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POSITIVE SOLUTIONS FOR A SYSTEM OF SINGULAR SECOND ORDER NONLOCAL BOUNDARY VALUE PROBLEMS

NASEER AHMAD ASIF, PAUL W. ELOE, AND RAHMAT ALI KHAN

Abstract. Sufficient conditions for the existence of positive solutions for a coupled system of nonlinear nonlocal boundary value problems of the type

\[-x''(t) = f(t, y(t)), \quad t \in (0, 1),
\]

\[-y''(t) = g(t, x(t)), \quad t \in (0, 1),
\]

\[x(0) = y(0) = 0, \quad x(1) = \alpha x(\eta), \quad y(1) = \alpha y(\eta),
\]

are obtained. The nonlinearities \(f, g : (0, 1) \times (0, \infty) \to (0, \infty)\) are continuous and may be singular at \(t = 0, t = 1, x = 0,\) or \(y = 0.\) The parameters \(\eta, \alpha\) satisfy \(\eta \in (0, 1), 0 < \alpha < 1/\eta.\) An example is provided to illustrate the results.

1. Introduction

Nonlocal boundary value problems (BVPs) arise in different areas of applied mathematics and physics. For example, the vibration of a guy wire composed of \(N\) parts with a uniform cross section and different densities in different parts can be modeled as a nonlocal boundary value problem [18]; problems in the theory of elastic stability can also be modeled as nonlocal boundary value problems [19].

The study of nonlocal BVPs for linear second order ordinary differential equations was initiated by Il’in and Moiseev in [10, 11] and extended to nonlocal linear elliptic boundary value problems by Bitsadze and Samarskií, [2, 3, 4]. Existence theory for nonlinear three-point boundary value problems was initiated by Gupta [9]. Since then the study of nonlinear regular multi-point BVPs has attracted the attention of many researchers; see for example, [5, 9, 13, 14, 15, 17, 18, 20] for scalar equations, and for systems of ordinary differential equations, see [6, 7, 12].

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Recently, the study of singular BVPs has also attracted some attention. An excellent resource with an extensive bibliography was produced by Agarwal and O’Regan [1]. Recently, S. Xie and J. Zhu [21] applied topological degree theory in a cone to study the following two point BVP for a coupled system of nonlinear fourth-order ordinary differential equations

\[-x^{(4)} = f_1(t, y), \quad t \in (0, 1),\]
\[-y'' = f_2(t, x), \quad t \in (0, 1),\]
\[x(0) = x(1) = x''(0) = x''(1) = 0,\]
\[y(0) = y(1) = 0.\]

(1.1)

In [21], the nonlinearities $f_i \in C((0, 1) \times \mathbb{R}^+, \mathbb{R}^+)$ satisfy $f_i(t, 0) \equiv 0$ ($i = 1, 2$) and may be singular at $t = 0$ or $t = 1$ only.

More recently, Y. Zhou and Y. Xu [23] studied the following nonlocal BVP for a system of second order regular ordinary differential equations

\[-x''(t) = f(t, y), \quad t \in (0, 1),\]
\[-y''(t) = g(t, x), \quad t \in (0, 1),\]
\[x(0) = 0, \quad x(1) = \alpha x(\eta),\]
\[y(0) = 0, \quad y(1) = \alpha y(\eta),\]

(1.2)

where $\eta \in (0, 1), 0 < \alpha < 1/\eta$, $f, g \in C([0, 1] \times [0, \infty), [0, \infty))$, $f(t, 0) \equiv 0$, $g(t, 0) \equiv 0$. The above system was extended to the singular case by B. Liu, L. Liu, and Y. Wu [16], where the nonlinearities $f, g$ were assumed to be singular at $t = 0$ or $t = 1$ together with the assumption that $f(t, 0) \equiv 0$, $g(t, 0) \equiv 0$, $t \in (0, 1)$.

In this paper, we generalize the system (1.2) by allowing $f, g$ to be singular at $t = 0$, $t = 1$, $x = 0$, or $y = 0$ and obtain sufficient conditions for the existence of a positive solution of the BVP for the system of singular equations, (1.2). By singularity we mean that the functions $f(t, u)$ or $g(t, u)$ are allowed to be unbounded at $t = 0$, $t = 1$, or $u = 0$. In general, the assumption that there exist singularities with respect to the dependent variable is not new; see [1, 6], for example. However, in the case of nonlocal boundary conditions and coupled systems of ordinary differential equations, we believe this assumption is new.

Throughout this paper, we shall assume that

\[f, g : (0, 1) \times (0, \infty) \rightarrow (0, \infty)\]

are continuous and may be singular at $t = 0$, $t = 1$, or $u = 0$. We also assume that $f(t, 0), g(t, 0)$ are not identically 0. Let $N > \max\{\frac{1}{\eta}, \frac{1}{1-\eta}, \frac{2-\alpha}{1-\alpha\eta}\}$ denote a fixed positive integer. Assume that the following conditions hold:

\[(A_1) \text{ there exist } K, L \in C((0, 1), (0, \infty)) \text{ and } F, G \in C((0, \infty), (0, \infty)) \text{ such that}\]
\[f(t, u) \leq K(t)F(u), \quad g(t, u) \leq L(t)G(u), \quad t \in (0, 1), \quad u \in (0, \infty).\]
and
\[ \int_0^1 t(1-t)K(t)dt < +\infty, \quad b := \int_0^1 t(1-t)L(t)dt < +\infty; \]

(A2) there exist \( \alpha_1, \alpha_2 \in (0, +\infty) \) with \( \alpha_1 \alpha_2 \leq 1 \) such that
\[ \lim_{u \to \infty} \frac{F(u)}{u^{\alpha_1}} \to 0, \quad \lim_{u \to \infty} \frac{G(u)}{u^{\alpha_2}} \to 0; \]

(A3) there exist \( \beta_1, \beta_2 \in (0, +\infty) \) with \( \beta_1 \beta_2 \geq 1 \) such that
\[ \lim \inf_{u \to 0^+} \min_{t \in [\eta, 1]} \frac{f(t, u)}{u^{\beta_1}} > 0, \quad \lim \inf_{u \to 0^+} \min_{t \in [\eta, 1]} \frac{g(t, u)}{u^{\beta_2}} > 0; \]

(A4) \( f(t, u), G(u) \) are non-increasing with respect to \( u \) and for each fixed \( n \in \{N, N+1, N+2, \ldots\} \), there exists a constant \( M_1 > 0 \) such that
\[ f \left( t, \frac{1}{n} + b \mu_n G \left( \frac{1}{n} \right) \right) \geq M_1 \left( \nu_n \int_\eta^{1-1/n} (s-\frac{1}{n})(1-\frac{1}{n}-s)ds \right)^{-1}; \]

(A5) \( F(u), g(t, u) \) are non-increasing with respect to \( u \) and for each fixed \( n \in \{N, N+1, N+2, \ldots\} \), there exists a constant \( M_2 > 0 \) such that
\[ F \left( \nu_n \int_\eta^{1-1/n} (s-\frac{1}{n})(1-\frac{1}{n}-s)g(s, M_2)ds \right) \leq \frac{M_2 - \frac{1}{n}}{\mu_n}. \]

The parameters \( \mu_n \) and \( \nu_n \) in (A4) and (A5) are given by
\[ \mu_n = \frac{\max\{1, \alpha\}}{1 - \frac{2}{n} + \frac{\alpha}{n} - \alpha \eta}, \quad \nu_n = \min\{1, \alpha\} \min\{\eta - \frac{1}{n}, 1 - \frac{1}{n} - \eta\} \cdot \frac{1 - \frac{2}{n} + \frac{\alpha}{n} - \alpha \eta}{1 - \frac{2}{n} + \frac{\alpha}{n} - \alpha \eta}. \]

Since \( N > \max\{\frac{1}{n}, \frac{1}{n}, \frac{2-n}{1-\alpha n}\} \), \( \mu_n, \nu_n > 0 \).

We state the main results of this paper here.

**Theorem 1.1.** Assume that (A1) – (A3) hold. Then the system (1.1) has at least one positive solution.

**Theorem 1.2.** Assume that (A1), (A2) and (A4) hold. Then the system (1.1) has at least one positive solution.

**Theorem 1.3.** Assume that (A1), (A3) and (A5) hold. Then the system (1.1) has at least one positive solution.

**Theorem 1.4.** Assume that (A1), (A4) and (A5) hold. Then the system (1.1) has at least one positive solution.
For each $x \in C[0, 1]$ we write $\|x\| = \max \{|x(t)| : t \in [0, 1]\}$. Clearly, $C[0, 1]$ with the norm $\|\cdot\|$ is a Banach space. For $n \geq N$, define a cone $P$, and a cone $K_n$ of $C[\frac{1}{n}, 1 - \frac{1}{n}]$ as follows:

\[ P = \{ x \in C[0, 1] : x(t) \geq 0, t \in [0, 1] \}, \]
\[ P_n = \{ x \in P : x \text{ is concave on } [0, 1], \min_{t \in [\frac{1}{n}, 1 - \frac{1}{n}]} x(t) \geq \frac{1}{n} \}, \]
\[ K_n = \{ x \in C[\frac{1}{n}, 1 - \frac{1}{n}] : x \text{ is concave on } [0, 1] \}. \]

For any real constant $r > 0$, define

\[ \Omega_r = \{ x \in C[0, 1] : \|x\| < r \} \]
as an open neighborhood of $0 \in C[0, 1]$ of radius $r$. $(x(t), y(t))$ is called a positive solution of (1.1) if

\[ (x, y) \in (C[0, 1] \cap C^2(0, 1)) \times (C[0, 1] \cap C^2(0, 1)), \]
\[ x(t) > 0, y(t) > 0 \text{ on } (0, 1) \text{ and } (x, y) \text{ satisfies (1.1)}. \]

The proofs of our main results (Theorems 1.1-1.4) are based on the Guo-Krasnosel’skii fixed-point theorem.

**Lemma 2.1** ([8, Guo Krasnosel’skii Fixed-Point Theorem]). Let $K$ be a cone of a real Banach space $E$, and let $\Omega_1, \Omega_2$ be bounded open neighborhoods of $0 \in E$, and assume $\Omega_1 \subset \Omega_2$. Suppose that $T : K \cap (\Omega_2 \setminus \Omega_1) \to K$ is completely continuous such that one of the following conditions holds:

(i) $\|Tx\| \leq \|x\|$ for $x \in \partial \Omega_1 \cap K$; $\|Tx\| \geq \|x\|$ for $x \in \partial \Omega_2 \cap K$;

(ii) $\|Tx\| \leq \|x\|$ for $x \in \partial \Omega_2 \cap K$; $\|Tx\| \geq \|x\|$ for $x \in \partial \Omega_1 \cap K$.

Then, $T$ has a fixed point in $K \cap (\Omega_2 \setminus \Omega_1)$.

For fixed $n \geq N$ and $z \in C[0, 1]$, the linear boundary value problem

\[ -u''(t) = z(t), \quad t \in [\frac{1}{n}, 1 - \frac{1}{n}], \]
\[ u(\frac{1}{n}) = \frac{1}{n}, \quad u(1 - \frac{1}{n}) = au(\eta) + \frac{1-a}{n}, \]

has a unique solution

\[ u(t) = \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t, s)z(s)ds, \]
where \( H_n : \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \times \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \rightarrow [0, \infty) \) is an associated Green’s function and is defined by

\[
H_n(t, s) = \begin{cases} 
(t - \frac{1}{n})(1 - \frac{1}{n} - \alpha (q - s)) - (t - s), & \frac{1}{n} \leq s \leq t \leq 1 - \frac{1}{n}, s \leq \eta, \\
(t - \frac{1}{n})(1 - \frac{1}{n} - \alpha (q - s)) - \alpha \eta \left( \eta - (t - \frac{1}{n}) \right), & \frac{1}{n} \leq s \leq t \leq 1 - \frac{1}{n}, s \leq \eta, \\
(t - \frac{1}{n})(1 - \frac{1}{n} - \alpha (q - s)) - \alpha \eta \left( (1 - \frac{1}{n} - (t - \frac{1}{n})) \right), & 0 \leq s \leq t \leq 1 - \frac{1}{n}, s \geq \eta, \\
(t - \frac{1}{n})(1 - \frac{1}{n} - \alpha (q - s)) - (t - s), & 0 \leq s \leq t \leq 1 - \frac{1}{n}, s \geq \eta.
\end{cases}
\]

We note that \( H_n(t, s) \rightarrow H(t, s) \) as \( n \rightarrow \infty \), where

\[
H(t, s) = \begin{cases} 
\frac{(1 - s)}{1 - \alpha \eta} - \frac{\alpha(t - s)}{1 - \alpha \eta} - (t - s), & 0 \leq s \leq t \leq 1, s \leq \eta, \\
\frac{(1 - s)}{1 - \alpha \eta} - \frac{\alpha(t - s)}{1 - \alpha \eta}, & 0 \leq t \leq s \leq 1, s \leq \eta, \\
\frac{(1 - s)}{1 - \alpha \eta}, & 0 \leq t \leq s \leq 1, s \geq \eta, \\
\frac{(1 - t)}{1 - \alpha \eta} - (t - s), & 0 \leq s \leq t \leq 1 - \frac{1}{n}, s \geq \eta.
\end{cases}
\]

is the Green’s function corresponding the boundary value problem

\[-u''(t) = z(t), \quad t \in [0, 1],
\]

\[u(0) = 0, \quad u(1) = \alpha u(\eta)\]

with

\[u(t) = \int_0^1 H(t, s)z(s)ds,\]

as its integral representation. We need the following properties of the Green’s function \( H_n \) in the sequel. For the proof, see [22].

**Lemma 2.2.** The function \( H_n \) can be written as

\[
H_n(t, s) = G_n(t, s) + \frac{\alpha \left( t - \frac{1}{n} \right)}{1 - \frac{2}{n} + \frac{n}{n - \alpha \eta}} G_n(\eta, s),
\]

where

\[
G_n(t, s) = \frac{n}{n - 2} \begin{cases} 
(s - \frac{1}{n}) \left( 1 - \frac{1}{n} - t \right), & \frac{1}{n} \leq s \leq t \leq 1 - \frac{1}{n}, \\
(t - \frac{1}{n}) \left( 1 - \frac{1}{n} - s \right), & \frac{1}{n} \leq t \leq s \leq 1 - \frac{1}{n}.
\end{cases}
\]
Lemma 2.3. Let 

\[ \mu_n = \frac{\max\{1, \alpha\}}{1 - \frac{2}{n} + \frac{2}{n} - \alpha n}, \quad \nu_n = \frac{\min\{1, \alpha\} \min\{\eta - \frac{1}{n}, 1 - \frac{1}{n} - \eta\}}{1 - \frac{2}{n} + \frac{2}{n} - \alpha n}. \]

Then 

(i) \( H_n(t, s) \leq \mu_n (s - \frac{1}{n}) (1 - \frac{1}{n} - s), \quad (t, s) \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \times \left[\frac{1}{n}, 1 - \frac{1}{n}\right], \)

(ii) \( H_n(t, s) \geq \nu_n (s - \frac{1}{n}) (1 - \frac{1}{n} - s), \quad (t, s) \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \times \left[\frac{1}{n}, 1 - \frac{1}{n}\right]. \)

Now consider the system of nonlinear non-singular BVPs

\[ \begin{aligned}
&-x''(t) = f(t, \max\{\frac{1}{n}, y(t)\}), \quad t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right], \\
&-y''(t) = g(t, \max\{\frac{1}{n}, x(t)\}), \quad t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right], \\
x(\frac{1}{n}) = \frac{1}{n}, \quad x(1 - \frac{1}{n}) = \alpha x(\eta) + \frac{1 - \alpha}{\eta}, \\
y(\frac{1}{n}) = \frac{1}{n}, \quad y(1 - \frac{1}{n}) = \alpha y(\eta) + \frac{1 - \alpha}{\eta},
\end{aligned} \tag{2.6} \]

where \( n > N \). Write (2.6) as an equivalent system of integral equations

\[ \begin{aligned}
x(t) &= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t, s)f(s, \max\{\frac{1}{n}, y(s)\})ds, \\
y(t) &= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t, s)g(s, \max\{\frac{1}{n}, x(s)\})ds.
\end{aligned} \tag{2.7} \]

Thus, \((x, y)\) is a solution of (2.6) if and only if

\[ (x, y) \in C\left[\frac{1}{n}, 1 - \frac{1}{n}\right] \times C\left[\frac{1}{n}, 1 - \frac{1}{n}\right] \]

and \((x, y)\) is a solution of (2.7).

Define operators \( A_n, B_n, T_n : K_n \to K_n \) by

\[ \begin{aligned}
(A_n y)(t) &= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t, s)f(s, \max\{\frac{1}{n}, y(s)\})ds, \\
(B_n x)(t) &= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t, s)g(s, \max\{\frac{1}{n}, x(s)\})ds, \\
(T_n x)(t) &= (A_n(B_n x))(t).
\end{aligned} \tag{2.8} \]

If \( u_n \in K_n \) is a fixed point of \( T_n \), then the system of BVPs (2.6) has a solution \((x_n, y_n)\) given by

\[
\begin{cases}
x_n(t) = u_n(t), \\
y_n(t) = (B_n u_n)(t).
\end{cases}
\]

By construction, the system of BVPs (2.6) is regular and so the following lemma is standard.

Lemma 2.4. Assume \( f, g : (0, 1) \times (0, \infty) \to [0, \infty) \) are continuous. Then \( T_n : K_n \to K_n \) is completely continuous.
3. Main results

Proof of Theorem 1.1. By (A2), there exist constants $C_1, C_2, N_1, N_2 > 0$ such that

\[
4^{\alpha_1}ab^{\alpha_1}\mu_n^{\alpha_1+1} C_1 C_2^{\alpha_1} < 1,
\]

and

\[
F(x) \leq C_1 x^{\alpha_1} + N_1, \quad G(x) \leq C_2 x^{\alpha_2} + N_2 \text{ for } x \geq \frac{1}{n}.
\]

Choose a constant $R > 0$ such that

\[
R \geq \frac{\frac{1}{n} + \frac{2^{\alpha_1} a\mu_n C_1}{n^{\alpha_1}} + a\mu_n N_1 + 4^{\alpha_1} ab^{\alpha_1} B_n^{\alpha_1+1} C_1 N_2^{\alpha_1}}{1 - 4^{\alpha_1} ab^{\alpha_1} \mu_n^{\alpha_1+1} C_1 C_2^{\alpha_1}}.
\]

For any $u \in \partial \Omega_R \cap K_n$, using (2.8) and (A1), we have

\[
(T_n u)(t) = (A_n(B_n u))(t) = \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) f(s, (B_n u)(s))ds
\]

\[
= \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau))d\tau)ds
\]

\[
\leq \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) K(s) f(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau))d\tau)ds.
\]

In view of (3.2) and (A2), it follows that

\[
(T_n u)(t) \leq \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t,s) K(s)(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau))d\tau)^{\alpha_1} + N_1)ds
\]

\[
= \frac{1}{n} + C_1 \int_{1/n}^{1-1/n} H_n(t,s) K(s)(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau))d\tau)^{\alpha_1} ds
\]

\[
+ N_1 \int_{1/n}^{1-1/n} H_n(t,s) K(s)ds
\]

\[
\leq \frac{1}{n} + C_1 \int_{1/n}^{1-1/n} H_n(t,s) K(s)(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) L(\tau) G(u(\tau))d\tau)^{\alpha_1} ds
\]

\[
+ N_1 \int_{1/n}^{1-1/n} H_n(t,s) K(s)ds
\]

\[
\leq \frac{1}{n} + C_1 \int_{1/n}^{1-1/n} H_n(t,s) K(s)
\]

\[
\cdot \left( \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) L(\tau)(C_2(u(\tau))^\alpha_2 + N_2) d\tau \right)^{\alpha_1} ds
\]
+ \frac{N_1}{n} \int_{1/n}^{1} H_n(t, s) K(s) ds.

Employing (i) of Lemma 2.3, we obtain

\[(T_n u)(t) \leq \frac{1}{n} + C_1 \mu_n \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) K(s) ds \]
\[\cdot \left( \frac{1}{n} + \mu_n \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau) L(\tau)(C_2(u(\tau)))^{\alpha_2} + N_2) d\tau \right)^{\alpha_1} \]
\[+ N_1 \mu_n \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) K(s) ds \]
\[\leq \frac{1}{n} + C_1 \mu_n \int_{1/n}^{1-1/n} s(1 - s) K(s) ds \]
\[\cdot \left( \frac{1}{n} + \mu_n \int_{1/n}^{1-1/n} \tau(1 - \tau) L(\tau)(C_2(u(\tau)))^{\alpha_2} + N_2) d\tau \right)^{\alpha_1} \]
\[+ N_1 \mu_n \int_{1/n}^{1-1/n} s(1 - s) K(s) ds. \]

Hence,

\[(T_n u)(t) \leq \frac{1}{n} + C_1 \mu_n \int_{1/n}^{1} s(1 - s) K(s) ds \]
\[\cdot \left( \frac{1}{n} + \mu_n \int_{1/n}^{1} \tau(1 - \tau) L(\tau)(C_2(u))^{\alpha_2} + N_2) d\tau \right)^{\alpha_1} \]
\[+ N_1 \mu_n \int_{1/n}^{1} s(1 - s) K(s) ds \]
\[\leq \frac{1}{n} + \mu_n C_1 \int_{0}^{1} s(1 - s) K(s) ds \]
\[\cdot \left( \frac{1}{n} + \mu_n \int_{0}^{1} \tau(1 - \tau) L(\tau) d\tau(C_2(u))^{\alpha_2} + N_2 \right)^{\alpha_1} \]
\[+ \mu_n N_1 \int_{0}^{1} s(1 - s) K(s) ds \]
\[\leq \frac{1}{n} + a \mu_n N_1 + 2^{\alpha_1} a \mu_n C_1 \left( \frac{1}{n} \right)^{\alpha_1} + b^{\alpha_1} \mu_n^{\alpha_1} (C_2(u))^{\alpha_2} + N_2)^{\alpha_1} \]
\[\leq \frac{1}{n} + \frac{2^{\alpha_1} a \mu_n C_1}{n^{\alpha_1}} + a \mu_n N_1 + 2^{2\alpha_1} a b^{\alpha_1} \mu_n^{\alpha_1 + 1} C_1 (C_2^{\alpha_2}(u)^{\alpha_1} + N_2^{\alpha_1}). \]
Using (3.3), we obtain
\[(3.4) \quad \|T_n u\| \leq \|u\| \text{ for all } u \in \partial \Omega_R \cap K_n.\]

Now, by \((A_3)\), there exist constants \(C_3, C_4 > 0\) and \(\rho \in (0, R)\) such that
\[(3.5) \quad f(t, x) \geq C_3 x^{\beta_3}, g(t, x) \geq C_4 x^{\beta_2} \quad \text{for } x \in [0, \rho] \text{ and } t \in [\eta, 1].\]

Choose
\[(3.6) \quad r_n = \min \left\{ \rho, \frac{C_3 C_4^{\beta_3} \nu_n^{\beta_2 + 1}}{n^{\beta_3 \beta_2}} \left( \int_\eta^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)ds \right)^{\beta_2 + 1} \right\}.\]

For any \(u \in \partial \Omega_{r_n} \cap K_n\), using (2.8), (3.5) and (ii) of Lemma 2.3, we have
\[\langle T_n u \rangle(t) = (A_n(B_n u))(t) \]
\[\geq \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t, s)f(s, \frac{1}{n}) + \int_{1/n}^{1-1/n} H_n(s, \tau)g(\tau, u(\tau))d\tau)ds \]
\[\geq \int_{1/n}^{1-1/n} H_n(t, s)f(s, \frac{1}{n}) + \int_{1/n}^{1-1/n} H_n(s, \tau)g(\tau, u(\tau))d\tau)ds \]
\[\geq \int_{\eta}^{1-1/n} H_n(t, s)f(s, \frac{1}{n}) + \int_{1/n}^{1-1/n} H_n(s, \tau)g(\tau, u(\tau))d\tau)ds \]
\[\geq C_3 \int_{\eta}^{1-1/n} H_n(t, s) \left( \int_{\eta}^{1-1/n} H_n(s, \tau)g(\tau, u(\tau))d\tau \right)^{\beta_2} ds \]
\[\geq C_3 \nu_n^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)ds \]
\[\cdot \left( \nu_n \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau)g(\tau, u(\tau))d\tau \right)^{\beta_2} \]
\[\geq C_3 \nu_n^{\beta_2 + 1} \int_{\eta}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)ds \]
\[\cdot \left( C_4 \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau)(u(\tau))^{\beta_2}d\tau \right)^{\beta_2} \]
\[\geq C_3 C_4^{\beta_3} \nu_n^{\beta_2 + 1} \left( \int_{\eta}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)ds \right)^{\beta_2 + 1}.\]

Thus, in view of (3.6), it follows that
\[(3.7) \quad \|T_n u\| \geq \|u\| \text{ for } u \in \partial \Omega_{r_n} \cap K_n.\]

By Lemma 2.1, \(T_n\) has a fixed point \(u_n \in K_n \cap (\overline{\Omega_R \setminus \Omega_{r_n}}).\)

Note that
\[(3.8) \quad r_n \leq u_n(t) \leq R \quad \text{for all } t \in [\frac{1}{n}, 1 - \frac{1}{n}]\]
Hence, for which implies that for 

\[ u_n \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \] and for \( m \geq n \), \( \{u_m\} \) is uniformly bounded on \( \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \).

To show that \( \{u_m\} \) for \( m \geq n \), is equicontinuous on \( \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \), consider for \( t \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \), the integral equation

\[ u_m(t) = u_m(\frac{1}{m}) + \int_{1/m}^{1-1/m} H_m(t, s)f(s, (B_m u_m)(s))ds. \]

Employ Lemma 2.2 to obtain

\[ u_m(t) = u_m(\frac{1}{m}) + \int_{1/m}^{1-1/m} \left[ G_m(t, s) + \frac{\alpha(t - \frac{1}{m})}{1 - \frac{2}{m} + \frac{2}{m} - \alpha \eta} G_m(\eta, s) \right] f(s, (B_m u_m)(s))ds \]

\[ = u_m(\frac{1}{m}) + \frac{m}{m - 2} \int_{1/m}^{1-1/m} (s - \frac{1}{m})(1 - \frac{m}{m - 1} - t)f(s, (B_m u_m)(s))ds \]

\[ + \frac{m}{m - 2} \int_{1/m}^{1-1/m} (t - \frac{1}{m})(1 - \frac{m}{m - 1} - s)f(s, (B_m u_m)(s))ds \]

\[ + \frac{\alpha(t - \frac{1}{m})}{1 - \frac{2}{m} + \frac{2}{m} - \alpha \eta} \int_{1/m}^{1-1/m} G_m(\eta, s)f(s, (B_m u_m)(s))ds. \]

Differentiate with respect to \( t \) to obtain

\[ u_m'(t) = -\frac{m}{m - 2} \int_{1/m}^{t} (s - \frac{1}{m})f(s, (B_m u_m)(s))ds \]

\[ + \frac{m}{m - 2} \int_{1/m}^{t} (1 - \frac{m}{m - 1} - s)f(s, (B_m u_m)(s))ds \]

\[ + \frac{\alpha(t - \frac{1}{m})}{1 - \frac{2}{m} + \frac{2}{m} - \alpha \eta} \int_{1/m}^{1-1/m} G_m(\eta, s)f(s, (B_m u_m)(s))ds, \]

which implies that for \( t \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \)

\[ |u_m'(t)| \leq \int_{1/m}^{1-1/m} f(s, (B_m u_m)(s))ds \]

\[ + \frac{\alpha}{1 - \frac{2}{m} + \frac{2}{m} - \alpha \eta} \int_{1/m}^{1-1/m} G_m(\eta, s)f(s, (B_m u_m)(s))ds. \]

Hence, for \( m \geq n \), \( \{u_m\} \) is equicontinuous on \( \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \).

For \( m \geq n \), define

\[ v_m = \begin{cases} 
  u_m(\frac{1}{m}), & \text{if } 0 \leq t \leq \frac{1}{m}, \\
  u_m(t), & \text{if } \frac{1}{m} \leq t \leq 1 - \frac{1}{m}, \\
  \alpha u_m(\eta), & \text{if } 1 - \frac{1}{m} \leq t \leq 1.
\end{cases} \]
Since \( v_m \) is a constant extension of \( u_m \) to \([0, 1]\), the sequence \( \{v_m\} \) is uniformly bounded and equicontinuous on \([0, 1]\). Thus, there exists a subsequence \( \{v_{n_k}\} \) of \( \{v_m\} \) converging uniformly on \([0, 1]\) to \( v \in P \cap (\overline{P_R}\setminus\Omega_r) \).

We introduce the notation
\[
x_{n_k}(t) = v_{n_k}(t), \quad y_{n_k}(t) = \frac{1}{n_k} + \int_{1/n_k}^{1-1/n_k} H_{n_k}(t, s)g(s, v_{n_k}(s))ds,
\]
\[
\overline{x}(t) = \lim_{n_k \to \infty} x_{n_k}(t), \quad \overline{y}(t) = \lim_{n_k \to \infty} y_{n_k}(t),
\]
and for \( t \in [0, 1] \) consider the integral equation
\[
x_{n_k}(t) = x_{n_k}(1/n_k) + \int_{1/n_k}^{1-1/n_k} H_{n_k}(t, s)f(t, y_{n_k}(s))ds.
\]
Letting \( n_k \to \infty \), we have
\[
\overline{x}(t) = \overline{x}(0) + \int_0^1 H(t, s)f(t, \overline{y}(s))ds,
\]
and
\[
\overline{y}(t) = \int_0^1 H(t, s)g(s, \overline{x}(s))ds, \quad t \in [0, 1].
\]
Moreover,
\[
\overline{x}(0) = 0, \quad x(1) = \alpha \overline{x}(\eta), \quad \overline{y}(0) = 0, \quad \overline{y}(1) = \alpha \overline{y}(\eta).
\]
Hence, \((\overline{x}(t), \overline{y}(t))\) is a solution of the system (1.2).

Since
\[
f, g : (0, 1) \times (0, \infty) \to (0, \infty), \quad f(t, 0), g(t, 0) \text{ are not identically } 0, \quad \text{and } H \text{ is of fixed sign on } (0, 1) \times (0, 1),
\]
it follows that \( \overline{x}, \overline{y} > 0 \) on \((0, 1)\).

Example 3.1. Let
\[
f(t, y) = \frac{1}{t(1-t)} \left( \frac{1}{y} + 3y^{1/3} \right), \quad g(t, x) = \frac{1}{t(1-t)} \left( \frac{1}{x} + 4x \right)
\]
and \( \alpha = 2, \eta = \frac{1}{7} \). Choose
\[
K(t) = L(t) = \frac{1}{t(1-t)}, \quad F(y) = \frac{1}{y} + 3y^{1/3}, \quad G(x) = \frac{1}{x} + 4x,
\]
and \( \alpha_1 = \frac{1}{2}, \alpha_2 = 2, \beta_1 = \beta_2 = 1 \). Then \((A_1) - (A_3)\) are satisfied. Hence, by Theorem 1.1, system (1.2) has a positive solution.
Proof of Theorem 1.2. For \( u \in \partial \Omega_{M_1} \cap K_n \), using (2.8), we obtain for \( t \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \)
\[
(T_n u)(t) = (A_n(B_n u))(t) = \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(t, s)f(s, (B_n u)(s))ds
\]
\[
\geq \int_{1/n}^{1-1/n} H_n(t, s)f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau)g(\tau, u(\tau))d\tau)ds
\]
\[
\geq \int_{1/n}^{1-1/n} H_n(t, s)f(s, \frac{1}{n} + \mu_n G(\frac{1}{n})\int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau)L(\tau)d\tau)ds
\]
\[
\geq \int_{1/n}^{1-1/n} H_n(t, s)f(s, \frac{1}{n} + b \mu_n G(\frac{1}{n}))ds
\]
\[
\geq M_1 \int_{1/n}^{1-1/n} H_n(t, s)ds(\nu_n \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau)d\tau)^{-1} \geq M_1,
\]
which implies that
\[
|T_n u| \geq |u|\] for all \( u \in \partial \Omega_{M_1} \cap K_n \).

In view of (A2), we can choose \( R > M_1 \) such that (3.4) holds. Hence, by Lemma 2.1, \( T_n \) has a fixed point \( u_n \in K_n \cap (\Omega_R \setminus \Omega_{M_1}) \). By the same process as done in Theorem 1.1, the system (1.2) has a positive solution. \[\Box\]

**Example 3.2.** Let
\[
f(t, y) = \frac{e^{\frac{1}{y}}}{t(1 - t)}, \quad g(t, x) = \frac{e^{\frac{1}{x}}}{t(1 - t)}
\]
and \( \alpha = 2, \eta = \frac{1}{4} \). Choose
\[
K(t) = L(t) = \frac{1}{t(1 - t)}, \quad F(y) = e^{\frac{1}{y}}, \quad G(x) = e^{\frac{1}{x}}.
\]
Choose constant \( M_1 \) such that
\[
M_1 \leq \frac{4(n-3)}{n}e^{\frac{2}{n}} \int_{1/3}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)ds.
\]
Then (A1), (A2) and (A4) are satisfied. Hence, by Theorem 1.2, system (1.2) has a positive solution.
Example 3.3. Let

\[ f(t, y) = \begin{cases} 
\frac{y^{1/2}}{\xi^{1/2}}, & y \leq 1, \\
\frac{x^{1/2}}{\xi^{1/2}}, & y > 1, 
\end{cases} \quad g(t, x) = \begin{cases} 
\frac{y^{1/2}}{\xi^{1/2}}, & x \leq 1, \\
\frac{x^{1/2}}{\xi^{1/2}}, & x > 1.
\end{cases} \]
and $\alpha = 2$, $\eta = \frac{1}{3}$. Choose

$$K(t) = L(t) = \frac{1}{t(1-t)}, \quad F(y) = \begin{cases} ye^{\frac{y}{2}}, & y \leq 1, \\ e, & y > 1, \end{cases}, \quad G(x) = \begin{cases} xe^{\frac{x}{2}}, & x \leq 1, \\ e, & x > 1, \end{cases}$$

and $\beta_1 = \beta_2 = 1$. Choose constant $M_2$ such that

$$M_2 \geq \max \left\{ \frac{1}{n} + 6F(e(1-3/n)) \int_{1/3}^{1-1/n} \frac{(s-1/n)(1-1/n-s)}{s(1-s)} ds \right\}.$$

Then $(A_1), (A_3)$ and $(A_5)$ are satisfied. Hence, by Theorem 1.3, system (1.2) has a positive solution.

**Proof of Theorem 1.4.** By $(A_1)$ and $(A_4)$, we obtain (3.10). By $(A_5)$ we can choose a constant $M_2 > M_1$ such that (3.11) holds. Then $T_n$ has a fixed point $u_n \in K_n \cap (\Omega_{M_2} \setminus \Omega_{M_1})$. By the same process as done in Theorem 1.1, the system (1.2) has a positive solution.

**Example 3.4.** Let

$$f(t, y) = \frac{1}{t(1-t)} \frac{1}{\sqrt{y}}, \quad g(t, x) = \frac{1}{t(1-t)} \frac{1}{x^2}$$

and $\alpha = 2$, $\eta = \frac{1}{3}$. Choose

$$K(t) = L(t) = \frac{1}{t(1-t)}, \quad F(y) = \frac{1}{\sqrt{y}}, \quad G(x) = \frac{1}{x^2}.$$

Choose constants $M_1$ and $M_2$ such that $M_1 \leq \frac{n^{-3}}{\sqrt{\pi(n^2-1)}} \int_{1/3}^{1-1/n} \frac{(s-1/n)(1-1/n-s)}{s(1-s)} ds$ and $M_2 \geq \frac{1}{6n} \left( \frac{1}{3} - \sqrt{\frac{2}{n-3}} \int_{1/3}^{1-1/n} \frac{(s-1/n)(1-1/n-s)}{s(1-s)} ds \right)^{-1/2}$. Then $(A_1), (A_4)$ and $(A_5)$ are satisfied. Hence, by Theorem 1.4, system (1.2) has a positive solution.

**References**


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