and shear forces at the rail ends are frequently included in the boundary conditions; for references, see Gupta [14] and Yang [27]. One derivation of this fourth order equation plus the two-point boundary conditions has resulted when the method of lines is used in the discretization over regions of certain partial differential equations describing the deflection of an elastic rail.

In close relation to the results of this paper is the recent work by Ma and Wang [20] in which they established the existence of at least one positive solution of the above fourth order equation satisfying $(2, 2)$ conjugate conditions for the cases when g is superlinear or sublinear.

It is noted by Meyer [21] that, strictly speaking, boundary value problems for higher order ordinary differential equations are a particular class of interface problems. One example in which this is exhibited is given by Keener [17] in determining the speed of a flagellate protozoan in a viscous fluid. Modelling the problem so as to describe an infinite swimming corrugated sheet, the relevant equations are the equations for a Stokes'

flow in the variables for the velocity vector for the fluid, the pressure and the viscosity of the fluid. A stream function is introduced which is shown to satisfy a fourth order partial differential equation known as the *biharmonic equation*. To find the fluid motion, the biharmonic equation is solved subject to the no-slip condition on the surface of the flagellum, with one of the spatial variables satisfying

$$\varphi^{(4)} - 2k^2\varphi'' + k^4\varphi = 0$$

subject to some nonhomogeneous (2, 2) conjugate boundary conditions.

Another particular case of a boundary value problem for a higher order ordinary differential equation arising as an interface problem is given by Wayner, *et al.* [26]. That paper deals with a study of perfectly wetting liquids. Using the concept of disjoining pressures a continuum description of thin film is characterized as having small interfacial slope and Reynold's numbers, which allowed for an application of a boundary layer approximation to describe the fluid mechanics in the thin film region. Combining the conservation equations of mass, momentum, and energy, a differential equation for the thin film thickness, h , results,

$$h^{(m)} = g(h, \dots, h^{(m-1)}, x),$$

where $m=2$ if only the disjoining pressure is included, while $m=4$ if the capillary pressure term is included.

We also mention in the context of boundary value problems for higher order ordinary differential equations, the so-called Blasius equation from laminar boundary layer theory,

$$u''' + u''u = 0,$$

where u' is the dimensionless velocity in the boundary layer of a flat plate in parallel laminar flow. The solution of this equation subject to the two-point boundary conditions, $u(0) = u'(0) = 0$, $u'(\infty) = 1$, is well-established in Evans [11].

A number of papers have been devoted to singular boundary value problems in which topological transversality methods were applied; see, for example [2, 3, 8, 18, 22, 23].

This paper arises as the completion of the work initiated by Eloë and Henderson [9] in which (1), (2) was dealt with for $k = n - 1$. The results and methods of [9] as well as this work are outgrowth of papers on second order singular boundary value problems by Gatica, Hernandez and Waltman [12] and Gatica, Olikier and Waltman [13] which in turn received some embellishment and generalization by Eloë and Henderson [7] and Henderson and Yin [15, 16]. In attempting to improve some of

these generalizations, the recent paper by Wang [25] did contain some flaws, however, that paper was corrected in a subsequent work by Agarwal and Wong [1].

We obtain solutions of (1), (2) by arguments involving positivity properties, an iteration, and a fixed point theorem due to [13] for mappings that are decreasing with respect to a cone in a Banach space. We remark that, for $n=2$, positive solutions of (1), (2) are concave. This concavity was exploited in [13], and later in the generalizations [1, 9, 15, 16, 25], in defining an appropriate subset of a cone on which a positive operator was defined to which the fixed point theorem was applied. The crucial property in defining this subset in [13] made use of an inequality that provides lower bounds on positive concave functions as a function of their maximum. Namely, this inequality may be stated as:

$$\begin{aligned} & \text{If } y \in C^{(2)}[0, 1] \text{ is such that } y(x) \geq 0, 0 \leq x \leq 1, \text{ and} \\ & \quad y''(x) \leq 0, 0 \leq x \leq 1, \text{ then} \\ & \quad y(x) \geq \frac{1}{4} \max_{0 \leq s \leq 1} |y(s)|, \quad \frac{1}{4} \leq x \leq \frac{3}{4}. \end{aligned} \quad (3)$$

Although (3) can be developed using concavity, it can also be obtained directly with the classical maximum principle. This observation was exploited by Eloe and Henderson [9]. Then in [10], a generalization of (3) was given for positive functions satisfying the boundary conditions (2).

In Section 2, we provide preliminary definitions and some properties of cones in a Banach space. We also state the fixed point theorem from [13] for mappings that are decreasing with respect to a cone. In that section, we state the generalization of (3) as it extends to solutions of (1), (2). An analogous inequality is also stated for a related Green's function.

In Section 3, we apply the generalization of (3) in defining a subset of a cone on which we define an operator which is decreasing with respect to the cone. A sequence of perturbations of f is constructed, with each term of the sequence lacking the singularity of f . In terms of this sequence, we define a sequence of decreasing operators to which the fixed point theorem yields a sequence of iterates. This sequence of iterates is shown to converge to a positive solution of (1), (2).

2. SOME PRELIMINARIES AND A FIXED POINT THEOREM

In this section, we first give definitions and some properties of cones in a Banach space. After that, we state a fixed point theorem due to [13] for operators that are decreasing with respect to a cone. We then state a

theorem from [10] generalizing (3) followed by an analogous inequality for a Green's function.

Let \mathcal{B} be a Banach space, and K a closed, nonempty subset of \mathcal{B} . K is a cone provided (i) $\alpha u + \beta v \in K$, for all $u, v \in K$ and all $\alpha, \beta \geq 0$, and (ii) $u, -u \in K$ imply $u = 0$. Given a cone K , a partial order, \leq , is induced on \mathcal{B} by $x \leq y$, for $x, y \in \mathcal{B}$ iff $y - x \in K$. (For clarity, we may sometimes write $x \leq y$ (wrt K).) If $x, y \in \mathcal{B}$ with $x \leq y$, let $\langle x, y \rangle$ denote the closed order interval between x and y given by, $\langle x, y \rangle = \{z \in \mathcal{B} \mid x \leq z \leq y\}$. A cone K is normal in B provided, there exists $\delta > 0$ such that $\|e_1 + e_2\| \geq \delta$, for all $e_1, e_2 \in K$, with $\|e_1\| = \|e_2\| = 1$.

Remark 1. If K is a normal cone in \mathcal{B} , then closed order intervals are norm bounded.

The following fixed point theorem can be found in [13].

THEOREM 1. *Let \mathcal{B} be a Banach space, K a normal cone in \mathcal{B} , $E \subseteq K$ such that, if $x, y \in E$ with $x \leq y$, then $\langle x, y \rangle \subseteq E$, and let $T: E \rightarrow K$ be a continuous mapping that is decreasing with respect to K , and which is compact on any closed order interval contained in E . Suppose there exists $x_0 \in E$ such that $T^2 x_0 = T(Tx_0)$ is defined, and furthermore, $Tx_0, T^2 x_0$ are order comparable to x_0 . If, either*

(I) $Tx_0 \leq x_0$ and $T^2 x_0 \leq x_0$, or $x_0 \leq Tx_0$ and $x_0 \leq T^2 x_0$, or

(II) The complete sequence of iterates $\{T^n x_0\}_{n=0}^\infty$ is defined, and there exists $y_0 \in E$ such that $Ty_0 \in E$ and $y_0 \leq T^n x_0$, for all $n \geq 0$,

then T has a fixed point in E .

We next observe that, if $y \in C^{(n)}[0, 1]$ such that $(-1)^{n-k} y^{(n)}(x) > 0$ on $(0, 1)$, and if y satisfies (2), then $y(x) = \int_0^1 G(x, s) y^{(n)}(s) ds$, where $G(x, s)$ is the Green's function for $y^{(n)} = 0$ and satisfying (2). It is well-known [6] that $(-1)^{n-k} G(x, s) > 0$ on $(0, 1) \times (0, 1)$, and hence it follows that $y(x) > 0$ on $(0, 1)$. It follows in turn, after successive applications of Rolle's Theorem, that $y(x)$ has one extreme point at, say $x_0 \in (0, 1)$. If we define a piecewise polynomial, p , by

$$p(x) = \begin{cases} \frac{|y|_\infty}{x_0^k} x^k, & 0 \leq x \leq x_0, \\ \frac{|y|_\infty}{(x_0 - 1)^{n-k}} (x - 1)^{n-k}, & x_0 \leq x \leq 1, \end{cases} \quad (4)$$

where $|y|_\infty = \sup_{0 \leq x \leq 1} |y(x)| = y(x_0)$, then Eloë and Henderson [10] established the following.

THEOREM 2. Assume $y \in C^{(n)}[0, 1]$ is such that $(-1)^{n-k} y^{(n)}(x) > 0$, $0 < x < 1$. Assume in addition that y satisfies the boundary conditions (2). Then $y(x) \geq p(x)$, $0 \leq x \leq 1$, where p is defined by (4).

Theorem 2 was used in [10] to give the following generalization of (3). This generalization will be fundamental in our future arguments.

THEOREM 3. Assume $y \in C^{(n)}[0, 1]$ is such that $(-1)^{n-k} y^{(n)}(x) \geq 0$, $0 \leq x \leq 1$, and y satisfies the boundary conditions (2). Then

$$y(x) \geq \frac{|y|_{\infty}}{4^m}, \quad \frac{1}{4} \leq x \leq \frac{3}{4}, \quad (5)$$

where $m = \max\{k, n-k\}$.

Remark 2. If y is a solution of (1), (2), then Theorems 2 and 3 apply to y .

For the sake of future reference, we restate if $G(x, s)$ is the Green's function for

$$y^{(n)} = 0, \quad 0 \leq x \leq 1, \quad (6)$$

satisfying (2), then

$$(-1)^{n-k} G(x, s) > 0 \quad \text{on } (0, 1) \times (0, 1), \quad (7)$$

and it is also known from [6] that both

$$(-1)^{n-k} \frac{\partial^k}{\partial x^k} G(0, s) > 0, \quad 0 < s < 1, \quad (8)$$

and

$$\frac{\partial^{n-k}}{\partial x^{n-k}} G(1, s) > 0, \quad 0 < s < 1. \quad (9)$$

For the remainder of the paper, for $0 < s < 1$, let $\tau(s) \in [0, 1]$ be defined by

$$(-1)^{n-k} G(\tau(s), s) = \sup_{0 \leq x \leq 1} (-1)^{n-k} G(x, s). \quad (10)$$

The following analogue of (5) for $G(x, s)$ was also obtained in [10].

THEOREM 4. *Let $G(x, s)$ denote the Green's function for (6), (2). Then, for $0 < s < 1$,*

$$(-1)^{n-k} G(x, s) \geq \frac{(-1)^{n-k}}{4^m} G(\tau(s), s), \quad \frac{1}{4} \leq x \leq \frac{3}{4}, \quad (11)$$

where $m = \max\{k, n - k\}$.

3. SOLUTIONS OF (1), (2)

In this section, we apply Theorem 1 to a sequence of operators that are decreasing with respect to a cone. The obtained fixed points provide a sequence of iterates which converges to a solution of (1), (2). Positivity of solutions and Theorems 2–4 are fundamental in this construction.

To that end, let the Banach space $\mathcal{B} = C[0, 1]$, with norm $\|y\| = |y|_\infty$, and let

$$K = \{y \in \mathcal{B} \mid y(x) \geq 0 \text{ on } [0, 1]\}.$$

K is a normal cone in \mathcal{B} .

To obtain a solution of (1), (2), we seek a fixed point of the integral operator,

$$T\varphi(x) = (-1)^{n-k} \int_0^1 G(x, s) f(s, \varphi(s)) ds,$$

where $G(x, s)$ is the Green's function for (6), (2). Due to the singularity of f given by (D), T is not defined on all of the cone K .

Next, define $g: [0, 1] \rightarrow [0, 1]$ by

$$g(x) = \begin{cases} (2x)^k, & 0 \leq x \leq \frac{1}{2}, \\ [2(1-x)]^{n-k}, & \frac{1}{2} \leq x \leq 1, \end{cases}$$

and for each $\theta > 0$, define $g_\theta(x) = \theta g(x)$. Then for the remainder of this work, assume the condition:

$$(F) \quad \text{For each } \theta > 0, \quad 0 < \int_0^1 f(x, g_\theta(x)) dx < \infty.$$

We remark, for each $\theta > 0$, that $g_\theta \in K$, $g_\theta(x) > 0$ on $(0, 1)$, and g_θ satisfies the boundary conditions (2).

Our first result of this section is a consequence of Theorem 2 and its proof in [10].

THEOREM 5. Let $y \in C^{(n)}[0, 1]$ be such that $(-1)^{n-k} y^{(n)} > 0$ on $(0, 1)$, and y satisfies (2). Then, there exists a $\theta > 0$ such that $g_\theta(x) \leq y(x)$ on $[0, 1]$.

Proof. Let y be as stated above and let $x_0 \in (0, 1)$ be the unique point in the statements preceding Theorem 2 such that $y(x_0) = |y|_\infty$. Then, by Theorem 2, $y(x) \geq p(x)$ on $[0, 1]$, where p is given by (4). Choosing $\theta = p(\frac{1}{2})$, then

$$p(x) \geq p(\frac{1}{2}) g(x) = g_\theta(x) \quad \text{on } [0, 1],$$

and so, $y(x) \geq g_\theta(x)$ on $[0, 1]$. ■

In view of Theorem 5, let $D \subseteq K$ be defined by

$$D = \{ \varphi \in \mathcal{B} \mid \text{there exists } \theta(\varphi) > 0 \text{ such that } g_\theta(x) \leq \varphi(x) \text{ on } [0, 1] \},$$

(i.e. $D = \{ \varphi \in \mathcal{B} \mid \text{there exists } \theta(\varphi) > 0 \text{ such that } g_\theta \leq \varphi(\text{wrt } K) \}$). Then, define $T: D \rightarrow K$ by

$$T(\varphi)(x) = (-1)^{n-k} \int_0^1 G(x, s) f(s, \varphi(s)) ds, \quad 0 \leq x \leq 1, \quad \varphi \in D.$$

Note that, from conditions (A)–(F) and properties of $G(x, s)$ in (7)–(9), if $\varphi \in D$, then $(-1)^{n-k} (T\varphi)^{(n)} > 0$ on $(0, 1)$, and $T\varphi$ satisfies the boundary conditions (2). Application of Theorem 5 yields that $T\varphi \in D$ so that $T: D \rightarrow D$. Moreover, if φ is a solution of (1), (2), then by Theorem 5 again, $\varphi \in D$. As a consequence, $\varphi \in D$ is a solution of (1), (2) if, and only if, $T\varphi = \varphi$.

Our next result establishes *a priori* bounds on solutions of (1), (2) which belong to D .

THEOREM 6. Assume that conditions (A)–(F) are satisfied. Then, there exists an $R > 0$ such that $\|\varphi\| = |\varphi|_\infty \leq R$, for all solutions, φ , of (1), (2) that belong to D .

Proof. Let $m = \max\{k, n-k\}$, and assume to the contrary that the conclusion is false. This implies there exists a sequence, $\{\varphi_l\} \subset D$, of solutions of (1), (2) such that $\lim_{l \rightarrow \infty} |\varphi_l| = \infty$. Without loss of generality, we may assume that, for each $l \geq 1$

$$|\varphi_l|_\infty \leq |\varphi_{l+1}|_\infty. \tag{12}$$

For each $l \geq 1$, let $x_l \in (0, 1)$ be the unique point from the statements preceding Theorem 2 such that

$$0 < \varphi_l(x_l) = |\varphi_l|_\infty,$$

and also from Theorem 3,

$$\varphi_l(x) \geq \frac{1}{4^m} \varphi_l(x_l), \quad \frac{1}{4} \leq x \leq \frac{3}{4}.$$

By the monotonicity in (12), $\varphi_l(x_l) \geq \varphi_1(x_1)$, for all l , and so

$$\varphi_l(x) \geq \frac{1}{4^m} \varphi_1(x_1), \quad \frac{1}{4} \leq x \leq \frac{3}{4} \quad \text{and} \quad l \geq 1. \quad (13)$$

Let $\theta = (1/4^m) \varphi_1(x_1)$. Then

$$g_\theta(x) \leq \frac{1}{4^m} \varphi_1(x_1) \leq \varphi_l(x), \quad \frac{1}{4} \leq x \leq \frac{3}{4} \quad \text{and} \quad l \geq 1. \quad (14)$$

We claim that $\varphi_l(x) \geq g_\theta(x)$, for $0 \leq x \leq \frac{1}{4}$. Let φ_l be given, and let p_l be the corresponding piecewise polynomial defined by (4) relative to φ_l and x_l . Then

$$\begin{aligned} p_l\left(\frac{1}{4}\right) &= \min \left\{ \frac{\varphi_l(x_l)}{x_l^k} \left(\frac{1}{4}\right)^k, \frac{\varphi_l(x_l)}{(1-x_l)^{n-k}} \left(\frac{1}{4}\right)^{n-k} \right\} \\ &\geq \frac{\varphi_l(x_l)}{4^m} \\ &\geq \frac{\varphi_1(x_1)}{4^m} \\ &\geq g_\theta\left(\frac{1}{4}\right). \end{aligned} \quad (15)$$

There are two cases for x_l :

- (i) Suppose $x_l \geq \frac{1}{4}$. Then, for $0 \leq x \leq \frac{1}{4}$,

$$\begin{aligned} p_l(x) &= \frac{|\varphi_l|_\infty}{x_l^k} x^k \\ &\geq |\varphi_l|_\infty x^k \\ &\geq |\varphi_1|_\infty x^k \\ &\geq \frac{|\varphi_1|_\infty}{4^m} x^k \\ &= g_\theta(x). \end{aligned} \quad (16)$$

(ii) Suppose $x_l < \frac{1}{4}$. Then, for $0 \leq x \leq x_l$, it follows exactly as in (16) that

$$p_l(x) \geq g_\theta(x).$$

On the other hand, on $[x_l, \frac{1}{4}]$, p_l is a decreasing function, g_θ is an increasing function, and $p_l(\frac{1}{4}) \geq g_\theta(\frac{1}{4})$ from (15). Thus, for $x_l \leq x \leq \frac{1}{4}$,

$$p_l(x) \geq p_l(\frac{1}{4}) \geq g_\theta(\frac{1}{4}) \geq g_\theta(x).$$

Thus, again for $0 \leq x \leq \frac{1}{4}$,

$$p_l(x) \geq g_\theta(x). \quad (17)$$

From (16) and (17), and recalling $\varphi_l(x) \geq p_l(x)$ on $[0, 1]$ by Theorem 2, it follows that

$$\varphi_l(x) \geq g_\theta(x), \quad 0 \leq x \leq \frac{1}{4},$$

and hence the claim. An analogous argument yields $\varphi_l(x) \geq g_\theta(x)$, $\frac{3}{4} \leq x \leq 1$. Thus, in conjunction with (14), we conclude

$$g_\theta(x) \leq \varphi_l(x), \quad 0 \leq x \leq 1 \quad \text{and} \quad l \geq 1.$$

Now, set

$$0 < M = \sup\{(-1)^{n-k} G(x, s) \mid (x, s) \in [0, 1] \times [0, 1]\}.$$

Then, assumptions (B) and (F) yield, for $0 \leq x \leq 1$ and all $l \geq 1$,

$$\begin{aligned} \varphi_l(x) &= T\varphi_l(x) \\ &= (-1)^{n-k} \int_0^1 G(x, s) f(s, \varphi_l(s)) ds \\ &\leq M \int_0^1 f(s, g_\theta(s)) ds \\ &= N, \end{aligned}$$

for some $0 < N < \infty$. In particular,

$$|\varphi_l|_\infty \leq N, \quad \text{for all } l \geq 1,$$

which contradicts $\lim_{l \rightarrow \infty} |\varphi_l|_\infty = \infty$. The proof is complete. \blacksquare

Remark 3. With R as in Theorem 6, $\varphi \leq R(\text{wrt } K)$, for all solutions $\varphi \in D$ of (1), (2).

Our next step in obtaining solutions of (1), (2) is to construct a sequence of nonsingular perturbations of f . For each $l \geq 1$, define $\psi_l: [0, 1] \rightarrow [0, \infty)$ by

$$\psi_l(x) = (-1)^{n-k} \int_0^1 G(x, s) f(s, l) ds.$$

By conditions (A)–(E), for $l \geq 1$,

$$0 < \psi_{l+1}(x) \leq \psi_l(x) \quad \text{on } (0, 1),$$

and

$$\lim_{l \rightarrow \infty} \psi_l(x) = 0 \quad \text{uniformly on } [0, 1]. \tag{18}$$

Now define a sequence of functions $f_l: (0, 1) \times [0, \infty) \rightarrow (0, \infty)$, $l \geq 1$, by

$$f_l(x, y) = f(x, \max\{y, \psi_l(x)\}).$$

Then, for each $l \geq 1$, f_l is continuous and satisfies (B). Furthermore, for $l \geq 1$,

$$\begin{aligned} f_l(x, y) &\leq f(x, y) && \text{on } (0, 1) \times (0, \infty), && \text{and} \\ f_l(x, y) &\leq f(x, \psi_l(x)) && \text{on } (0, 1) \times (0, \infty). \end{aligned} \tag{19}$$

THEOREM 7. *Assume that conditions (A)–(F) are satisfied. Then the boundary value problem (1), (2) has a solution $y \in D$.*

Proof. We begin by defining a sequence of operators $T_l: K \rightarrow K$, $l \geq 1$, by

$$T_l \varphi(x) = (-1)^{n-k} \int_0^1 G(x, s) f_l(s, \varphi(s)) ds.$$

Note that, for $l \geq 1$ and $\varphi \in K$, $(-1)^{n-k} (T_l \varphi)^{(n)}(x) > 0$ on $(0, 1)$, $T_l \varphi$ satisfies the boundary conditions (2), and $T_l \varphi(x) > 0$ on $(0, 1)$; in particular, $T_l \varphi \in D$. Since each f_l satisfies (B), it follows that, if $\varphi_1, \varphi_2 \in K$ with $\varphi_1 \leq \varphi_2$ (wrt K), then for $l \geq 1$, $T_l \varphi_2 \leq T_l \varphi_1$ (wrt K); that is, each T_l is decreasing with respect to K . It is also clear that $0 \leq T_l(0)$ and $0 \leq T_l^2(0)$ (wrt K), for each l .

Hence when we apply Theorem 1, for each l , there exists a $\varphi_l \in K$ such that $T_l \varphi_l = \varphi_l$. The above note implies, for $l \geq 1$, that $(-1)^{n-k} \varphi_l^{(n)}(x) > 0$ on $(0, 1)$, φ_l satisfies (2), and $\varphi_l(x) > 0$ on $(0, 1)$. In addition, inequality (19), coupled with the positivity of $(-1)^{n-k} G(x, s)$, yields $T_l \varphi \leq T \psi_l$ (wrt K), for each $\varphi \in K$ and $l \geq 1$. Thus,

$$\varphi_l = T_l \varphi_l \leq T \psi_l \text{ (wrt } K), \quad l \geq 1. \tag{20}$$

By essentially the same argument as in Theorem 6, in conjunction with inequality (14), it can be shown that there exists an $R > 0$ such that, for each $l \geq 1$,

$$\varphi_l \leq R(\text{wrt}K). \quad (21)$$

Our next claim is that there exists a $\kappa > 0$ such that $\kappa \leq |\varphi_l|_\infty$, for all l . We assume this claim to be false. Then, by passing to a subsequence and relabeling, we assume with no loss of generality that $\lim_{l \rightarrow \infty} |\varphi_l|_\infty = 0$. This implies

$$\lim_{l \rightarrow \infty} \varphi_l(x) = 0 \quad \text{uniformly on } [0, 1]. \quad (22)$$

Next set

$$0 < m = \inf\{(-1)^{n-k} G(x, s) \mid (x, s) \in [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}]\}.$$

By condition (D), there exists a $\delta > 0$ such that, for $\frac{1}{4} \leq x \leq \frac{3}{4}$ and $0 < y < \delta$,

$$f(x, y) > \frac{2}{m}.$$

The limit (22) implies there exists an $l_0 \geq 1$ such that, for $l \geq l_0$,

$$0 < \varphi_l(x) < \frac{\delta}{2}, \quad \text{for } 0 < x < 1.$$

Also, from (18), there exists an $l_1 \geq l_0$ such that, for $l \geq l_1$,

$$0 < \psi_l(x) < \frac{\delta}{2}, \quad \text{for } \frac{1}{4} \leq x \leq \frac{3}{4}.$$

Thus, for $l \geq l_1$ and $\frac{1}{4} \leq x \leq \frac{3}{4}$,

$$\begin{aligned} \varphi_l(x) &= (-1)^{n-k} \int_0^1 G(x, s) f_l(s, \varphi_l(s)) ds \\ &\geq (-1)^{n-k} \int_{1/4}^{3/4} G(x, s) f_l(s, \varphi_l(s)) ds \\ &\geq m \int_{1/4}^{3/4} f(s, \max\{\varphi_l(s), \psi_l(s)\}) ds \\ &\geq m \int_{1/4}^{3/4} f\left(s, \frac{\delta}{2}\right) ds \\ &\geq 1. \end{aligned}$$

But this contradicts the uniform limit (22). Hence, our claim is verified. That is, there exists a $\kappa > 0$ such that

$$\kappa \leq |\varphi_l|_\infty \leq R, \quad \text{for all } l.$$

Applying Theorem 3,

$$\begin{aligned} \varphi_l(x) &\geq \frac{1}{4^m} |\varphi_l|_\infty \\ &\geq \frac{\kappa}{4^m}, \quad \frac{1}{4} \leq x \leq \frac{3}{4}, \quad l \geq 1. \end{aligned}$$

One can mimic part of the proof of Theorem 6 to show, if $\theta = (\kappa/4^m)$, then

$$g_\theta(x) \leq \varphi_l(x) \quad \text{on } [0, 1], \quad \text{for } l \geq 1.$$

By (21), we now have

$$g_\theta \leq \varphi_l \leq R(\text{wrt } K), \quad \text{for } l \geq 1;$$

that is, the sequence $\{\varphi_l\}$ belongs to the closed order interval $\langle g_\theta, R \rangle \subset D$. When restricted to this closed order interval, T is a compact mapping, and so, there is a subsequence of $\{T\varphi_l\}$ which converges to some $\varphi^* \in K$. We relabel the subsequence as the original sequence so that $\lim_{l \rightarrow \infty} \|T\varphi_l - \varphi^*\| = 0$.

The final part of the proof is to establish that $\lim_{l \rightarrow \infty} \|T\varphi_l - \varphi_l\| = 0$. To this end, let $\theta = \kappa/4^m$ be as above, and set

$$0 < M = \sup\{(-1)^{n-k} G(x, s) \mid (x, s) \in [0, 1] \times [0, 1]\}.$$

Let $\varepsilon > 0$ be given. By the integrability condition (F), there exists $0 < \delta < 1$ such that

$$2M \left[\int_0^\delta f(s, g_\theta(s)) ds + \int_{1-\delta}^1 f(s, g_\theta(s)) ds \right] < \varepsilon.$$

Further, by (18), there exists an l_0 such that, for $l \geq l_0$,

$$\psi_l(x) \leq g_\theta(x) \quad \text{on } [\delta, 1 - \delta],$$

so that

$$\psi_l(x) \leq g_\theta(x) \leq \varphi_l(x) \quad \text{on } [\delta, 1 - \delta].$$

Observe also that, for $\delta \leq s \leq 1 - \delta$ and $l \geq l_0$,

$$f_l(s, \varphi_l(s)) = f(s, \varphi_l(s)).$$

Hence, for $l \geq l_0$ and $0 \leq x \leq 1$,

$$\begin{aligned} T\varphi_l(x) - \varphi_l(x) &= T\varphi_l(x) - T_l\varphi_l(x) \\ &= (-1)^{n-k} \int_0^\delta G(x, s) [f(s, \varphi_l(s)) - f_l(s, \varphi_l(s))] ds \\ &\quad + (-1)^{n-k} \int_{1-\delta}^1 G(x, s) [f(s, \varphi_l(s)) - f_l(s, \varphi_l(s))] ds. \end{aligned}$$

So, for $l \geq l_0$ and $0 \leq x \leq 1$,

$$\begin{aligned} |T\varphi_l(x) - \varphi_l(x)| &\leq M \left[\int_0^\delta [f(s, \varphi_l(s)) + f(s, \max\{\varphi_l(s), \psi_l(s)\})] ds \right. \\ &\quad \left. + \int_{1-\delta}^1 [f(s, \varphi_l(s)) + f(s, \max\{\varphi_l(s), \psi_l(s)\})] ds \right] \\ &\leq 2M \left[\int_0^\delta f(s, \varphi_l(s)) ds + \int_{1-\delta}^1 f(s, \varphi_l(s)) ds \right] \\ &\leq 2M \left[\int_0^\delta f(s, g_\theta(s)) ds + \int_{1-\delta}^1 f(s, g_\theta(s)) ds \right] \\ &< \varepsilon. \end{aligned}$$

In particular,

$$\lim_{l \rightarrow \infty} \|T\varphi_l - \varphi_l\| = 0.$$

In turn, we have $\lim_{l \rightarrow \infty} \|\varphi_l - \varphi^*\| = 0$, and thus

$$\varphi^* \in \langle g_\theta, R \rangle \subset D,$$

and

$$\varphi^* = \lim_{l \rightarrow \infty} T\varphi_l = T(\lim_{l \rightarrow \infty} \varphi_l) = T\varphi^*,$$

which is sufficient for the conclusion of the theorem. ■

REFERENCES

1. R. P. Agarwal and P. J. W. Wong, Existence of solutions for singular boundary value problems for higher order differential equations, *Nonlin. Anal.*, in press.
2. J. V. Baxley, A singular boundary value problem: Membrane response of a spherical cap, *SIAM J. Appl. Math.* **48** (1988), 497–505.
3. L. E. Bobisud, Existence of positive solutions to some nonlinear singular boundary value problems on finite and infinite intervals, *J. Math. Anal. Appl.* **173** (1993), 69–83.
4. A. Callegari and A. Nachman, Some singular nonlinear differential equations arising in boundary layer theory, *J. Math. Anal. Appl.* **64** (1978), 96–105.
5. A. Callegari and A. Nachman, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, *SIAM J. Appl. Math.* **38** (1980), 275–281.
6. W. A. Coppel, “Disconjugacy,” Lecture Notes in Mathematics, Vol. 220, Springer-Verlag, Berlin/New York, 1971.
7. P. W. Elloe and J. Henderson, Singular nonlinear boundary value problems for higher order ordinary differential equations, *Nonlin. Anal.* **17** (1991), 1–10.
8. P. W. Elloe and J. Henderson, Existence of solutions of some singular higher order boundary value problems, *Zeit. Angew. Math. Mech.* **73** (1993), 315–323.
9. P. W. Elloe and J. Henderson, Singular nonlinear $(n - 1, 1)$ conjugate boundary value problems, *Georgian Math. J.*, in press.
10. P. W. Elloe and J. Henderson, Inequalities based on a generalization of concavity, *Proc. Amer. Math. Soc.*, in press.
11. H. Evans, “Laminar Boundary Layer Theory,” Addison-Wesley, Reading, MA, 1968.
12. J. A. Gatica, G. E. Hernandez, and P. Waltman, Radially symmetric solutions of a class of singular elliptic problems, *Proc. Edinburgh Math. Soc.* **33** (1990), 169–180.
13. J. A. Gatica, V. Oliker, and P. Waltman, Singular nonlinear boundary value problems for second order ordinary differential equations, *J. Differential Equations* **79** (1989), 62–78.
14. C. P. Gupta, Existence and uniqueness results for the bending of an elastic beam equation at resonance, *J. Math. Anal. Appl.* **135** (1988), 208–225.
15. J. Henderson and K. C. Yin, Singular boundary value problems, *Bull. Inst. Math. Acad. Sinica* **19** (1991), 229–242.
16. J. Henderson and K. C. Yin, Singular focal boundary value problems, *Nonlinear Differential Equations* **1** (1995), 127–150.
17. J. P. Keener, “Principles of Applied Mathematics,” Addison-Wesley, Redwood City, CA, 1988.
18. L. B. Kong and J. Y. Wang, A remark on some singular nonlinear higher order boundary value problems, *Zeit. Angew. Math. Mech.*, in press.
19. C. D. Luning and W. L. Perry, Positive solutions of negative exponent generalized Emden-Fowler boundary value problems, *SIAM J. Appl. Math.* **12** (1981), 874–879.
20. R. Ma and H. Wang, On the existence of positive solutions of fourth-order ordinary differential equations, *Appl. Anal.* **59** (1995), 225–231.
21. G. H. Meyer, “Initial Value Methods for Boundary Value Problems,” Academic Press, New York, 1973.

22. D. O'Regan, Existence of positive solutions to some singular and non-singular second order boundary value problems, *J. Differential Equations* **84** (1990), 228–251.
23. D. O'Regan, "Theory of Singular Boundary Value Problems," World Scientific, Singapore, 1994.
24. S. Taliaferro, A nonlinear singular boundary value problem, *Nonlinear Anal.* **3** (1979), 897–904.
25. J. Y. Wang, A singular nonlinear boundary value problem for a higher order ordinary differential equation, *Nonlinear Anal.* **22** (1994), 1051–1056.
26. P. C. Wayner, Y. K. Kao, and L. V. LaCroix, The interline heat transfer coefficient of an evaporating wetting film, *Int. J. Heat Mass Transfer* **19** (1976), 487–492.
27. Y. S. Yang, Fourth-order, two-point boundary value problems, *Proc. Amer. Math. Soc.* **104** (1988), 175–180.