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OPTIMAL SELLING RULES IN A REGIME-SWITCHING EXPONENTIAL GAUSSIAN DIFFUSION MODEL*

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Abstract. This paper develops optimal selling rules in asset trading using a regime-switching exponential Gaussian diffusion model. The optimization problem is solved by a combined approach of boundary value problems and probabilistic analysis. A system of linear differential equations with variable coefficients and two-point boundary conditions, satisfied by the objective function of the problem, is derived. The existence and uniqueness of the solution are proved. A closed-form solution in terms of Weber functions is obtained for one-dimensional cases. For m-dimensional cases, a stochastic recursive algorithm for numerically searching the optimal value is developed. Numerical results are reported.

Key words. optimal selling rule, Markov chain, regime-switching, Gaussian diffusion, boundary value problem, stochastic recursive algorithm

AMS subject classifications. 91B26, 91B28, 60J27, 62L20

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1. Introduction.

This paper develops an optimal selling rule in asset trading using a regime-switching exponential Gaussian diffusion model for asset price. A selling rule is specified by two threshold levels—an upper level (greater than the purchase price) for the profit target and a lower level (less than the purchase price) for the stop-loss limit. The asset is sold once its price hits either level. Our objective in this study is to obtain a pair of optimal threshold levels that maximize a prespecified objective function which reflects the investment goal and/or risk attitude of investors.

Recently, considerable attention has been drawn to regime-switching models in financial mathematics which aim to include the influence of macroeconomic factors on the individual asset price behavior. In this setting, asset prices are dictated by a number of stochastic differential equations coupled by a finite-state Markov chain, which represents various randomly changing economical factors. Model parameters (drift and volatility coefficients) are assumed to depend on the Markov chain. Regime-switching models have been used in derivative pricing (see Buffington and Elliott [2], Guo [12], Guo and Zhang [13], and Yao, Zhang, and Zhou [22] among others), for interest rates and bond prices (see Bansal and Zhou [1] and Dai, Singleton, and Yang [6] among others), and in modeling commodity and electricity prices (see Clewlow and Strickland [3], Erlwein, Benth, and Mamon [9], Kluge [16], and Lucia and Schwartz [18] among others).

Along another line, Zhang [26] studied an optimal selling rule for stock liquidation using a regime-switching geometric Brownian motion (GBM) model. In [26], a method that combines differential equation with probabilistic analysis was developed;

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an analytical solution for the two-regime case was obtained, and optimization techniques for deterministic functions were used to find the optimal thresholds. However, when the number of regimes exceeds two, the analytical solutions are difficult to obtain, and thus the deterministic optimization approaches are not applicable anymore. To find a feasible solution, Yin, Liu, and Zhang [23] took a different approach, namely, using stochastic approximation algorithms. By focusing on threshold-type strategies, recursive algorithms using Monte Carlo simulation were developed in [23]. Convergence and the rates of convergence of the algorithms were proved. The stochastic algorithms were tested by using both simulations and market data (see Yin, Liu, and Zhang [23], and Yin et al. [25] for more details).

In this work, we extend the aforementioned optimal selling rule study for the regime-switching GBM model to a class of regime-switching exponential Gaussian diffusion models that include the GBM and regime-switching GBM models as special cases. The new mathematical model is presented first, and its connection with other models is then noted. An objective function associated with the optimization problem is defined next. Consequently, a system of linear differential equations with two boundary conditions, satisfied by the objective function, is derived. We point out a significant difference between the system considered in this paper and that of Zhang [26]. That is, the coefficients of the differential equations are no longer constant. Therefore, solutions from [26] cannot be used in this paper. We develop a different approach. The existence of a solution to the variable coefficient boundary value problem is proved by adopting a method of upper and lower solutions that use the Green's function of the associated homogeneous system. The uniqueness of the solution is established by applying Dynkin's formula. In addition, a numerical method to construct a sequence of increasing functions (lower solution approximation) and a sequence of decreasing functions (upper solution approximation) is developed. The second part of the paper is concerned with stochastic optimization methods. We develop a recursive algorithm which provides a feasible solution for searching the best selling rules and is particularly applicable to models with large state spaces.

The rest of the paper is organized as follows. Section 2 presents the regime-switching model and the precise formulation of the selling rule problem. The differential equations and boundary values satisfied by the objective function of the optimization problem is derived. Section 3 establishes the existence and uniqueness of the solution to the problem. Section 4 is concerned with stochastic recursive algorithms. The selling rule problem is reformulated as a stochastic optimization problem. A recursive algorithm for searching the optimal thresholds using gradient estimation and projection procedure is developed. Conditions for convergence of the algorithm are provided. Numerical results are reported. Finally, the paper is concluded with further remarks in section 5.

2. Problem formulation. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be the underlying probability space, upon which all stochastic processes are defined. Let $\alpha(t)$ be a continuous-time Markov chain taking values in $\mathcal{M} := \{1, \ldots, m\}$, a finite state space. The states represent general market trends and other economic factors (called "state of the world" or "regime") and are labeled by integers 1 to m, where m is the total number of regimes considered for the economy. For example, with $m = 2$, $\alpha(t) = 1$ may stand for an up market and $\alpha(t) = 2$ a down market. Let $B(t)$ be a real-valued standard Brownian motion. Assume that $\alpha(t)$ is independent of $B(t)$.

Let $S(t)$ be the asset price at time $t \geq 0$,

$$S(t) = S_0 \exp(X(t)), \quad t \geq 0, \tag{2.1}$$

where $S_0 > 0$ denotes the asset price at $t = 0$ (i.e., $S(0) = S_0$), and $X(t)$ is the solution of the stochastic differential equation

$$\begin{cases} dX(t) = [b(\alpha(t)) + \mu(\alpha(t))X(t)]dt + \sigma(\alpha(t))dB(t), \\ X(0) = 0. \end{cases} \tag{2.2}$$

Note that the coefficients $b(\alpha(t))$, $\mu(\alpha(t))$, and $\sigma(\alpha(t))$ in (2.2) all depend on $\alpha(t)$, indicating that they can take different values for different regimes. We assume that $b(i) \geq 0$ and $\sigma(i) > 0$ for each $i \in \mathcal{M}$. Before introducing the optimal selling rule problem, we make three remarks regarding the model given by (2.1) and (2.2).

Remark 2.1. Consider a special case in which there is only one state for $\alpha(t)$, i.e., $m = 1$. Then $\alpha(t) = 1$ for all $t \geq 0$. In this case, these is no regime switching, and we can write $b(\alpha(t)) = b$, $\mu(\alpha(t)) = \mu$, and $\sigma(\alpha(t)) = \sigma$, where b, μ, and σ are constants. Then (2.2) becomes an Ornstein–Uhlenbeck process,

$$dX(t) = [b + \mu X(t)]dt + \sigma dB(t). \tag{2.3}$$

In particular, if we assume that $\mu < 0$ and let $\kappa = -\mu$ and $\theta = b/\kappa$, then we have the well-known Vasicek model [21] for interest rates, namely,

$$dr(t) = \kappa[\theta - r(t)]dt + \sigma dB(t),$$

where $r(t) := X(t)$ denotes the instantaneous spot rate at time $t \geq 0$, θ is the mean-reverting level, κ is the rate at which $r(t)$ is pulled back to the level θ, and σ is the volatility of $r(t)$. Also note that the solution of (2.3) is given by

$$X(t) = \frac{b}{\mu}(e^{\mu t} - 1) + \sigma \int_0^t e^{\mu(t-s)} dB(s), \tag{2.4}$$

which is a Gaussian process. Consequently, the asset price $S(t) = S_0 \exp(X(t))$ becomes an exponential Gaussian process. However, when there is more than one state for $\alpha(t)$, i.e., $m \geq 2$, then $X(t)$ will no longer be a Gaussian process. Instead, it is a mixture of m Gaussian processes. We use the term regime-switching exponential Gaussian diffusion model for (2.1) and (2.2) in this paper that generalizes the Ornstein–Uhlenbeck process.

Remark 2.2. Set $\mu(i) \equiv 0$ and let $b(i) = \nu(i) - \frac{1}{2}\sigma^2(i)$ for $i \in \mathcal{M}$ in (2.2). Then the model given by (2.1) and (2.2) is reduced to the regime-switching GBM model considered in [26] with drift $\nu(\alpha(t))$ and volatility $\sigma(\alpha(t))$, i.e.,

$$\frac{dS(t)}{S(t)} = \nu(\alpha(t))dt + \sigma(\alpha(t))dB(t),$$

which includes the commonly used log-normal model as a special case (when $m = 1$). Thus the model we consider in this paper further generalizes the (regime-switching) log-normal model. The selling rule problem for the case of $\mu(i) \equiv 0$ for $i \in \mathcal{M}$ has already been handled in Zhang [26]. In what follows we will focus on $\mu(i) \neq 0$; see also Remark 2.6.

Remark 2.3. Note that if $m = 1$, i.e., without regime switching, then the model given by (2.1) and (2.2) becomes a particular member of the class of affine diffusion models (see Duffie, Filipović, and Schachermayer [8] for definition of affine models). For $m > 1$, the model generalizes the affine model by adding a Markovian regime switching.

To continue our studies, consider a threshold-type selling rule specified by a pair of numbers z_1 and z_2 with $-\infty < z_1 \le 0 \le z_2 < \infty$. Define a stopping time τ by

$$\tau = \inf\{t > 0 : \ X(t) \notin (z_1, z_2)\}. \tag{2.5}$$

Let $S_L = S_0 e^{z_1}$ and $S_U = S_0 e^{z_2}$. Then $0 < S_L \le S_0 \le S_U < \infty$, and τ can be equivalently defined in terms of $S(t)$, i.e.,

$$\tau = \inf\{t > 0 : \ S(t) \notin (S_L, S_U)\}. \tag{2.6}$$

We call τ the selling time and S_L and S_U the lower and upper thresholds, respectively, for asset $S(t)$. That means we sell the asset at time τ either to take a profit (if S_U is reached) or to prevent further loss (if S_L is reached).

The optimal selling rule problem is to find a pair of numbers (z_1, z_2) that maximize the objective function:

$$V(z_1, z_2) = E\left\{\Phi(X(\tau)) \exp(-\rho\tau)\right\}, \tag{2.7}$$

where $\Phi(x)$ is a prespecified utility function and $\rho > 0$ is a discount factor.

Remark 2.4. Depending on the investment purpose and/or the risk attitude of an investor (risk-neutral or risk-averse), an appropriate utility function $\Phi(x)$ can be used in the objective (2.7). For instance, if we choose $\Phi(x) = e^x - 1$, then we can rewrite the objective function (2.7) as

$$V(z_1, z_2) = E\left\{\exp(-\rho\tau)\frac{S(\tau) - S_0}{S_0}\right\},$$

which gives the expectation of the discounted percentage return. By maximizing this objective function, one seeks the maximum percentage return, a common index used in evaluating investment performance.

Remark 2.5. The selling rule problem we consider in this paper is an optimal stopping problem. Note that the objective function (2.7) is determined by the first hitting time τ of process $X(t)$ at the double barriers z_1, z_2. When the log-normal model (without regime-switching) is specified for the underlying asset price, a probabilistic approach can be used to obtain the distribution function of the stopping time τ (see Karatzas and Shreve [15] and Steele [20] for extensive discussions on the probabilistic methods and results), and, consequently, an analytical objective function can be derived. However, when the new regime-switching model is used, it is difficult to obtain the distribution function of τ; thus the "pure" probabilistic approach does not work. We resort to methods of differential equations together with probabilistic approaches to solve the problem.

To proceed, we derive a two-point boundary value problem associated with (2.7). For a given real number z, consider the process $\xi(t)$ that is the solution of

$$d\xi(t) = [b(\alpha(t)) + \mu(\alpha(t))\xi(t)]dt + \sigma(\alpha(t))dB(t), \quad \xi(0) = z.$$

Then $\xi(t) = X(t)$ if $z = 0$. For each $z \in [z_1, z_2]$, define a stopping time:

$$\tau(z) = \inf\{t > 0 : \ \xi(t) \notin (z_1, z_2)\}.$$

Note we use $\tau(z)$ to indicate the z dependence of the stopping time. Let

$$v(z, i) = E\left\{\Phi(\xi(\tau(z))) \exp(-\rho\tau(z))\Big|\alpha(0) = i, \xi(0) = z\right\}. \tag{2.8}$$

Then the objective function (2.7) can be written in terms of $v(z,i)$ as

$$V(z_1,z_2)=\sum_{i=1}^{m}p_i v(0,i), \tag{2.9}$$

where $p_i = P\{\alpha(0)=i\}$, $i=1,\dots,m$, assumed given, is the initial probability distribution of the Markov chain $\alpha(\cdot)$.

Let matrix $Q=(q_{ij})_{m\times m}$ be the generator of the Markov chain $\alpha(\cdot)$. From Markov chain theory (see, for example, Yin and Zhang [24]), the entries q_{ij} of Q satisfy (i) $q_{ij}\ge 0$ if $j\neq i$; (ii) $\sum_{j=1}^{m}q_{ij}=0$ for each $i=1,\dots,m$. Moreover,

$$\lim_{\Delta t\to 0^+}\frac{P(\Delta t)-I}{\Delta t}=Q, \tag{2.10}$$

where $P(\Delta t)=(p_{ij}(\Delta t))_{m\times m}=(P\{\alpha(\Delta t)=j|\alpha(0)=i)\})_{m\times m}$ is the transition probability matrix of $\alpha(\cdot)$, and I denotes the $m\times m$ identity matrix.

Consider a small interval Δt. Since $\xi(t)$ and $\alpha(t)$ are jointly Markovian, it follows that

$$v(z,i)=\sum_{j=1}^{m}E\left\{v(\xi(\Delta t),j)\exp(-\rho\Delta t)\right\}P\{\alpha(\Delta t)=j|\alpha(0)=i\}.$$

Expanding $v(\xi(\Delta t),j)\exp(-\rho\Delta t)$ at 0, using Itô's formula, sending $\Delta t\to 0$, and using the limit (2.10), we obtain the following system of differential equations associated with the value functions $v(z,i)$, $i=1,\dots,m$:

$$\frac{\sigma^2(i)}{2}\frac{d^2v(z,i)}{dz^2}+[b(i)+\mu(i)z]\frac{dv(z,i)}{dz}-\rho v(z,i)+\sum_{j=1}^{m}q_{ij}v(z,j)=0 \tag{2.11}$$

for $z\in(z_1,z_2)$. The boundary conditions are given by

$$v(z_1,i)=\Phi(z_1),\quad v(z_2,i)=\Phi(z_2). \tag{2.12}$$

If the boundary value problem (2.11) and (2.12) has a smooth solution $v(z,i)$, $i=1,\dots,m$, then, using Dynkin's formula (see, for example, Oksendal [19]), we can show that it must be given by (2.8), which implies the uniqueness of the solution. Therefore, it is necessary to establish the existence of a C^2 solution to (2.11) and (2.12). This is the task of the next section.

Remark 2.6. While a system of *constant* coefficient linear differential equations was obtained in Zhang [26] based on the regime-switching log-normal model for asset price, what we have here for the new model is a system of differential equations with *variable* coefficients. Therefore, methods used in [26] for constant coefficient systems are not applicable and we need a new approach for the analysis of (2.11) and (2.12). One of the major contributions of this paper (in the next section) is that we employ a new method and successfully prove the existence of a C^2 solution of the variable coefficient boundary value problem (2.11) and (2.12).

In what follows, we use f_x and f_{xx} to denote the first- and second-order derivatives of f with respect to x, respectively, where f is either a real-valued or a vector-valued function of x. Using this notation, we rewrite the system (2.11)–(2.12) in the following matrix form:

$$\begin{cases} AV_{zz}(z)+[B_1+B_0z]V_z(z)+CV(z)=FV(z) \ \text{ for } z\in(z_1,z_2),\\ V(z_1)=\Phi(z_1)\mathbb{1}_m,\ V(z_2)=\Phi(z_2)\mathbb{1}_m, \end{cases} \tag{2.13}$$

where $V(z) = (v(z,1), \dots, v(z,m))^T$, $\mathbb{1}_m = (1, \dots, 1)^T$, $A = \frac{1}{2}\operatorname{diag}(\sigma^2(1), \dots, \sigma^2(m))$, $B_0 = \operatorname{diag}(\mu(1), \dots, \mu(m))$, $B_1 = \operatorname{diag}(b(1), \dots, b(m))$, $C = Q_d - \rho I = \operatorname{diag}(q_{11} - \rho, \dots, q_{mm} - \rho)$, and $F = Q_d - Q$ where $Q_d = \operatorname{diag}(q_{11}, \dots, q_{mm})$.

3. Solution of the boundary value problem. In this section, we assume that $\mu(i) > 0$ for $i \in \mathcal{M}$. The case of $\mu(i) < 0$ can be handled similarly. We first study the scalar system (the one-dimensional case) and derive an explicit solution. Then we prove the existence of a solution for multidimensional systems, using the one-dimensional result.

When $m = 1$, (2.13) reduces to a second-order scalar linear differential equation subject to two boundary conditions:

$$\begin{cases} \dfrac{\sigma^2}{2} V_{zz}(z) + [b + \mu z] V_z(z) - \rho V(z) = 0 \quad \text{for } z \in (z_1, z_2), \\ V(z_1) = \Phi(z_1),\ V(z_2) = \Phi(z_2), \end{cases} \tag{3.1}$$

where $V(z) = v(z,1)$, $\mu = \mu(1)$, $b = b(1)$, and $\sigma = \sigma(1)$. Set $x = \kappa_1 + \kappa_0 z$, where $\kappa_0 = \frac{\sqrt{2\mu}}{\sigma}$ and $\kappa_1 = \frac{b}{\sigma}\sqrt{\frac{2}{\mu}}$. Let $\widetilde{V}(x) = V(z)$. Then (3.1) is transformed into

$$\begin{cases} \widetilde{V}_{xx}(x) + x\widetilde{V}_x(x) - \lambda \widetilde{V}(x) = 0 \quad \text{for } x \in (\kappa_1 + \kappa_0 z_1, \kappa_1 + \kappa_0 z_2), \\ \widetilde{V}(\kappa_1 + \kappa_0 z_1) = \Phi(z_1),\ \widetilde{V}(\kappa_1 + \kappa_0 z_2) = \Phi(z_2), \end{cases} \tag{3.2}$$

where $\lambda := \rho/\mu$. To solve the homogeneous equation (3.2), we use the following transform:

$$\widetilde{V}(x) = \exp\left(-\frac{x^2}{4}\right) D(x).$$

Then $D(x)$ satisfies

$$D_{xx}(x) + \left[\frac{1}{2} - \frac{x^2}{4} - \bar{\lambda}\right] D(x) = 0, \tag{3.3}$$

where $\bar{\lambda} := 1 + \lambda > 0$. From the results presented in Darling and Siegert [7] and Finch [10], we have the following proposition.

PROPOSITION 3.1. *The function $D^\nu(x)$ defined below (known as the parabolic cylinder function or the Weber function) satisfies the equation*

$$D^\nu_{xx}(x) + \left[\frac{1}{2} - \frac{x^2}{4} + \nu\right] D^\nu(x) = 0, \tag{3.4}$$

where

$$D^\nu(x) = \begin{cases} \sqrt{\dfrac{2}{\pi}} \exp\left(\dfrac{x^2}{4}\right) \displaystyle\int_0^\infty t^\nu \exp\left(-\frac{t^2}{2}\right) \cos\left(xt - \frac{\pi\nu}{2}\right)\, dt, & \nu > -1, \\ \dfrac{1}{\Gamma(-\nu)} \exp\left(-\dfrac{x^2}{4}\right) \displaystyle\int_0^\infty t^{-\nu-1} \exp\left(-\frac{t^2}{2} - xt\right)\, dt, & \nu < 0, \end{cases} \tag{3.5}$$

and $\Gamma(\cdot)$ is the Gamma function. The two branches in (3.5) *agree for $-1 < \nu < 0$.*

Comparing (3.3) with (3.4), we see that one solution of (3.3) is given by

$$D(x) = D^{-\bar{\lambda}}(x) = D^{-(1+\lambda)}(x) = \frac{1}{\Gamma(1+\lambda)} \exp\left(-\frac{x^2}{4}\right) \int_0^\infty t^\lambda \exp\left(-\frac{t^2}{2} - xt\right)\, dt.$$

The second independent solution is given by

$$D(-x) = \frac{1}{\Gamma(1+\lambda)} \exp\left(-\frac{x^2}{4}\right) \int_0^\infty t^\lambda \exp\left(-\frac{t^2}{2} + xt\right)\, dt.$$

It follows that the solution to (3.2) is

$$\widetilde{V}(x) = C_1 \int_0^\infty t^\lambda \exp\left(-\frac{(t+x)^2}{2}\right)\, dt + C_2 \int_0^\infty t^\lambda \exp\left(-\frac{(t-x)^2}{2}\right)\, dt,$$

where C_1 and C_2 are constants to be determined using the given boundary conditions.

Consider the scalar boundary value problem defined below:

$$\text{(3.6)} \qquad \begin{cases} D_{xx}(x) + \left[\dfrac{1}{2} - \dfrac{x^2}{4} - (1+\gamma)\right] D(x) = 0 \quad \text{for } x \in (x_1, x_2), \\ D(x_1) = 0,\ D(x_2) = 0, \end{cases}$$

where $\gamma > 0$ is a fixed constant. Set

$$D_1(x) = \exp\left(-\frac{x^2}{4}\right) \int_0^\infty t^\gamma \exp\left(-\frac{t^2}{2} - xt\right)\, dt$$

and

$$D_2(x) = \exp\left(-\frac{x^2}{4}\right) \int_0^\infty t^\gamma \exp\left(-\frac{t^2}{2} + xt\right)\, dt.$$

Then D_1 and D_2 form a Descartes system of solutions for the homogeneous equation in (3.6), since $D_1 > 0$, $D_2 > 0$, and $W(D_1, D_2) > 0$ on $[x_1, x_2]$, where

$$W(D_1, D_2) = \det \begin{pmatrix} D_1 & D_2 \\ D_{1,x} & D_{2,x} \end{pmatrix}$$

denotes the Wronskian of D_1 and D_2. Thus, the equation in (3.6) is disconjugate on $[x_1, x_2]$ (see Coppel [5]). This result, coupled with the observation that the boundary conditions in (3.6) (i.e., $D(x_1) = 0$ and $D(x_2) = 0$) are two-point conjugate boundary conditions, implies two immediate corollaries which we shall employ below to establish the existence of a solution of (2.13) and to provide numerical approximations that converge monotonically to the appropriate C^2 solution.

COROLLARY 3.2. *There exists a Green's function $G(\gamma; x, s)$ for the boundary value problem* (3.6) *satisfying*

$$G(\gamma; x, s) < 0 \text{ for } (x, s) \in (x_1, x_2) \times (x_1, x_2).$$

Moreover, $G_x(\gamma; x_1, s) < 0$ for $s \in (x_1, x_2)$ and $G_x(\gamma; x_2, s) > 0$ for $s \in (x_1, x_2)$, where G_x denotes the partial derivative of G with respect to x.

Note that the Green's function G plays the role that

$$D(x) = \int_{x_1}^{x_2} G(x, s) f(s) ds$$

is the unique solution of

$$\begin{cases} D_{xx}(x) + \left[\dfrac{1}{2} - \dfrac{x^2}{4} - (1+\gamma)\right] D(x) = f(x) \quad \text{for } x \in (x_1, x_2), \\ D(x_1) = 0,\ D(x_2) = 0, \end{cases}$$

where $f(x)$ is a continuous function on $[x_1, x_2]$.

COROLLARY 3.3. *The solution of the boundary value problem*

$$\begin{cases} D_{xx}(x) + \left[\frac{1}{2} - \frac{x^2}{4} - (1+\gamma)\right] D(x) = 0 \quad \text{for } x \in (x_1, x_2), \\ D(x_1) > 0,\ D(x_2) > 0 \end{cases}$$

is positive on $[x_1, x_2]$.

Proof. The disconjugacy of $D_{xx}(x) + [\frac{1}{2} - \frac{x^2}{4} - (1+\gamma)]\, D(x) = 0$ on $[x_1, x_2]$ means that any nontrivial solution has at most one root (counting multiplicities) on $[x_1, x_2]$. Since the solution is strictly positive at each boundary, the desired result follows. □

Having done with the one-dimensional case, now we address the existence of a C^2 solution to the m-dimensional ($m > 1$) boundary value system (2.13). To carry out the analysis, the following assumption is needed.

ASSUMPTION 3.4.

$$\frac{\mu(1)}{\sigma^2(1)} = \frac{\mu(2)}{\sigma^2(2)} = \cdots = \frac{\mu(m)}{\sigma^2(m)}$$

and

$$\frac{b^2(1)}{\sigma^2(1)\mu(1)} = \frac{b^2(2)}{\sigma^2(2)\mu(2)} = \cdots = \frac{b^2(m)}{\sigma^2(m)\mu(m)}\ .$$

THEOREM 3.5. *Under Assumption* 3.4*, there exists a unique* C^2 *solution to the boundary value problem* (2.13).

Proof. We employ the method of upper and lower solutions to obtain existence. Let $x = \kappa_1 + \kappa_0 z$, where $\kappa_0 = \frac{\sqrt{2\mu(i)}}{\sigma(i)}$ and $\kappa_1 = \frac{b(i)}{\sigma(i)}\sqrt{\frac{2}{\mu(i)}}$ are two constants due to Assumption 3.4. For notational brevity, in what follows, we introduce

$$\bar{\kappa}_1 = \kappa_1 + \kappa_0 z_1\ ,\ \bar{\kappa}_2 = \kappa_1 + \kappa_0 z_2. \tag{3.7}$$

Let $\widetilde{V}(x) = V(z)$. Then (2.13) is converted to the following problem:

$$\begin{cases} \widetilde{V}_{xx}(x) + x\widetilde{V}_x(x) - \widetilde{C}\widetilde{V}(x) = \widetilde{F}\widetilde{V}(x) \quad \text{for } x \in (\bar{\kappa}_1, \bar{\kappa}_2), \\ \widetilde{V}(\bar{\kappa}_1) = \Phi(z_1)\mathbb{1}_m,\ V(\bar{\kappa}_2) = \Phi(z_2)\mathbb{1}_m, \end{cases} \tag{3.8}$$

where

$$\widetilde{C} = \operatorname{diag}(\lambda_1, \ldots, \lambda_m), \quad \lambda_i = \frac{\rho - q_{ii}}{\mu(i)}, \quad i = 1, \ldots, m, \tag{3.9}$$

and

$$\widetilde{F} = \begin{pmatrix} 0 & -q_{12}/\mu(1) & \cdots & -q_{1m}/\mu(1) \\ -q_{21}/\mu(2) & 0 & \cdots & -q_{2m}/\mu(2) \\ \vdots & \vdots & \cdots & \vdots \\ -q_{m1}/\mu(m) & -q_{m2}/\mu(m) & \cdots & 0 \end{pmatrix}. \tag{3.10}$$

Note that $\rho > 0$, $\mu(i) > 0$, and $q_{ii} \le 0$. Hence $\lambda_i > 0$ for $i = 1, \ldots, m$.

We use the (vector) transform $\widetilde{V}(x) = \exp(-\frac{x^2}{4})D(x)$, where

$$D(x) = (D_1(x), \dots, D_m(x))^T.$$

Then (3.8) is transformed into

$$\begin{cases} D_{xx}(x) + \bar{C}D(x) = \widetilde{F}D(x) \ \text{ for } x \in (\bar{\kappa}_1, \bar{\kappa}_2), \\ D(\bar{\kappa}_1) = \exp\left(\dfrac{(\bar{\kappa}_1)^2}{4}\right)\Phi(z_1)\mathbb{1}_m, \ \ D(\bar{\kappa}_2) = \exp\left(\dfrac{(\bar{\kappa}_2)^2}{4}\right)\Phi(z_2)\mathbb{1}_m, \end{cases} \tag{3.11}$$

where

$$\bar{C} = \operatorname{diag}\left(\left[\frac{1}{2} - \frac{x^2}{4} - (1+\lambda_1)\right], \dots, \left[\frac{1}{2} - \frac{x^2}{4} - (1+\lambda_m)\right]\right). \tag{3.12}$$

Note that the left-hand side of the vector equation (3.11) is decoupled and, hence, diagonal. For each $i = 1, \dots, m$, the corresponding homogeneous scalar boundary value problem is given by

$$\begin{cases} D_{i,xx}(x) + \left[\dfrac{1}{2} - \dfrac{x^2}{4} - (1+\lambda_i)\right] D_i(x) = 0 \ \text{ for } x \in (\bar{\kappa}_1, \bar{\kappa}_2), \\ D_i(\bar{\kappa}_1) = 0, \ D_i(\bar{\kappa}_2) = 0. \end{cases} \tag{3.13}$$

Let $G(\lambda_i; x, s)$ be the associated Green's function as given by Corollary 3.2. Define

$$G(x,s) = \operatorname{diag}\Big(G(\lambda_1; x, s), \dots, G(\lambda_m; x, s)\Big).$$

Then $G(x,s)$ is a Green's function of the system (3.11).

Next, define a Banach space C_m by

$$C_m[\bar{\kappa}_1, \bar{\kappa}_2] = \Big\{U = (u_1, \dots, u_m)^T : [\bar{\kappa}_1, \bar{\kappa}_2] \to \mathbb{R}^m, u_i \in C[\bar{\kappa}_1, \bar{\kappa}_2], i = 1, \dots, m\Big\}$$

with norm $\|U\| = \max_{1 \le i \le m}\{\|u_i\|_0\}$, where $\|\cdot\|_0$ denotes the usual supremum norm. Consider the partial order on $\mathbb{R}^m$:

$$V \le U \Longleftrightarrow v_i \le u_i, \quad i = 1, \dots, m, \ \text{where } U, V \in \mathbb{R}^m.$$

Using this partial order, we define a partial order on $C_m[\bar{\kappa}_1, \bar{\kappa}_2]$:

$$V \le U \Longleftrightarrow V(x) \le U(x), x \in [\bar{\kappa}_1, \bar{\kappa}_2], \ \text{where } U, V \in C_m.$$

Let $D_\Phi \in C_m$ denote the solution of the following homogeneous equation with non-homogeneous boundary conditions:

$$\begin{cases} D_{xx}(x) + \bar{C}D(x) = 0 \quad \text{for } x \in (\bar{\kappa}_1, \bar{\kappa}_2), \\ D(\bar{\kappa}_1) = \exp\left(\dfrac{(\bar{\kappa}_1)^2}{4}\right)\Phi(z_1)\mathbb{1}_m, \ \ D(\bar{\kappa}_2) = \exp\left(\dfrac{(\bar{\kappa}_2)^2}{4}\right)\Phi(z_2)\mathbb{1}_m. \end{cases} \tag{3.14}$$

The existence of D_Φ is ensured by Corollary 3.3. Define an operator $\mathbf{K}$ on C_m by

$$(\mathbf{K}D)(x) = D_\Phi(x) + \int_{\bar{\kappa}_1}^{\bar{\kappa}_2} G(x,s)\widetilde{F}D(s)\, ds, \tag{3.15}$$

where $\widetilde{F}$ is given by (3.10).

Remark 3.6. Let $\mathbf{K}$ be defined by (3.15). Then $\mathbf{K} : C_m[\bar{\kappa}_1, \bar{\kappa}_2] \longrightarrow C_m^2[\bar{\kappa}_1, \bar{\kappa}_2]$.

The remark follows by standard properties of the diagonal structure of the Green's matrix $G(x, s)$ (see Coddington and Levinson [4, p. 192]). In fact, each scalar-valued function $G(\lambda_i; x, s)$ is continuous on triangles, $x < s$, $s < x$, and satisfies the differential equation

$$D_{i,xx}(x) + \left[\frac{1}{2} - \frac{x^2}{4} - (1 + \lambda_i)\right] D_i(x) = 0$$

on triangles, $x < s$, $s < x$, and

$$\lim_{x \to s^+} G_x(x, s) - \lim_{x \to s^-} G_x(x, s) = 1.$$

If $D \in C_m[\bar{\kappa}_1, \bar{\kappa}_2]$, then it is standard to show that $\mathbf{K}D \in C_m^2[\bar{\kappa}_1, \bar{\kappa}_2]$.

The following remark is also immediate from Corollary 3.2 and (3.15) (see Coddington and Levinson [4, p. 192] and Jackson [14, p. 99]).

Remark 3.7. $D \in C_m^2$ is a solution of the boundary value problem (3.11) if and only if $D \in C_m$ and $\mathbf{K}D = D$.

In view of Corollary 3.2 and (3.10), we have $G(x, s)\widetilde{F} \geq 0$ elementwise. Therefore, $\mathbf{K}$ is a monotonic operator; that is,

$$V \leq U \Longrightarrow \mathbf{K}V \leq \mathbf{K}U, \;\; U, V \in C_m.$$

We establish upper and lower solutions of the boundary value problem (3.11), respectively. That is, (see Jackson [14]), we seek $U_0 \in C_m^2$ and $V_0 \in C_m^2$ satisfying

$$V_0 \leq U_0, \quad V_0 \leq \mathbf{K}V_0, \quad \mathbf{K}U_0 \leq U_0, \tag{3.16}$$

and

$$V_0(\bar{\kappa}_i) \leq \exp\left(\frac{(\bar{\kappa}_i)^2}{4}\right) \Phi(z_i)\mathbb{1}_m \leq U_0(\bar{\kappa}_i), \quad i = 1, 2. \tag{3.17}$$

Once we obtain the upper and lower solutions, the proof for existence of a solution is complete. To see this, define a closed and convex region $\Omega \subset C_m$ by

$$D \in \Omega \Longleftrightarrow V_0(x) \leq D(x) \leq U_0(x), \quad \bar{\kappa}_1 \leq x \leq \bar{\kappa}_2 \; .$$

The inequalities (3.16) and (3.17), coupled with the fact that $\mathbf{K}$ is monotone, imply that $\mathbf{K} : \Omega \to \Omega$. Thus, the existence of a solution D, satisfying

$$V_0(x) \leq D(x) \leq U_0(x), \qquad \bar{\kappa}_1 \leq x \leq \bar{\kappa}_2, \tag{3.18}$$

follows as an application of the Schauder fixed point theorem (Jackson [14, p. 102]).

It can be shown, using the definition (3.15) and Corollary 3.2, that V_0 is a lower solution if

$$\begin{cases} V_{0,xx}(x) + \bar{C}V_0(x) \geq \widetilde{F}V_0(x) \quad \text{for } x \in (\bar{\kappa}_1, \bar{\kappa}_2), \\ V_0(\bar{\kappa}_1) \leq \exp\left(\dfrac{(\bar{\kappa}_1)^2}{4}\right) \Phi(z_1)\mathbb{1}_m, \;\; V_0(\bar{\kappa}_2) \leq \exp\left(\dfrac{(\bar{\kappa}_2)^2}{4}\right) \Phi(z_2)\mathbb{1}_m, \end{cases} \tag{3.19}$$

where $\bar{C}$ is the diagonal matrix defined in (3.12). Similarly, U_0 is a upper solution if the above inequalities are reversed, i.e.,

$$\begin{cases} U_{0,xx}(x) + \bar{C}U_0(x) \le \widetilde{F}U_0(x) \quad \text{for } x \in (\bar{\kappa}_1, \bar{\kappa}_2), \\ U_0(\bar{\kappa}_1) \ge \exp\left(\dfrac{(\bar{\kappa}_1)^2}{4}\right)\Phi(z_1)\mathbb{1}_m, \;\; U_0(\bar{\kappa}_2) \ge \exp\left(\dfrac{(\bar{\kappa}_2)^2}{4}\right)\Phi(z_2)\mathbb{1}_m. \end{cases} \tag{3.20}$$

Consider the solution D_Φ of (3.14). Since D_Φ satisfies the homogeneous equation (3.14), $0 \ge \widetilde{F}D_\Phi$, and D_Φ satisfies the boundary conditions, it is readily seen that D_Φ satisfies (3.19). Thus we can choose $V_0 = D_\Phi$.

On the other hand, the upper solution can be chosen as $U_0 = (K, \dots, K)^T \in \mathbb{R}^m$, where K is a constant satisfying

$$K \ge \max\left\{\exp\left(\frac{(\bar{\kappa}_1)^2}{4}\right)\Phi(z_1), \exp\left(\frac{(\bar{\kappa}_2)^2}{4}\right)\Phi(z_2)\right\}. \tag{3.21}$$

To show that the so-chosen U_0 is indeed an upper solution, we need only to verify the first inequality in (3.20). In fact, in view of (3.9) and (3.10), substituting the constant vector $(K, \dots, K)^T$ into the inequality yields, for $i = 1, \dots, m$,

$$\frac{1}{2} - \frac{x^2}{4} - (1 + \lambda_i) = \frac{1}{2} - \frac{x^2}{4} - \left(1 + \frac{\rho}{\mu(i)} - \frac{q_{ii}}{\mu(i)}\right) \le -\sum_{j \ne i} \frac{q_{ij}}{\mu(i)},$$

which is true in view of $\sum_{j=1}^m q_{ij} = 0$ and $\mu(i) > 0$. This completes the proof of the existence of a C^2 solution. □

Remark 3.8. Define $V_{k+1} = \mathbf{K}V_k$, $U_{k+1} = \mathbf{K}U_k$, $k = 0, 1, 2 \dots$. Then it follows that

$$V_k \le V_{k+1} \le U_{k+1} \le U_k, \quad k \ge 0.$$

This string of inequalities is immediate from the monotonicity of $\mathbf{K}$. Consequently, there exist functions $\bar{V}, \bar{U}$ such that $\{V_k\} \uparrow \bar{V}$, $\{U_k\} \downarrow \bar{U}$ (pointwise and componentwise) as $k \to \infty$. Moreover, by Dini's theorem, the convergence is uniform in x. So $\bar{V}, \bar{U} \in C_m$. Applying operator (3.15) to V_k (resp., U_k) and letting $k \to \infty$, we have $\mathbf{K}\bar{V} = \bar{V}$ and $\mathbf{K}\bar{U} = \bar{U}$. Therefore, both $\bar{V}$ and $\bar{U}$ are the solutions of (3.11). From Remark 3.6, we know both $\bar{V}$ and $\bar{U}$ are C_m^2 functions. The uniqueness of the solution implies that $\bar{V} = \bar{U}$.

Remark 3.9. From the proof of Theorem 3.5, we also see that the C^2 solution of the system (2.13) (and therefore the objective function (2.9)) is continuous with respect to the boundary points z_1 and z_2.

Now we study the optimality of the objective function (2.9). We make the following assumption on z_1 and z_2; see Zhang [26] for further discussions.

ASSUMPTION 3.10.

$$a_1 \le z_1 \le b_1, \quad a_2 \le z_2 \le b_2,$$

where a_1, b_1, a_2, b_2 *are prespecified constants satisfying* $-\infty < a_1 < b_1 < 0 < a_2 < b_2 < \infty$.

THEOREM 3.11. *Under Assumptions* 3.4 *and* 3.10, *the following assertions hold:*

1. *For each* $1 \le i \le m$, $v(z, i) \in C^2$ *and is the unique solution to* (2.11) *and* (2.12).

2. *For each fixed pair* (z,i), $v(z,i)$ *is a continuous function of* (z_1,z_2) *on* $[a_1,b_1]\times[a_2,b_2]$.
3. *There exists an optimal pair* $(z_1^*,z_2^*)\in[a_1,b_1]\times[a_2,b_2]$ *that maximizes the objective function* (2.9).

Proof. Parts 1 and 2 are obtained by Theorem 3.5 together with Dynkin's formula. Part 3 follows from the compactness of $[a_1,b_1]\times[a_2,b_2]$. □

We provide a numerical example to demonstrate the approximation process proposed in Remark 3.8.

Example 3.12. Consider a two-dimensional system ($m=2$) and construct the two sequences of approximation solutions (upper and lower) by iteratively solving the corresponding boundary value problems. We numerically solve these equations and graphically display the convergence of the two sequences.

When $m=2$, the system (3.11) can be written componentwise as

$$\text{(3.22)}\quad \begin{cases} D_{1,xx}(x)+\left(\dfrac{1}{2}-\dfrac{x^2}{4}-(1+\lambda_1)\right)D_1(x)=-\dfrac{q_{12}}{\mu(1)}D_2(x),\\ D_{2,xx}(x)+\left(\dfrac{1}{2}-\dfrac{x^2}{4}-(1+\lambda_2)\right)D_2(x)=-\dfrac{q_{21}}{\mu(2)}D_1(x)\quad \text{for } x\in(\bar\kappa_1,\bar\kappa_2),\\ D_i(\bar\kappa_1)=\exp\left(\dfrac{(\bar\kappa_1)^2}{4}\right)\Phi(z_1),\quad D_i(\bar\kappa_2)=\exp\left(\dfrac{(\bar\kappa_2)^2}{4}\right)\Phi(z_2),\ \ i=1,2. \end{cases}$$

We first find the solution $D_\Phi=(D_{1,\Phi},D_{2,\Phi})^T$ of the associated homogeneous equations with nonhomogeneous boundary conditions, i.e.,

$$\text{(3.23)}\quad \begin{cases} D_{1,xx}(x)+\left(\dfrac{1}{2}-\dfrac{x^2}{4}-(1+\lambda_1)\right)D_1(x)=0,\\ D_{2,xx}(x)+\left(\dfrac{1}{2}-\dfrac{x^2}{4}-(1+\lambda_2)\right)D_2(x)=0\quad \text{for } x\in(\bar\kappa_1,\bar\kappa_2),\\ D_i(\bar\kappa_1)=\exp\left(\dfrac{(\bar\kappa_1)^2}{4}\right)\Phi(z_1),\quad D_i(\bar\kappa_2)=\exp\left(\dfrac{(\bar\kappa_2)^2}{4}\right)\Phi(z_2),\ \ i=1,2. \end{cases}$$

To make the expression compact, let

$$W_\lambda(x)=\exp\left(-\frac{x^2}{4}\right)\int_0^\infty t^\lambda\exp\left(-\frac{t^2}{2}-xt\right)\,dt.$$

Then we have

$$\text{(3.24)}\qquad D_{1,\Phi}(x)=C_1W_{\lambda_1}(x)+C_2W_{\lambda_1}(-x),$$

where the two constants C_1,C_2 are determined by the given pair of boundary conditions,

$$C_1=\frac{W_{\lambda_1}(-\bar\kappa_2)\exp\left(\frac{(\bar\kappa_1)^2}{4}\right)\Phi(z_1)-W_{\lambda_1}(-\bar\kappa_1)\exp\left(\frac{(\bar\kappa_2)^2}{4}\right)\Phi(z_2)}{W_{\lambda_1}(\bar\kappa_1)W_{\lambda_1}(-\bar\kappa_2)-W_{\lambda_1}(-\bar\kappa_1)W_{\lambda_1}(\bar\kappa_2)},$$

$$C_2=\frac{W_{\lambda_1}(\bar\kappa_1)\exp\left(\frac{(\bar\kappa_2)^2}{4}\right)\Phi(z_2)-W_{\lambda_1}(\bar\kappa_2)\exp\left(\frac{(\bar\kappa_1)^2}{4}\right)\Phi(z_1)}{W_{\lambda_1}(\bar\kappa_1)W_{\lambda_1}(-\bar\kappa_2)-W_{\lambda_1}(-\bar\kappa_1)W_{\lambda_1}(\bar\kappa_2)}.$$

Replacing λ_1 in the equations for $D_{1,\Phi}$ with λ_2 yields $D_{2,\Phi}$. Thus we obtain an analytical lower solution $V_0=D_\Phi$.

Starting at $V_0 = (V_{0,1}, V_{0,2})^T$, the approximate sequence $V_k = (V_{k,1}, V_{k,2})^T$, $k \geq 1$, can be constructed by iteratively solving the following two-point boundary value problem:

(3.25)
$$\begin{cases} V_{k+1,1,xx}(x) + \left(\frac{1}{2} - \frac{x^2}{4} - (1+\lambda_1)\right) V_{k+1,1}(x) = -\frac{q_{12}}{\mu(1)} V_{k,2}(x) \quad \text{for } x \in (\bar\kappa_1, \bar\kappa_2), \\ V_{k+1,1}(\bar\kappa_1) = \exp\left(\frac{(\bar\kappa_1)^2}{4}\right)\Phi(z_1), \quad V_{k+1,1}(\bar\kappa_2) = \exp\left(\frac{(\bar\kappa_2)^2}{4}\right)\Phi(z_2), \end{cases}$$

(3.26)
$$\begin{cases} V_{k+1,2,xx}(x) + \left(\frac{1}{2} - \frac{x^2}{4} - (1+\lambda_2)\right) V_{k+1,2}(x) = -\frac{q_{21}}{\mu(2)} V_{k,1}(x) \quad \text{for } x \in (\bar\kappa_1, \bar\kappa_2), \\ V_{k+1,2}(\bar\kappa_1) = \exp\left(\frac{(\bar\kappa_1)^2}{4}\right)\Phi(z_1), \quad V_{k+1,2}(\bar\kappa_2) = \exp\left(\frac{(\bar\kappa_2)^2}{4}\right)\Phi(z_2)\ . \end{cases}$$

The same process, starting at the upper solution $U_0 = (K, K)^T$, will produce the other sequence U_k, $k \geq 0$. In view of (3.21), we choose

$$K = \max\left\{\exp\left(\frac{(\bar\kappa_1)^2}{4}\right)\Phi(z_1),\ \exp\left(\frac{(\bar\kappa_2)^2}{4}\right)\Phi(z_2)\right\}.$$

We used a box method (see Zwillinger [27]) to solve (3.25) and (3.26). Various parameters for the numerical experiment were chosen as follows:

$$\mu(1) = 0.1, \quad \mu(2) = 0.2, \quad b(1) = 0, \quad b(2) = 0, \quad \sigma^2(1) = 0.25, \quad \sigma^2(2) = 0.5,$$

$$Q = (q_{ij}) = \begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix}, \quad \rho = 1, \quad z_1 = -1, \quad z_2 = 1, \quad \Phi(x) \equiv 1.$$

Figure 1 displays a number of upper and lower approximation solutions. It demonstrates that the upper and lower approximate sequences converge to a common solution, which is the unique solution to the boundary value system.

4. Stochastic optimization method. Except for the special one-dimensional case, it is very difficult to obtain the analytical representation of the objective function (2.7). Thus finding a systematic way of obtaining the optimal threshold values becomes an important task. To search for the optimal thresholds, we develop stochastic recursive approximation algorithms in this section. To this end, we reformulate the task of finding optimal thresholds as a stochastic approximation or stochastic optimization problem. For a general approach to stochastic approximation methods, the reader is referred to Kushner and Yin [17] for an up-to-date account of stochastic approximation.

4.1. Optimization problem and stochastic approximation algorithms. In lieu of using the differential equation method, we convert the optimal stopping problem to a stochastic optimization problem. The rationale is based on using a threshold-type strategy, and the underlying problem can be stated as

$$\text{Problem } \mathcal{P}: \begin{cases} \text{Find argmax } \varphi(z) = E\left\{\Phi(X(\tau))\exp(-\rho\tau)\right\}, \\ z = (z_1, z_2)^T \in [a_1, b_1] \times [a_2, b_2], \end{cases} \tag{4.1}$$

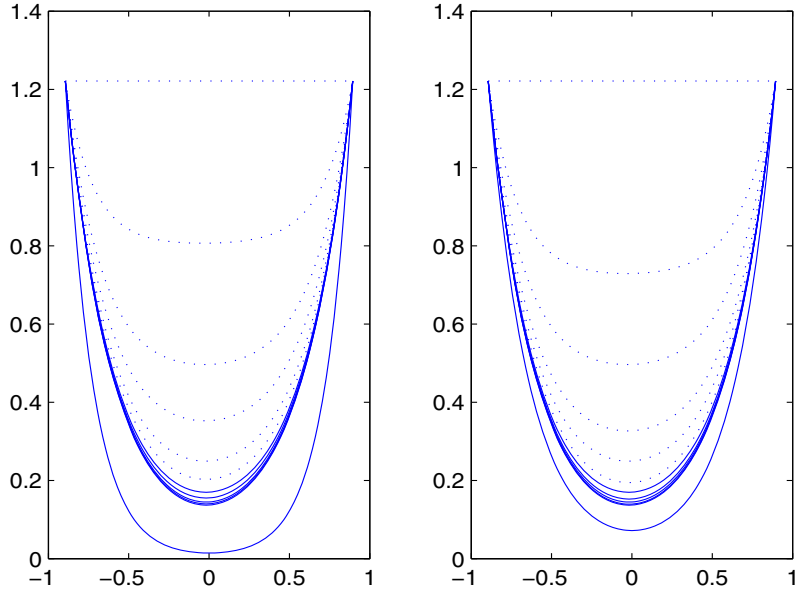

FIG. 1. *Approximation sequences and convergence. The dotted lines are the upper approximation sequences, and the solid lines are the lower approximation sequences. The left graph is for* $D_1(x)$*, and the right graph is for* $D_2(x)$*.*

where we use $\varphi(z)$ for the objective function $V(z_1, z_2)$ defined in (2.7), and τ is the stopping time defined by (2.5). Our objective is to find the optimal vector-valued threshold value for the constraint optimization problem $\mathcal{P}$.

To approximate the optimal threshold value $z^* = (z_1^*, z_2^*)^T$, we construct a recursive algorithm

$$z_{n+1} = z_n + \{\text{step size}\} \cdot \{\text{gradient estimate of } \varphi(z)\}, \tag{4.2}$$

where $z_n = (z_{n,1}, z_{n,2})^T$ denote the threshold values at the nth iteration. The step size is typically a decreasing sequence of real numbers satisfying certain conditions.

To implement (4.2), we need to construct gradient estimates of the objective function $\varphi(z)$ either by observing the real data with noisy measurements or by using a simulation. We use ξ to denote the collective random factors (including the Brownian motion, the Markov chain, and other observation noise or simulation of random effects from random seeds) so that each realization of ξ uniquely determines a sample path of the asset price dynamics (2.2) as well as the stopping time τ (2.5) for a fixed value of z. At the nth iteration, suppose the threshold values are $z_n = (z_{n,1}, z_{n,2})^T$. Let $\widetilde{\varphi}(z_n, \xi_n)$ denote the value of the discounted utility function either observed or simulated using the sample path associated with ξ_n. We assume that $E\{\widetilde{\varphi}(z, \xi_n)\} = \varphi(z)$.

Let $\Delta\widetilde{\varphi}(z_n, \xi_n) = (\Delta_1\widetilde{\varphi}(z_n, \xi_n), \Delta_2\widetilde{\varphi}(z_n, \xi_n))^T$ denote the sample path gradient estimates using a finite difference approximation, where, for $i = 1, 2$,

$$\Delta_i\widetilde{\varphi}(z_n, \xi_n) = \frac{\widetilde{\varphi}(z_n + \delta_n e_i, \xi_n) - \widetilde{\varphi}(z_n - \delta_n e_i, \xi_n)}{2\delta_n}, \tag{4.3}$$

$e_1 = (1, 0)^T$ and $e_2 = (0, 1)^T$ are the standard unit vectors, and $\{\delta_n\}$ is a sequence of positive real numbers tending to 0 and satisfying certain conditions.

Remark 4.1. The following points are worth noting.

(1) In (4.3), we use the same sample path generated by ξ_n for calculations of the function $\widetilde{\varphi}$ at different z values. This is because when Monte Carlo simulation is used to calculate a finite difference gradient approximation, using the same random numbers in calculating the two function values can reduce the variance of the estimator (see, for example, Glasserman [11]). The common random number generators can be effectively used in conjunction with stochastic approximation methods; see Kushner and Yin [17, pp. 15, 143].

(2) In the above construction of gradient estimates, instead of one simulation run, we could use multiple replications. We could use (2.2) to generate n_0 independent sample paths of $X(t)$. For each sample path, we find the value of τ, i.e., the first exit time of $X(t)$ from the interval $(z_{n,1}, z_{n,2})$. Then we construct the gradient estimates using n_0 different random seeds and then average them out. In lieu of one replication, we then use the average of n_0 replications as the gradient estimator. The advantage is that the result will be smoother. However, if we deal with real data, this idea cannot be implemented. For simplicity, we do not write the expression but refer the reader to [23] for further details.

The stochastic recursive algorithm (4.2) takes the form

$$z_{n+1} = z_n + \varepsilon_n \Delta\widetilde{\varphi}(z_n, \xi_n), \tag{4.4}$$

where $\{\varepsilon_n\}$ is a sequence of real numbers known as step sizes satisfying $0 \le \varepsilon_n \to 0$ and $\varepsilon_n/\delta_n \to 0$ as $n \to \infty$, and $\sum_n \varepsilon_n = \infty$. To ensure the boundedness of the iterates, similarly to Yin, Liu, and Zhang [23] (see also [17, p. 121]), we use the following modified stochastic approximation algorithm for the constrained problem $\mathcal{P}$:

$$z_{n+1} = \Pi[z_n + \varepsilon_n \Delta\widetilde{\varphi}(z_n, \xi_n)], \tag{4.5}$$

or, in a component form,

$$z_{n+1,i} = \Pi_{[a_i,b_i]}[z_{n,i} + \varepsilon_n \Delta_i \widetilde{\varphi}(z_n, \xi_n)] \quad \text{for} \quad i = 1, 2,$$

where the projection Π is defined as, for each real value x,

$$\Pi_{[a_i,b_i]}(x) = \begin{cases} a_i & \text{if } x < a_i, \\ b_i & \text{if } x > b_i, \\ x & \text{otherwise.} \end{cases}$$

The idea is as follows: For each component i, after the update $z_{n,i} + \varepsilon_n \Delta_i \widetilde{\varphi}(z_n, \xi_n)$ is obtained, we compare this value with the bounds a_i and b_i. If the updated value is smaller than the lower value a_i, reset the value to a_i; if it is greater than the upper value b_i, reset it to b_i; otherwise keep the value as it was. Note that in view of the techniques in [17, Chapter 5], the projection algorithm may be rewritten as

$$z_{n+1} = z_n + \varepsilon_n \Delta\widetilde{\varphi}(z_n, \xi_n) + \varepsilon_n R_n, \tag{4.6}$$

where $\varepsilon_n R_n = z_{n+1} - z_n - \varepsilon_n \Delta\widetilde{\varphi}(z_n, \xi_n)$, known as reflection term, is the minimal force needed to bring the iterates back to the constrained region if they ever escape from there.

In what follows, we present sufficient conditions guaranteeing the convergence of the algorithm. For analysis purposes only, define

$$\begin{aligned} &\psi_n = \Delta\widetilde{\varphi}(z_n, \xi_n) - E_n \Delta\widetilde{\varphi}(z_n, \xi_n), \\ &\zeta_{n,i} = E_n \Delta_i \widetilde{\varphi}(z_n, \xi_n) - [\varphi(z_n + \delta_n e_i) - \varphi(z_n - \delta_n e_i)], \quad i = 1, 2, \\ &b_{n,i} = \frac{\varphi(z_n + \delta_n e_i) - \varphi(z_n - \delta_n e_i)}{2\delta_n} - \frac{\partial \varphi(z_n)}{\partial z^i}, \quad i = 1, 2, \end{aligned} \tag{4.7}$$

where E_n denotes the conditional expectation with respect to $\mathcal{F}_n$, the σ-algebra generated by $\{z_0, \xi_j : j < n\}$, and $\varphi_z(z) = ((\partial/\partial z^1)\varphi(z), (\partial/\partial z^2)\varphi(z))^T$ denotes the gradient of $\varphi(\cdot)$. Above, $\zeta_{n,i}$ and $b_{n,i}$ for $i = 1, 2$ represent the noise and bias, and $\{\psi_n\}$ is a martingale difference sequence. This separation together with the expanded form of the recursion is for analysis purposes. As far as computation is concerned, only (4.5) is needed.

Write $\zeta_n = (\zeta_{n,1}, \zeta_{n,2})^T$ and $\beta_n = (b_{n,1}, b_{n,2})^T$ and note that $\zeta_n = \zeta_n(z_n, \xi_n)$. With the noise $\zeta_n(z_n, \xi_n)$ and the bias β_n defined above, algorithm (4.5) becomes

$$z_{n+1} = z_n + \varepsilon_n \varphi_z(z_n) + \varepsilon_n \frac{\psi_n}{2\delta_n} + \varepsilon_n \beta_n + \varepsilon_n \frac{\zeta_n}{2\delta_n} + \varepsilon_n R_n. \tag{4.8}$$

Denote $t_n = \sum_{i=1}^{n-1} \varepsilon_i$ and

$$m(t) = \begin{cases} n : t_n \le t < t_{n+1}, & t \ge 0, \\ 0, & t < 0. \end{cases}$$

To study the convergence of the algorithm, define a piecewise constant interpolation by $z^0(t) = z_n$ for $t \in [t_n, t_{n+1})$ and $z^n(t) = z^0(t + t_n)$ for $n > 0$. Similarly, define the interpolation for R_n. Let $\{\Delta_n\}$ be a sequence of positive real numbers tending to 0 as $n \to \infty$ such that $\sup_{j \ge n} \varepsilon_j/\Delta_n \to 0$. Select an increasing sequence $n = m_1 < m_2 < \cdots$ such that $\sum_{k=m_l}^{m_{l+1}-1} \varepsilon_k/\Delta_n \to 1$ as $n \to \infty$ uniformly in l. Then we have the following convergence result.

PROPOSITION 4.2. *Assume that $\varphi_{zz}(\cdot)$, the second partial derivative of $\varphi(\cdot)$, is continuous, that $\sup_n E|\widetilde{\varphi}(z, \xi_n)|^2 < \infty$ for each z, that the projected ordinary differential equation*

$$\dot{z}(t) = \varphi_z(z(t)) + r(t), \ r(t) \in C(z(t)) \tag{4.9}$$

has a unique solution for each initial condition, and that there is a unique stationary point z_* of (4.9) in $(a_1, b_1) \times (a_2, b_2)$ that is globally asymptotically stable in the sense of Liapunov. In addition, for each z in the constraint set, $\{\zeta_n(z, \xi)\}$ is uniformly integrable, and $\sum_{k=m_l}^{m_{l+1}-1} \varepsilon_k E_{m_l} \zeta(z, \xi_k)/\delta_k \to 0$ in probability. Then $z^n(\cdot)$ converges to $z(\cdot)$, the solution of the projected ordinary differential equation (4.9). Assume that $\{s_n\}$ is a sequence of real numbers satisfying $s_n \to \infty$ as $n \to \infty$. Then $z^n(s_n + \cdot)$ converges to z_* with probability 1.

In Proposition 4.2, $r(t)$ satisfies $R(t) = \int_0^t r(s)ds$, with $R(t)$ being the limit of the interpolation sequence of the projection term R_n. The set $C(z)$ is defined as follows: If z is inside $(a_1, b_1) \times (a_2, b_2)$, then $C(z)$ contains only the zero element. If z is on the boundary, then $C(z)$ is the infinite convex cone generated by the outer normal at z of the faces on which z lies; see [17, section 4.3] for more discussions. The proof of the proposition is based on a combined use of a probabilistic approach and analytic results on differential equations. For explanations on the conditions needed together with a proof, we refer the reader to [23]. In addition to the convergence, we may also study the rates of convergence and obtain large deviation-type bounds as was done in [25]. However, these are not the main concerns of the current paper. We are more interested in the numerical performance of the algorithm, which is discussed next.

4.2. Numerical results. In this section we provide two numerical examples and compare the results. We study a two-dimensional problem with variable parameters (i.e., regime-dependent parameters) in the first example and constant parameters in

TABLE 1
Optimal thresholds using the stochastic approximation algorithm.

Initial z	$-0.05, 0.05$	$-0.10, 0.20$	$-0.20, 0.40$	$-0.10, 0.60$	$-0.05, 0.85$
$z^*(n_0 = 100)$	$-0.36, 0.421$	$-0.36, 0.422$	$-0.36, 0.422$	$-0.36, 0.422$	$-0.36, 0.423$
$z^*(n_0 = 10)$	$-0.36, 0.409$	$-0.36, 0.418$	$-0.36, 0.429$	$-0.36, 0.418$	$-0.36, 0.419$
$z^*(n_0 = 1)$	$-0.36, 0.409$	$-0.36, 0.416$	$-0.36, 0.425$	$-0.359, 0.417$	$-0.359, 0.423$

the second. In both cases, the Markov chain $\alpha(t)$ takes two states, whose generator is given by

$$Q = \begin{pmatrix} -6.04 & 6.04 \\ 8.90 & -8.90 \end{pmatrix}.$$

The probability distribution of the initial Markov chain $\alpha(0)$ is given by $p_1 = p_2 = \frac{1}{2}$. We use the utility function $\Phi(x) = e^x - 1$ (see Remark 2.4).

Example 4.3. We choose the following parameter values for the regime-switching model: $\mu(1) = 0.01$, $\mu(2) = 0.02$, $b(1) = b(2) = 0$, $\sigma^2(1) = 0.25$, $\sigma^2(2) = 0.5$, and $\rho = 1$. We first implement the stochastic recursive algorithm developed in section 4.1. For the search region for $z = (z_1, z_2)$, we choose $(z_1, z_2) \in [a_1, b_1] \times [a_2, b_2] = [-0.36, -0.01] \times [0.01, 1.0]$. The sequence $\{\varepsilon_n\}$ for step sizes in (4.4) and the sequence $\{\delta_n\}$ used in the gradient estimation (4.3) are chosen to be $\varepsilon_n = 1/(n + k_0)$ and $\delta_n = 1/(n^{1/6} + k_1)$, respectively, where k_0 and k_1 are some positive integers, e.g., $k_0 = k_1 = 1$. The search stops whenever $\varepsilon_n < 0.001$. In what follows, we use n_0 replications, as presented in Remark 4.1. Table 1 reports the search results by using the stochastic recursive algorithm for five different initial values of z and for three different n_0 for gradient estimation. Note that the last row in the table ($n_0 = 1$) gives the results obtained by using a single path gradient estimate in the recursion.

Next we numerically solve the differential equations (2.11) with boundary conditions (2.12). For this example, they become

$$\begin{cases} \dfrac{\sigma^2(1)}{2}\dfrac{d^2v(z,1)}{dz^2} + \mu(1)z\dfrac{dv(z,1)}{dz} + (q_{11} - \rho)v(z,1) + q_{12}v(z,2) = 0, \\ \dfrac{\sigma^2(2)}{2}\dfrac{d^2v(z,2)}{dz^2} + \mu(2)z\dfrac{dv(z,2)}{dz} + (q_{22} - \rho)v(z,2) + q_{21}v(z,1) = 0, \\ v(z_1, i) = e^{z_1} - 1, \quad v(z_2, i) = e^{z_2} - 1, \quad i = 1, 2. \end{cases} \tag{4.10}$$

We use a grid size 0.01 to divide the region $[-0.36, -0.01] \times [0.01, 1.0]$ for (z_1, z_2). This results in 36 points along z_1, 100 points along z_2, and totally 3600 different pairs for (z_1, z_2). For each pair, which specifies the boundary values, a finite difference scheme is used to solve the system (4.10). The objective function $V(z_1, z_2)$ is then calculated by $V(z_1, z_2) = [v(0, 1) + v(0, 2)]/2$. Figure 2 plots the surface $V(z_1, z_2)$ using the 3600 values. The numerical results show that the maximum value for $V(z_1, z_2)$ is achieved at $(-0.36, 0.41)$ and $(-0.36, 0.42)$. This suggests that the optimal threshold (z_1^*, z_2^*) is very close to these two points. It is consistent with the estimates obtained in Table 1 by using the stochastic optimization algorithms. Note that numerically solving the differential equations is time consuming, while the stochastic recursive algorithms produce the optimal estimates in much less computation. This efficiency becomes more eminent when a small number of sample paths is used in gradient estimation. From Table 1 we notice that even a single sample path yields pretty good approximations to the optimal thresholds.

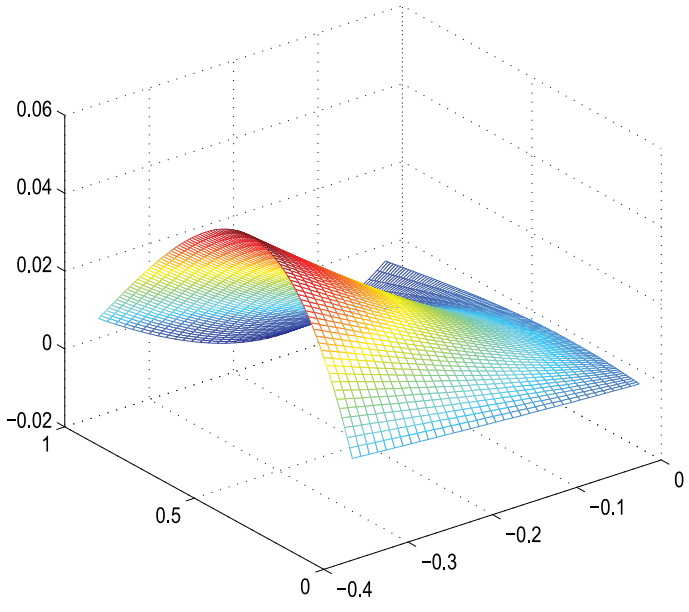

FIG. 2. *Surface of the value function* $V(z_1, z_2)$ *over the region* $(z_1, z_2) \in [-0.36, -0.01] \times [0.01, 1.0]$. *Grid size* 0.01 *is used.*

TABLE 2
Comparison of optimal selling rules in different markets.

	Optimal threshold (z_1^*, z_2^*)	Percentage increase in asset price	Percentage decrease in asset price
Bear market (Case I, Ex. 3)	$(-0.36, 0.33)$	39%	30%
Bull market (Case II, Ex. 3)	$(-0.36, 0.55)$	73%	30%
Mixed market (Ex. 2)	$(-0.36, 0.42)$	52%	30%

Based on the results, we may conclude that the optimal threshold for this specific example is given by $(z_1^*, z_2^*) = (-0.36, 0.42)$ with double-digit precision. This pair of values corresponds to a 52% increase and a 30% decrease in asset price, respectively. Following the selling rule, an investor would sell the asset he or she has bought whenever the price goes up by 52% or down by 30%.

Example 4.4. In this example we assume that the model parameters do not change across regimes, i.e., $\mu(1) = \mu(2) = \mu$, $\sigma(1) = \sigma(2) = \sigma$, while keeping other values the same, as in the last example. We report two cases: one uses regime 1 parameters and another uses regime 2 parameters from Example 4.3.

Case 1. $\mu = 0.01$, $\sigma^2 = 0.25$. The optimal thresholds are $(z_1^*, z_2^*) = (-0.36, 0.33)$, which correspond to a 39% increase and a 30% decrease in asset price.

Case 2. $\mu = 0.02$, $\sigma^2 = 0.50$. The optimal thresholds are $(z_1^*, z_2^*) = (-0.36, 0.55)$, which correspond to a 73% increase and a 30% decrease in asset price.

For comparison, in Table 2, we summarize the results from Examples 4.4 and 4.3. We may call Case 2 in Example 4.4 a bull market since a bigger μ value is used and Case 1 a bear market since a smaller μ value is used. Then we call Example 4.3 a mixed market because of the switching between the two μ numbers. Note that the optimal selling rules change in a manner that agrees with common investment practice. If a 30% drop in asset price is used by investors for the stop-loss limit, then

the upper threshold for achieving maximum profit is higher (73%) in the bull market than that in the (more realistic) mixed market (52%), which in turn is higher than that in the bear market (39%).

5. Concluding remarks. In this paper we developed an optimal selling rule using a regime-switching exponential Gaussian diffusion model. The optimal selling can be characterized by two threshold levels. We designed a numerical algorithm for searching these threshold levels.

Note that our results in this paper rely crucially on Assumption 3.4. It is interesting and practically useful to relax these conditions. In addition, we assumed the market mode to be completely observable. In order to apply our results in practice, one needs to estimate the system mode using nonlinear filtering techniques. The Wonham filter, in which the hidden Markov chain $\alpha(t)$ is observed in noise, is a good candidate; it provides sound conditional probability estimates given the stock price up to time t.

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