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Extending Uniqueness Implies Existence Results to Fractional Differential Equations

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Abstract

In 1967, Andrzej Lasota and Zdzisław Opial proved that under sufficient conditions, uniqueness of solutions for boundary value problems for a second-order ordinary differential equation implies their existence. Lloyd Jackson and Keith Schrader then proved an extension of this result for boundary value problems of third order. In proving the third-order case, this compactness theorem is applied as a key part of the proof. It states that under sufficient conditions, uniform boundedness of a sequence of solutions on a compact domain implies existence of a subsequence which converges uniformly with respect to its zeroth, first, and second derivatives. We present an extension of this compactness theorem to a fractional differential equation of all orders in the interval (2,3].

Dedication or Acknowledgements

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1 Introduction

In [6], Andrzej Lasota and Zdzisław Opial considered a nonlinear ordinary differential equation of the form

$$y''(x) = f(x, y(x), y'(x))$$

with the boundary conditions

$$y(x_1) = r_1, \quad y(x_2) = r_2$$

for some $x_1, x_2 \in (a,b) \subseteq \mathbb{R}$. They showed continuity of f, existence, uniqueness, and extension of solutions to initial-value problems over (a,b), and uniqueness of solutions to all boundary-value problems implies existence of solutions to all boundary-value problems. In [5], this result was generalized to a third-order differential equation. A complete treatment of this theory is found in [2]. We seek to extend this work to fractional differential equations of order $\beta \in (2,3]$. Here, we prove a generalization of an important compactness result due to Jackson and Schrader in [5]. We provide the basic definitions of fractional calculus below but refer to [1], [7], and [8] for a more detailed introduction.

2 Preliminary Notation

First, define the fractional integral of f of order α centered at x_1 by

$$(I_{x_1}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x} (x-s)^{\alpha-1} f(s) ds$$

when it exists. From here, we define the Riemann-Liouville fractional derivative of order α centered at x_1 by

$$D_{x_1}^{\alpha} f = D^{\lceil \alpha \rceil} I^{\lceil \alpha \rceil - \alpha} f$$

when this expression exists. Finally, define the Caputo fractional derivative of order α centered at x_1 by

$$D_{*x_1}^{\alpha} f = D_{x_1}^{\alpha} (f - T_{m-1})$$

where $m = \lceil \alpha \rceil$ and T_{m-1} is the Taylor approximation to f centered at x_1 of order $m-1$. Again, we only define this when the right hand expression exists and emphasize that the differential operator to the right of the equal sign is the Riemann-Liouville definition. We use the asterisk in the subscript to denote the Caputo definition. We now make definitions regarding our differential equation of interest.

Let $a < b$ and $\beta \in (2,3]$, which remain fixed for the remainder of this paper. For $x_1 \in (a,b)$ define $(1)_{x_1}$ by

$$D^{\beta}_{*x_1} y = f(x,y,y',y'')$$

where the $*$ in the subscript of the differential operator denotes the Caputo derivative. For $x_1 < x_2 < x_3$, $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in (a,b)\times\mathbb{R}$, define the boundary conditions $(2)_{(x_1,y_1),(x_2,y_2),(x_3,y_3)}$ by

$$y(x_1) = y_1, \quad y(x_2) = y_2, \quad y(x_3) = y_3$$

and the initial-value problem $(3)_{y_1,y_2,y_3}$ by

$$y(x_1) = y_1, \quad y'(x_1) = y_2, \quad y''(x_1) = y_3$$

Define condition (A) by

$$f \in C^0((a,b)\times\mathbb{R}^3).$$

Define condition (B) by

$$\forall (x_1,y_1,y_2,y_3), \text{ one and only one solution to } (1)_{x_1} - (3)_{y_1,y_2,y_3} \text{ exists.}$$

Define condition (C) to be that there exists at most one solution to the initial-value problem $(1)_{x_1} - (2)_{(x_1,y_1),(x_2,y_2),(x_3,y_3)}$.

3 Lemmas

Lemma 3.1. *Assume condition (A). Let $[c,d] \subseteq (a,b)$ and $M > 0$. Then there exists $\delta > 0$ such that for any $[x_1,x_2] \subseteq [c,d]$ with $x_2 - x_1 < \delta$ implies that for all $\alpha \in [-M,M]$, there exists a solution to $(1)_{x_1}$ satisfying the boundary conditions*

$$y(x_1) = y(x_2) = \alpha, \quad y'(x_1) = 0$$

or the boundary conditions

$$y(x_1) = y(x_2) = \alpha, \quad y'(x_2) = 0$$

Moreover, any such solution satisfies the condition

$$\forall x \in [x_1,x_2], \quad |y'(x)| \le 1, \quad |y''(x)| \le 1$$

Proof. Let $[c,d] \subseteq (a,b)$. Let $[x_1,x_2] \subseteq [c,d]$. Let $M > 0$. Define the operator $T : C^2[x_1,x_2] \to C^2[x_1,x_2]$ by

$$(Ty)(x) = \alpha + \int_{x_1}^{x_2} G(x,s) f(s,y(s),y'(s),y''(s))ds$$

where

$$G(x,s) = \begin{cases} \frac{1}{\Gamma(\beta)}\left((x-s)^{\beta-1} - \frac{(x-x_1)^2(x_2-s)^{\beta-1}}{(x_2-x_1)^2}\right) & s \leq x \\ \frac{-(x-x_1)^2(x_2-s)^{\beta-1}}{\Gamma(\beta)(x_2-x_1)^2} & x \leq s \end{cases}$$

Calculation 6.3 (see pp. 11) gives that $\forall x \in [x_1,x_2]$,

$$\int_{x_1}^{x_2} |G(x,s)|ds \leq K_{1,\beta}(x_2-x_1)^{\beta}$$

where $K_{1,\beta} = \frac{2}{\Gamma(\beta+1)}(x_2-x_1)^{\beta}$. For G_x, Calculation 6.3 gives

$$\int_{x_1}^{x_2} |G_x(x,s)|ds \leq K_{2,\beta}(x_2-x_1)^{\beta-1}$$

where $K_{2,\beta} = \frac{1+\frac{2}{\beta}}{\Gamma(\beta)} > 0$. For G_{xx}, Calculation 6.3 gives

$$\int_{x_1}^{x_2} |G_{xx}(x,s)|ds \leq K_{3,\beta}(x_2-x_1)^{\beta-2}$$

where $K_{3,\beta} = \frac{\beta-1+\frac{2}{\beta}}{\Gamma(\beta)} > 0$. Define

$$K = \{y \in C^2[x_1,x_2] : \|y\|_\infty \leq 2M, \|y'\|_\infty \leq 1, \|y''\|_\infty \leq 1\}.$$

Let $Q = max\{|f(x,u_1,u_2,u_3)| : x \in [x_1,x_2], |u_1| \leq 2M, |u_2| \leq 1, |u_3| \leq 1\}$, which is well-defined from condition (A) and compactness of the domain space in the definition of K. Now let

$$\delta = min\{(QK_{1,\beta})^{-\frac{1}{\beta}} M^{\frac{1}{\beta}}, (QK_{2,\beta})^{-\frac{1}{\beta-1}}, (QK_{3,\beta})^{-\frac{1}{\beta-2}}\} > 0.$$

Suppose $x_2 - x_1 < \delta$, $y \in K$. Now note

$$\begin{aligned} |(Ty)(x)| &\leq |\alpha| + Q\int_{x_1}^{x_2} |G(x,s)|ds \\ &\leq M + Q(K_{1,\beta}\delta^{\beta}) \\ &\leq M + M = 2M. \end{aligned}$$

Also,

$$
\begin{aligned}
|(Ty)'(x)| &\leq Q \int_{x_1}^{x_2} |G_x(x,s)| ds \\
&\leq Q K_{2,\beta} \delta^{\beta-1} \\
&\leq 1.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
|(Ty)''(x)| &\leq Q \int_{x_1}^{x_2} |G_{xx}(x,s)| ds \\
&\leq Q K_{3,\beta} \delta^{\beta-2} \\
&\leq 1.
\end{aligned}
$$

Therefore, T maps K into K. Since K is bounded and $T(K) \subseteq K$, $T(K)$ is bounded. Also, the zero function is in K, and K is clearly convex. Thus, we have all the assumptions needed to apply the Schauder fixed-point theorem except for the compactness condition, which we now prove. Let $\{y_n : n \in \mathbb{N}\} \subseteq K$. By continuity of G, G_x on $[x_1,x_2] \times [x_1,x_2]$ and compactness of $[x_1,x_2] \times [x_1,x_2]$, it follows that $\{Ty_n\}$, $\{(Ty_n)'\}$ are equicontinuous families of functions. By Calculation 6.4 (see pp. 13), $\{(Ty_n)''\}$ is an equicontinuous family of functions. By the Arzela-Ascoli Theorem, let $\{y_{n_j}\}$ be such that $\{Ty_{n_j}\}$ converges in the supremum norm. By the Arzela-Ascoli Theorem, let $\{y_{n_{j_k}}\}$ be such that $\{(Ty_{n_{j_k}})'\}$ converges in the supremum norm. By the Arzela-Ascoli Theorem, let $\{y_{n_{j_{k_l}}}\}$ be such that $\{(Ty_{n_{j_{k_l}}})''\}$ converges in the supremum norm. Therefore, $\{Ty_{n_{j_{k_l}}}\}$ converges in the norm $\|\cdot\|$ defined by $\|y\| = \|y\|_\infty + \|y'\|_\infty + \|y''\|_\infty$. Therefore, $T(K)$ is compact and is thus clearly contained in a compact subset of K. Therefore, we can apply the Schauder fixed-point theorem to see that T has a fixed point. Note that if $Ty = y$, we can rewrite as

$$
\begin{aligned}
y(x) &= \alpha + c_1(x-x_1) + c_2(x-x_1)^2 + I_{x_1}^{\beta} f \\
&\Rightarrow (D_{*x_1}^{\beta} y)(x) = 0 + 0 + 0 + f = f
\end{aligned}
$$

where the polynomial terms vanish since $\lceil \beta \rceil = 3 > 2$. Thus y is a solution to $(1)_{x_1}$. Note also that by Calculation 6.1 (see pp. 8), y satisfies the boundary conditions

$$
y(x_1) = y(x_2) = \alpha, \quad y'(x_1) = 0
$$

The condition bounding the function, derivative, and second derivative follows from the construction of K. Now consider the boundary conditions

$$y(x_1) = y(x_2) = \alpha, \quad y'(x_2) = 0.$$

This leads to a different Green's function G (see Calculation 6.2 for derivation on pp. 10) defined by

$$G(x,s) = \begin{cases} \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} + g(x,s) \text{ for } s \leq x \\ g(x,s) \text{ for } x \leq s \end{cases}$$

where

$$g(x,s) = (x-x_1)\left(\frac{(x_2-s)^{\beta-2}}{\Gamma(\beta-1)} - \frac{2(x_2-s)^{\beta-1}}{\Gamma(\beta)(x_2-x_1)}\right) + (x-x_1)^2\left(\frac{(x_2-s)^{\beta-1}}{\Gamma(\beta)(x_2-x_1)^2} - \frac{(x_2-s)^{\beta-2}}{\Gamma(\beta-1)(x_2-x_1)}\right).$$

This leads to

$$G_{xx}(x,s) = \begin{cases} \frac{(\beta-1)(\beta-2)}{\Gamma(\beta)}(x-s)^{\beta-3} + g_{xx}(x,s) \text{ for } s \leq x \\ g_{xx}(x,s) \text{ for } x \leq s \end{cases}$$

Now note that since g is a second order polynomial in x, we have that g_{xx} does not depend on x. This gives the same cancellation as with Calculation 6.4, i.e. the $\frac{(\beta-1)(\beta-2)(x-s)^{\beta-3}}{\Gamma(\beta)}$ term is the only one that does not vanish. Thus, proving equicontinuity for this case is the same since we will end up with similar cancellations. Following the exact same outline with this Green's function will yield the result for this set of boundary conditions and completes the proof. □

Lemma 3.2. *Let $y \in C^2[\alpha,\beta]$. Assume $\|y\|_\infty \leq M$, $M > 0$. Then there exists $K > 0$ depending on M and $\beta - \alpha$ such that if max$\{y'(x), y''(x)\} > K$, $\forall x \in [\alpha,\beta]$, then y' has a zero on (α,β).*

Proof. The details of the proof can be found in [2]. □

4 Generalization of the Kamke Theorem

Here we state a result, whose proof appears in [3]. The result is analogous to the Kamke Theorem for an arbitrary-order initial-value problem.

Theorem 4.1. *Assume $E \subset \mathbb{R} \times \mathbb{R}^n$, E open, and let $f_k : E \to \mathbb{R}$ denote a sequence of continuous functions that converge uniformly to a function f on every compact subset of E. For each $k \geq 1$, consider an initial value problem*

$$D^{\alpha}_{*x_k^*} y(x) = f_k(x, y(x), y'(x), \ldots, y^{(n-1)}(x)), \quad a < x_k^* < x < \omega_k, \tag{4.1}$$

$$y^{(i-1)}(x_k^*) = y_{ki}, \quad i = 1, \ldots, n, \tag{4.2}$$

and let $y_k(x)$ denote the solution of (4.1)-(4.2) on a maximal right interval I_k. Further, assume x_k^* is an increasing sequence and $x_k^* \to x^{*-}$ and assume there is $(y_1, \ldots, y_n) \in \mathbb{R}^n$ such that $(x_k^*, y_{k1}, \ldots, y_{kn}) \to (x^*, y_1, \ldots, y_n)$ as $k \to \infty$. Then there exists a solution $y(x)$ of the initial value problem (4.1)-(4.2) on a maximal right interval $I = [x^*, \omega)$ and there exists a subsequence $\{y_{k_l}\}$ of $\{y_k\}$ such that for each compact subset $J \subset [x^*, \omega)$, $\|y_{k_l} - y\|_{[n-1,J]} \to 0$ as $l \to \infty$.

We now state and prove a corollary to this theorem for the specific case $n = 3$, which is used in proving the main compactness theorem.

Corollary 4.1. *Assume $E \subset \mathbb{R} \times \mathbb{R}^2$, E open, and let $f : E \to \mathbb{R}$ satisfy (A)-(C). For each $k \geq 1$, consider an initial value problem*

$$D^{\alpha}_{*x_k^*} y(x) = f(x, y(x), y'(x), y''(x)), \quad a < x_k^* < x < b, \tag{4.3}$$

$$y^{(i-1)}(x_k^*) = y_{ki}, \quad i = 1, 2, 3, \tag{4.4}$$

and let $y_k(x)$ denote the solution of (4.3)-(4.4) on $[x^*, b)$. Further, assume x_k^* is an increasing sequence and $x_k^* \to x^{*-}$ and assume there is $(y_1, y_2, y_3) \in \mathbb{R}^3$ such that $(x_k^*, y_{k1}, y_{k2}, y_{k3}) \to (x^*, y_1, y_2, y_3)$ as $k \to \infty$. Then there exists a solution $y(x)$ of the initial value problem (4.3)-(4.4) on $I = [x^*, b)$ and there exists a subsequence $\{y_{k_l}\}$ of $\{y_k\}$ such that for each compact subset $J \subset [x^*, b)$, $\|y_{k_l} - y\|_{[n-1,J]} \to 0$ as $l \to \infty$.

Proof. Since condition (A) is satisfied, f satisfies the desired continuity property to apply the theorem. Since condition (B) is satisfied, it must be the case that $[x^*, \omega) = [x^*, b)$. □

5 Compactness Theorem

Theorem 5.1. *Let $(a,b) \subseteq \mathbb{R}$, and assume that for some $x^* \in (a,b)$, $(1)_{x^*}$ satisfies $(A)-(C)$. Let $[c,d] \subseteq [x^*,b)$, $\{y_n\}$ be a sequence of solutions of $(1)_{x^*}$ such that $\|y_n\|_\infty \leq M\ \forall n \in \mathbb{N}$. Then $\{y_n\}$ has a subsequence $\{y_{n_j}\}$ such that $\{y_{n_j}\}$ converges in the Banach space $(C^2[c,d], \|\cdot\|)$ where $\|\cdot\|$ is defined by $\|y\| = \|y\|_\infty + \|y'\|_\infty + \|y''\|_\infty$.*

Proof. Assume for the sake of contradiction that there does not exist a subsequence convergent in $(C^2[c,d], \|\cdot\|)$. Suppose for the sake of contradiction that $|y_n(x)| + |y_n'(x)| + |y_n''(x)|$ does not diverge to ∞ uniformly. Thus let $\varepsilon > 0$, $y_{n_j} \in \{y_n : n \in \mathbb{N}\}$, $x_{n_j} \in [c,d]$ be such that $\forall j \in \mathbb{N}$,

$$|y_{n_j}(x_{n_j})| + |y_{n_j}'(x_{n_j})| + |y_{n_j}''(x_{n_j})| \leq \varepsilon.$$

Since $[c,d]$ is compact, let $x_{n_{j_k}} \to x_1 \in [c,d]$. Note $\{y_{n_{j_k}}(x_{n_{j_k}}) : k \in \mathbb{N}\} \subseteq [-\varepsilon, \varepsilon]$, $\{y_{n_{j_k}}'(x_{n_{j_k}}) : k \in \mathbb{N}\} \subseteq [-\varepsilon, \varepsilon]$, $\{y_{n_{j_k}}''(x_{n_{j_k}}) : k \in \mathbb{N}\} \subseteq [-\varepsilon, \varepsilon]$. Thus relabeling if necessary, we apply the Bolzano-Weierstrass Theorem three more times to see that

$$\lim_{k\to\infty} x_{n_{j_k}} = x \in [c,d]$$
$$\lim_{k\to\infty} y_{n_{j_k}}(x_{n_{j_k}}) = y_1 \in [-\varepsilon, \varepsilon]$$
$$\lim_{k\to\infty} y_{n_{j_k}}'(x_{n_{j_k}}) = y_2 \in [-\varepsilon, \varepsilon]$$
$$\lim_{k\to\infty} y_{n_{j_k}}''(x_{n_{j_k}}) = y_3 \in [-\varepsilon, \varepsilon].$$

Therefore, it must be the case that $|y_n(x)| + |y_n'(x)| + |y_n''(x)| \to \infty$ uniformly. Since we assumed that $\{\|y_n\|_\infty\} \subseteq [0,M]$, $|y_n'(x)| + |y_n''(x)| \to \infty$ uniformly. The proof now follows the exact same argument as that provided in [5], but we supply the proof here to remain self-contained. Let $c \leq x_1 < x_2 < x_3 < x_4 \leq d$ be such that $x_4 - x_1 < \delta$, where δ is as stated in Lemma 3.2. By Lemma 3.2, we have that there exists $K > 0$ such that if $max\{y_n'(x), y_n''(x)\} > K$ for each $x \in [c,d]$, then y_n' has a zero on (x_1,x_2), on (x_2,x_3), and on (x_3,x_4) (taking $[\alpha,\beta] = [x_1,x_2], [\alpha,\beta] = [x_2,x_3]$, and $[\alpha,\beta] = [x_3,x_4]$ in three different cases). Consider first the case when $K > 1$. Applying the fact that $|y_n'(x)| + |y_n''(x)| \to \infty$ uniformly, we let $N \in \mathbb{N}$ be such that

$$max\{|y_N'(x)|, |y_N''(x)|\} > K$$

on $[c,d]$. Let $x_1 < t_1 < x_2 < t_2 < x_3 < t_3 < x_4$ be such that $y_N'(t_1) = y_N'(t_2) = y_N'(t_3) = 0$. Thus $|y_N''(t_i)| > K > 1$ for $i = 1,2,3$. We now consider two cases. First, consider the case when $y_N(t_i) = y_N(t_j)$ with $t_i < t_j$. Then we have that y_N is a solution to $(1)_{x^*}$ satisfying the boundary conditions

$$y(t_i) = y(t_j) = y_N(t_i), \quad y'(t_i) = 0.$$

Since by construction of δ and the fact that $t_j - t_i \leq t_4 - t_1 < \delta$, we can apply Lemma 3.1 and see that $\|y_N'\|_\infty \leq 1$, $\|y_N''\|_\infty \leq 1$. But this contradicts the fact that $\|y_N''\| \geq |y_N''(t_i)| > K > 1$, finishing this case. Now consider the case when $y_N(t_i) \neq y_N(t_j)$ for $t_i \neq t_j$. We assume that $y_N(t_1) < y_N(t_2) < y_N(t_3)$ since the exact same argument can be applied to any reordering of the three points. If $y_N''(t_2) > K$, then t_2 is a local minimum for y_N. By the intermediate value theorem and the fact that $y_N(t_1) < y_N(t_2)$, let $\tau_2 \in (t_1,t_2)$ be such that $y_N(\tau_2) = y_N(t_2)$. Thus we have a solution to $(1)_{x^*}$ satisfying the boundary conditions $y(\tau_2) = y(t_2) = y_N(t_2), \quad y'(t_2) = 0$, so we apply Lemma 3.2 in the same way to produce a contradiction. If $y_N''(t_2) < -K$, then t_2 is a local maximum for y_N. Since $y_N(t_2) < y_N(t_3)$, we again apply the intermediate value theorem to obtain $\tau_3 \in (t_2,t_3)$ such that $y_N(\tau_3) = y_N(t_2)$, producing the same contradiction. For the case $K < 1$, let $\phi = (2/K)y$ so that $|\phi''(t_2)| > 2 > 1$. Then this produces the same contradiction for a new differential equation with function $g = (2/K)f$. □

6 Calculations

Calculation 6.1. *Green's function for Lemma 2*

Proof. Remember that we are looking for $y(x_1) = y(x_2) = \alpha$, $y'(x_1) = 0$.

$$\begin{aligned} D^{\beta}_{*x_1} y &= f(x,y(x),y'(x),y''(x)) \\ &\Rightarrow y(x) = c_0 + c_1(x-x_1) + c_2(x-x_1)^2 + \\ &\frac{1}{\Gamma(\beta)} \int_{x_1}^{x} (x-s)^{\beta-1} f(s,y(s),y'(s),y''(s))ds \\ &\Rightarrow \alpha = y(x_1) = c_0 + 0 + 0 + 0 = c_0 \Rightarrow c_0 = \alpha \end{aligned}$$

Taking a derivative, we obtain

$$y'(x) = c_1 + 2c_2(x-x_1) + \frac{1}{\Gamma(\beta-1)} \int_{x_1}^{x} (x-s)^{\beta-2} f(s,y(s),y'(s),y''(s))ds$$

Since $y'(x_1) = 0$, this forces $c_1 = 0$, given that $\beta - 2 > 0 \Rightarrow (x-s)^{\beta-2}$ is continuous implying that the integral term vanishes. Now returning to the original integral equation and using the fact that $y(x_2) = \alpha$, we have

$$\begin{aligned}
\alpha &= y(x_2) \\
&= \alpha + c_2(x_2 - x_1)^2 + \frac{1}{\Gamma(\beta)} \int_{x_1}^{x} (x-s)^{\beta-1} f(s, y(s), y'(s), y''(s))ds \\
&\Rightarrow c_2 = \frac{-1}{\Gamma(\beta)(x_2-x_1)^2} \int_{x_1}^{x_2} (x_2-s)^{\beta-1} f(s, y(s), y'(s), y''(s)ds \\
&\Rightarrow y(x) = \alpha - \frac{1}{\Gamma(\beta)(x_2-x_1)^2} \int_{x_1}^{x_2} (x-s)^{\beta-1} f(s, y(s), y'(s), y''(s))ds \\
&\qquad\qquad + \frac{1}{\Gamma(\beta)} \int_{x_1}^{x} (x-s)^{\beta-1} f(s, y(s), y'(s), y''(s))ds \\
&\Rightarrow y(x) = \alpha + \int_{x_1}^{x_2} G(x,s) f(s, y(s), y'(s), y''(s))ds
\end{aligned}$$

where

$$G(x,s) = \begin{cases} \frac{1}{\Gamma(\beta)} \left((x-s)^{\beta-1} - \frac{(x-x_1)^2(x_2-s)^{\beta-1}}{(x_2-x_1)^2} \right) & s \le x \\ \frac{-(x-x_1)^2(x_2-s)^{\beta-1}}{\Gamma(\beta)(x_2-x_1)^2} & x < s \end{cases}$$

This implies that G_x and G_{xx} are given by

$$G_x(x,s) = \begin{cases} \frac{1}{\Gamma(\beta)} \left((\beta-1)(x-s)^{\beta-2} - \frac{2(x-x_1)(x_2-s)^{\beta-1}}{(x_2-x_1)^2} \right) & s \le x \\ \frac{-2(x-x_1)(x_2-s)^{\beta-1}}{\Gamma(\beta)(x_2-x_1)^2} & x < s \end{cases}$$

$$G_{xx}(x,s) = \begin{cases} \frac{1}{\Gamma(\beta)} \left((\beta-1)(\beta-2)(x-s)^{\beta-3} - \frac{2(x_2-s)^{\beta-1}}{(x_2-x_1)^2} \right) & s \le x \\ \frac{-2(x_2-s)^{\beta-1}}{\Gamma(\beta)(x_2-x_1)^2} & x < s \end{cases}$$

□

Calculation 6.2. *Second Green's Function*

Proof. We are now looking for $y(x_1) = y(x_2) = \alpha$, $y'(x_2) = 0$. Now we have

$$\begin{aligned}(D^{\beta}_{*x_1})(x) &= f(x, y(x), y'(x), y''(x)) \\ &\Rightarrow y(x) = c_0 + c_1(x - x_1) + c_2(x - x_1)^2 \\ &+ \frac{1}{\Gamma(\beta)} \int_{x_1}^{x} (x-s)^{\beta-1} f(s, y(s), y'(s), y''(s)) ds \\ &\Rightarrow c_0 = y(x_1) = \alpha + 0 + 0 + 0 = \alpha \\ &\Rightarrow c_0 = \alpha\end{aligned}$$

So rewriting, we have

$$\begin{aligned}y(x) &= \alpha + c_1(x - x_1) + c_2(x - x_1)^2 \\ &+ \frac{1}{\Gamma(\beta)} \int_{x_1}^{x} (x-s)^{\beta-1} f(s, y(s), y'(s), y''(s)) ds\end{aligned}$$

Now plugging in $y(x_2) = \alpha$, we have

$$\begin{aligned}\alpha = y(x_2) &= \alpha + c_1(x_2 - x_1) + c_2(x_2 - x_1)^2 \\ &+ \frac{1}{\Gamma(\beta)} \int_{x_1}^{x_2} (x_2-s)^{\beta-1} f(s, y(s), y'(s), y''(s)) ds \qquad (*)\end{aligned}$$

Now taking a derivative we have

$$y'(x) = c_1 + 2c_2(x - x_1) + \frac{1}{\Gamma(\beta-1)} \int_{x_1}^{x} (x-s)^{\beta-2} f(s, y(s), y'(s), y''(s)) ds$$

Thus plugging in $y'(x_2) = 0$, we have

$$\begin{aligned}0 = y'(x_2) &= c_1 + 2c_2(x_2 - x_1) \\ &+ \frac{1}{\Gamma(\beta-1)} \int_{x_1}^{x_2} (x_2-s)^{\beta-2} f(s, y(s), y'(s), y''(s)) ds \\ &\Rightarrow 0 = c_1(x_2 - x_1) + 2c_2(x_2 - x_1)^2 \\ &+ \frac{x_2 - x_1}{\Gamma(\beta-1)} \int_{x_1}^{x_2} (x_2-s)^{\beta-2} f(s, y(s), y'(s), y''(s)) ds \qquad (**)\end{aligned}$$

Subtracting equation $(*)$ from $(**)$ gives

$$c_2 = \frac{1}{(x_2 - x_1)^2} \int_{x_1}^{x_2} \left(\frac{x_2 - s}{\Gamma(\beta)} - \frac{x_2 - x_1}{\Gamma(\beta-1)} \right) (x_2-s)^{\beta-2} f(s, y(s), y'(s), y''(s)) ds$$

Multiplying equation $(*)$ by 2 and then subtracting $(**)$ from it gives

$$c_1 = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \left(\frac{x_2 - x_1}{\Gamma(\beta - 1)} - \frac{2(x_2 - s)}{\Gamma(\beta)} \right) (x_2 - s)^{\beta - 2} f(s, y(s), y'(s), y''(s)) ds$$

Plugging these constants into our original equation for y gives

$$y(x) = \alpha + \int_{x_1}^{x_2} g(x,s) ds + \int_{x_1}^{x} \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} ds$$

where

$$\begin{aligned} g(x,s) = (x - x_1) &\left(\frac{(x_2 - s)^{\beta-2}}{\Gamma(\beta - 1)} - \frac{2(x_2 - s)^{\beta-1}}{\Gamma(\beta)(x_2 - x_1)} \right) \\ &+ (x - x_1)^2 \left(\frac{(x_2 - s)^{\beta-1}}{\Gamma(\beta)(x_2 - x_1)^2} - \frac{(x_2 - s)^{\beta-2}}{\Gamma(\beta - 1)(x_2 - x_1)} \right) \end{aligned}$$

Thus we have that

$$y(x) = \alpha + \int_{x_1}^{x_2} G(x,s) f(s, y(s), y'(s), y''(s)) ds$$

where

$$G(x,s) = \begin{cases} \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} + g(x,s) \text{ for } s \leq x \\ g(x,s) \text{ for } x \leq s \end{cases}$$

□

Calculation 6.3. *Derivation of* $K_{1,\beta}$, $K_{2,\beta}$, *and* $K_{3,\beta}$

Proof. We begin with $K_{1,\beta}$.

$$
\begin{aligned}
\int_{x_1}^{x_2} |G(x,s)|ds &= \frac{1}{\Gamma(\beta)}\left(\int_{x_1}^{x}(x-s)^{\beta-1}ds + \int_{x_1}^{x_2}\frac{(x-x_1)^2(x_2-s)^{\beta-1}}{(x_2-x_1)^2}ds\right)\\
&\le \frac{1}{\Gamma(\beta)}\left(\int_{x_1}^{x_2}(x_2-s)^{\beta-1}ds + \int_{x_1}^{x_2}\frac{(x_2-x_1)^2}{(x_2-x_1)^2}(x_2-s)^{\beta-1}ds\right)\\
&= \frac{2}{\Gamma(\beta)}\int_{x_2}^{x_1} -(x_2-s)^{\beta-1}ds\\
&= \frac{2}{\beta\Gamma(\beta)}(x_2-s)^{\beta}\Big|_{x_2}^{x_1}\\
&= \frac{2}{\Gamma(\beta+1)}(x_2-x_1)^{\beta}\\
&= K_{1,\beta}(x_2-x_1)^{\beta}
\end{aligned}
$$

Now for $K_{2,\beta}$, we have

$$
\begin{aligned}
\int_{x_1}^{x_2} |G_x(x,s)|ds &= \frac{\beta-1}{\Gamma(\beta)}\int_{x_1}^{x}(x-s)^{\beta-2}ds + \frac{1}{\Gamma(\beta)}\int_{x_1}^{x_2}\frac{2(x-x_1)(x_2-s)^{\beta-1}}{(x_2-x_1)^2}ds\\
&\le \frac{\beta-1}{\Gamma(\beta)}\int_{x_1}^{x_2}(x_2-s)^{\beta-2}ds + \frac{1}{\Gamma(\beta)}\int_{x_1}^{x_2}\frac{2(x_2-x_1)(x_2-s)^{\beta-1}}{(x_2-x_1)^2}ds\\
&= \frac{1}{\Gamma(\beta)}\left(\frac{\beta-1}{\beta-1}(x_2-s)^{\beta-1}\Big|_{x_2}^{x_1} + \frac{2}{\beta(x_2-x_1)}(x_2-s)^{\beta}\Big|_{x_2}^{x_1}\right)\\
&= \frac{1}{\Gamma(\beta)}(1+\frac{2}{\beta})(x_2-x_1)^{\beta-1}\\
&= K_{2,\beta}(x_2-x_1)^{\beta-1}
\end{aligned}
$$

Finally, for $K_{3,\beta}$, we have

$$\begin{aligned}
\int_{x_1}^{x_2} |G_{xx}(x,s)|ds &= \frac{(\beta-1)(\beta-2)}{\Gamma(\beta)} \int_{x}^{x_1} -(x-s)^{\beta-3} ds \\
&\quad + \frac{2}{\Gamma(\beta)(x_2-x_1)^2} \int_{x_2}^{x_1} -(x_2-s)^{\beta-1} ds \\
&= \frac{1}{\Gamma(\beta)} \left((\beta-1)(\beta-2) \frac{(x-s)^{\beta-2}}{\beta-2} \Big|_{x}^{x_1} + \frac{2}{(x_2-x_1)^2} \frac{(x_2-x_1)^{\beta}}{\beta} \right) \\
&= \frac{1}{\Gamma(\beta)} \left((\beta-1)(x-x_1)^{\beta-2} + \frac{2}{\beta}(x_2-x_1)^{\beta-2} \right) \\
&\leq \frac{1}{\Gamma(\beta)} \left((\beta-1)(x_2-x_1)^{\beta-2} + \frac{2}{\beta}(x_2-x_1)^{\beta-2} \right) \\
&= \frac{\beta-1+\frac{2}{\beta}}{\Gamma(\beta)} (x_2-x_1)^{\beta-2} \\
&= K_{3,\beta}(x_2-x_1)^{\beta-2}
\end{aligned}$$

□

Calculation 6.4. *Equicontinuity of* $\{\ (Ty)'' : y \in K\ \}$

Proof. Let $\varepsilon > 0$. Let $\delta_1, \delta_2 > 0$ be such that

$$|z_0 - z_1| < \delta_1 \Rightarrow (z_1 - x_1)^{\beta-2} - (z_0 - x_1)^{\beta-2} < \frac{\Gamma(\beta)\varepsilon}{4Q(\beta-1)}$$

$$|z_0 - z_1| < \delta_2 \Rightarrow (z_1 - z_0)^{\beta-2} < \frac{\Gamma(\beta)\varepsilon}{4Q(\beta-1)}$$

which exists given that $\beta - 2 > 0$. Let $\delta = min\{\delta_1, \delta_2\} > 0$. Let $x_1 \leq z_0 \leq z_1 \leq x_2$ with $|z_1 - z_0| = z_1 - z_0 < \delta > 0$. Now note for any $y \in K$, we can use the construction of Q to obtain

$$
\begin{aligned}
&|(Ty)''(z_1)-(Ty)''(z_0)| = \left|\int_{x_1}^{x_2} G_{xx}(z_1,s)-G_{xx}(z_0,s)\right| \\
&\leq \int_{x_1}^{x_2} |G_{xx}(z_1,s)-G_{xx}(z_0,s)||f(s,y(s),y'(s),y''(s))|ds \\
&\leq \frac{Q(\beta-1)(\beta-2)}{\Gamma(\beta)}\left(\int_{x_1}^{z_0}(z_0-s)^{\beta-3}-(z_1-s)^{\beta-3}ds+\int_{z_0}^{z_1}(z_1-s)^{\beta-3}ds\right) \\
&+\frac{Q(\beta-1)(\beta-2)}{\Gamma(\beta)}\int_{z_1}^{x_2} 0ds \\
&=\frac{Q(\beta-1)(\beta-2)}{\Gamma(\beta)}\left(\frac{1}{\beta-2}\right)\left((z_1-s)^{\beta-2}-(z_0-s)^{\beta-2}\Big|_{x_1}^{z_0}\right) \\
&+\frac{Q(\beta-1)(\beta-2)}{\Gamma(\beta)}\left(\frac{1}{\beta-2}(z_1-s)^{\beta-2}\Big|_{z_1}^{z_0}\right) \\
&=\frac{Q(\beta-1)}{\Gamma(\beta)}\left((z_1-z_0)^{\beta-2}-((z_1-x_1)^{\beta-2}-(z_0-x_1)^{\beta-2})+(z_1-z_0)^{\beta-2}\right) \\
&\leq \frac{Q(\beta-1)}{\Gamma(\beta)}\left(2(z_1-z_0)^{\beta-2}+((z_1-x_1)^{\beta-2}-(z_0-x_1)^{\beta-2})\right) \\
&\leq \frac{Q(\beta-1)}{\Gamma(\beta)}\left(2(\frac{\Gamma(\beta)\varepsilon}{4Q(\beta-1)})+\frac{\Gamma(\beta)\varepsilon}{4Q(\beta-1)}\right) \\
&=\frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\
&=\varepsilon.
\end{aligned}
$$

Thus $\{(Ty)'' : y \in K\}$ is equicontinuous. □

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