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## CONJUGATE TYPE BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS

P. W. ELOE AND L. J. GRIMM

Dedicated to Professor Lloyd K. Jackson  
on the occasion of his sixtieth birthday.

**1. Introduction and preliminaries.** Two-point boundary value problems (BVP's) for delay differential equations have been studied extensively, beginning with the work of G. A. Kamenskii, S. B. Norkin and others (see [5], [7]) which was motivated by variational problems and problems in oscillation theory. L. J. Grimm and K. Schmitt [4] and Ju. I. Kovač and L. I. Savčenko [6] employed solutions of various differential inequalities for the study of two-point problems with retarded argument. In this paper, we show how a bilateral iteration procedure can be developed to yield existence and inclusion theorems for multipoint boundary value problems of conjugate type for nonlinear functional-differential equations.

Let  $n > 1$ ,  $I = [a, b]$  be a real compact interval, let  $a = x_1 < x_2 < \dots < x_k = b$ , let  $p_1(x), p_2(x), \dots, p_n(x)$  be continuous on  $I$ , and define the linear differential operator  $L$  by

$$(1.1) \quad Ly = y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_n(x)y.$$

A Ju. Levin (see Coppel [1]) has obtained the following result which will play a central role in our work.

**THEOREM 1.1.** *Let  $L$  and  $I$  be as above, and suppose that  $L$  is disconjugate on  $I$ . Then the Green's function  $G(x, s)$  for the  $k$ -point conjugate type boundary value problem*

$$(1.2) \quad Ly = 0,$$

$$(1.3) \quad y^{(i)}(x_j) = 0, \quad i = 0, \dots, n_j - 1, \quad j = 1, \dots, k,$$

where  $\sum_{j=1}^k n_j = n$ , satisfies the inequality

$$(1.4) \quad G(x, s)(x - x_1)^{n_1}(x - x_2)^{n_2} \dots (x - x_k)^{n_k} \geq 0, \quad x_1 < s < x_k.$$

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**2. Multipoint problems.** Let  $I$  be as above, with  $L$  defined by (1.1) and disconjugate on  $I$ ; let  $f: I \times \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $g: I \rightarrow \mathbb{R}$  be continuous, and let  $c_{ij}, i = 0, \dots, n_j - 1, j = 1, \dots, k$ , be real constants, where  $\sum_{j=1}^k n_j = n$ . Define  $\alpha = \min(\min_{x \in I} g(x), a)$ ,  $\beta = \max(\max_{x \in I} g(x), b)$ ,  $J_1 = [\alpha, a]$ , and  $J_2 = [b, \beta]$ .

Consider the conjugate type BVP

$$(2.1) \quad Ly(x) = f(x, y(x), y(g(x))),$$

$$(2.2) \quad \begin{aligned} y^{(i)}(x_j) &= c_{ij}, \quad 0 \leq i \leq n_j - 1, j = 1, \dots, k, \\ y(x) &\equiv \phi_\nu(x), \quad x \in J_\nu, \nu = 1, 2, \end{aligned}$$

where  $\phi_\nu(x)$  is continuous on  $J_\nu$  and  $\phi_1(a) = c_{01}, \phi_2(b) = c_{0k}$ . We shall denote (2.1) by

$$(2.3) \quad Ly = f[x, y],$$

and the boundary conditions (2.2) by

$$(2.4) \quad Ty = \begin{Bmatrix} c \\ \phi \end{Bmatrix}.$$

Assume that  $f$  satisfies the uniform Lipschitz condition

$$(2.5) \quad |f(x, y_1, z_1) - f(x, y_2, z_2)| \leq P(|y_1 - y_2| + |z_1 - z_2|)$$

for all  $(x, y_1, z_1), (x, y_2, z_2)$  in  $I \times \mathbb{R}^2$ , where  $P$  is a constant. Suppose there exist functions  $v_1(x)$  and  $w_1(x)$  continuous on  $J_1 \cup I \cup J_2$  and  $n$  times continuously differentiable on  $I$ , such that

$$Tv_1 = Tw_1 = \begin{Bmatrix} c \\ \phi \end{Bmatrix},$$

and such that, for  $x \in I$ ,

$$(2.6) \quad \begin{aligned} Lv_1 - f[x, v_1] + A_1(x) &\leq 0, \\ Lw_1 - f[x, w_1] - A_1(x) &\geq 0, \end{aligned}$$

where  $A_1(x) \equiv P(|v_1(x) - w_1(x)| + |v_1(g(x)) - w_1(g(x))|)$ . Let  $l_c(x)$  denote the unique solution of the problem  $Lu = 0, u^{(i)}(x_j) = c_{ij}, i = 0, \dots, n_j - 1, j = 1, \dots, k$ , and construct sequences  $\{v_m(x)\}$  and  $\{w_m(x)\}$  as follows:

$$(2.7) \quad \begin{aligned} v_{m+1}(x) &= \begin{cases} \phi_1(x), & x \in J_1, \\ \zeta_c(x) + \int_I G(x, s)(f[s, v_m] - A_m(s))ds, & x \in I, \\ \phi_2(x), & x \in J_2; \end{cases} \\ w_{m+1}(x) &= \begin{cases} \phi_1(x), & x \in J_1, \\ \zeta_c(x) + \int_I G(x, s)(f[s, w_m] + A_m(s))ds, & x \in I, \\ \phi_2(x), & x \in J_2, \end{cases} \end{aligned}$$

where

$$(2.8) \quad A_m(x) = P(|v_m(x) - w_m(x)| + |v_m(g(x)) - w_m(g(x))|), m \geq 1.$$

THEOREM 2.1. Let  $L$  be given by (1.1) and be disconjugate on  $I = [a, b]$ . Let  $f: I \times \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $g: I \times \mathbb{R} \rightarrow \mathbb{R}$  be continuous and let  $f$  satisfy (2.5). Suppose there exist functions  $v_1(x)$  and  $w_1(x)$  which satisfy (2.4) and (2.6), and define the sequences  $\{v_m(x)\}$  and  $\{w_m(x)\}$  by (2.7). Then the BVP (2.1)–(2.2) has a solution  $y(x)$  such that, for each  $m \geq 1$ ,

$$(2.9) \quad \begin{aligned} v_m(x) &\geq v_{m+1}(x) \geq y(x) \geq w_{m+1}(x) \geq w_m(x), \quad x \in I_1, \\ v_m(x) &\leq v_{m+1}(x) \leq y(x) \leq w_{m+1}(x) \leq w_m(x), \quad x \in I_2, \end{aligned}$$

where  $I_1 = \{x \in I: G(x, s) \leq 0\}$  and  $I_2 = \{x \in I: G(x, s) \geq 0\}$ .

PROOF. Set  $u_m(x) = v_m(x) - w_m(x)$ ,  $m \geq 1$ . By (2.6),  $Lu_1 \leq 0$  for  $x \in I$ , and

$$Tu_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix};$$

thus,  $u_1(x) = \int_I G(x, s)Lu_1(s)ds$  has sign opposite to that of  $G(x, s)$  for  $x \in I$ . Similarly, for each  $m > 1$ ,

$$u_{m+1}(x) = \int_I G(x, s)(f[s, v_m] - f[s, w_m] - 2A_m(s))ds,$$

for each  $x \in I$ . Noting that  $f[x, v_m] - f[x, w_m] - 2A_m(x) \leq 0$  for  $x \in I$ , it follows that, for each  $m \geq 1$ ,

$$(2.10) \quad v_m(x) \geq w_m(x), \quad x \in I_1; \quad v_m(x) \leq w_m(x), \quad x \in I_2.$$

We now show the monotonicity of the sequences  $\{v_m(x)\}$  and  $\{w_m(x)\}$  on  $I_1$  and on  $I_2$ . From (2.6), note that  $L(v_1 - v_2) \leq 0$  and  $L(w_1 - w_2) \geq 0$  for  $x \in I$ . Since

$$T(v_1 - v_2) = T(w_1 - w_2) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

$(v_1 - v_2)(x) \geq 0 \geq (w_1 - w_2)(x)$ ,  $x \in I_1$  and  $(v_1 - v_2)(x) \leq 0 \leq (w_1 - w_2)(x)$ ,  $x \in I_2$ . For each  $m \geq 2$ ,

$$\begin{aligned} L(v_m - v_{m+1}) &= f[x, v_{m-1}] - f[x, v_m] - A_{m-1}(x) + A_m(x) \\ &= \begin{cases} f[x, v_{m-1}] - f[x, v_m] - P(v_{m-1}(x) - v_m(x)) \\ \quad + P(w_{m-1}(x) - w_m(x)) - P|v_{m-1}(g(x)) - w_{m-1}(g(x))| \\ \quad + P|v_m(g(x)) - w_m(g(x))|, \quad x \in I_1; \\ f[x, v_{m-1}] - f[x, v_m] + P(v_{m-1}(x) - v_m(x)) \\ \quad - P(w_{m-1}(x) - w_m(x)) - P|v_{m-1}(g(x)) - w_{m-1}(g(x))| \\ \quad + P|v_m(g(x)) - w_m(g(x))|, \quad x \in I_2. \end{cases} \end{aligned}$$

(2.11)

$$\begin{aligned}
 L(w_m - w_{m+1}) &= f[x, w_{m-1}] - f[x, w_m] + A_{m-1}(x) - A_m(x) \\
 &= \begin{cases} f[x, w_{m-1}] - f[x, w_m] - P(w_{m-1}(x) - w_m(x)) \\ \quad + P(v_{m-1}(x) - v_m(x)) + P|v_{m-1}(g(x)) - w_{m-1}(g(x))| \\ \quad - P|v_m(g(x)) - w_m(g(x))|, x \in I_1; \\ f[x, w_{m-1}] - f[x, w_m] + P(w_{m-1}(x) - w_m(x)) \\ \quad - P(v_{m-1}(x) - v_m(x)) + P|v_{m-1}(g(x)) - w_{m-1}(g(x))| \\ \quad - P|v_m(g(x)) - w_m(g(x))|, x \in I_2. \end{cases}
 \end{aligned}$$

Assume now, as induction hypothesis, that for  $m > 1$ ,

$$(v_{m-1} - v_m)(x) \geq 0 \geq (w_{m-1} - w_m)(x), x \in I_1,$$

$$(v_{m-1} - v_m)(x) \leq 0 \leq (w_{m-1} - w_m)(x), x \in I_2.$$

Consider  $Lv_m - Lv_{m+1}$ , for  $x \in I$ . Suppose first that  $x \in I_1$ . From (2.11), it follows that

$$\begin{aligned}
 Lv_m - Lv_{m+1} &\leq P|v_{m-1}(g(x)) - v_m(g(x))| + P(w_{m-1}(x) - w_m(x)) \\
 &\quad - P|v_{m-1}(g(x)) - w_{m-1}(g(x))| + P|v_m(g(x)) - w_m(g(x))|.
 \end{aligned}$$

If  $g(x)$  is in  $J_1$  or  $J_2$ , then

$$Lv_m - Lv_{m+1} \leq P(w_{m-1}(x) - w_m(x)) \leq 0.$$

If  $g(x)$  is in  $I_1$ , then

$$Lv_m - Lv_{m+1} \leq P(w_{m-1}(x) - w_m(x)) + P(w_{m-1}(g(x)) - w_m(g(x))) \leq 0.$$

If  $g(x)$  is in  $I_2$ , then

$$Lv_m - Lv_{m+1} \leq P(w_{m-1}(x) - w_m(x)) + P(w_m(g(x)) - w_{m-1}(g(x))) \leq 0.$$

Thus, for  $x \in I_1$ ,  $Lv_m - Lv_{m+1} \leq 0$ . Similarly, for  $x \in I_2$ ,  $Lv_m - Lv_{m+1} \leq 0$ . Since

$$T(v_m - v_{m+1}) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

$v_m - v_{m+1} \geq 0$ ,  $x \in I_1$  and  $v_m - v_{m+1} \leq 0$ ,  $x \in I_2$ . Analogously, we find that  $Lw_m - Lw_{m+1} \geq 0$  on  $I$  and that  $w_m - w_{m+1} \leq 0$ ,  $x \in I_1$  and  $w_m - w_{m+1} \geq 0$ ,  $x \in I_2$ . Hence,

$$v_m(x) \geq v_{m+1}(x) \geq w_{m+1}(x) \geq w_m(x), x \in I_1,$$

$$v_m(x) \leq v_{m+1}(x) \leq w_{m+1}(x) \leq w_m(x), x \in I_2, m \geq 1.$$

It remains to show that there is a solution  $y(x)$  of (2.1)–(2.2) which satisfies (2.9). Note that, on  $I_1$ ,  $I_2$ ,  $J_1$  and  $J_2$ , the sequences  $\{v_m(x)\}$  and  $\{w_m(x)\}$  are monotonic, bounded, and equicontinuous. By Ascoli's

theorem, they have uniform limits  $v(x)$  and  $w(x)$  with  $v(x) \geq w(x), x \in I_1, v(x) \leq w(x), x \in I_2$ , and  $v(x) \equiv w(x) \equiv \phi_\nu(x)$  on  $J_\nu, \nu = 1, 2$ . It follows from (2.7) that, for  $x \in I$ ,

$$Lv(x) = f[x, v] - A(x),$$

$$Lw(x) = f[x, w] + A(x),$$

where  $A(x) = P(|v(x) - w(x)| + |v(g(x)) - w(g(x))|)$ , and that

$$Tv = Tw = \begin{Bmatrix} c \\ \phi \end{Bmatrix}.$$

Now, for each function  $y(x) \in C(J_1 \cup I \cup J_2)$ , define  $\bar{y}$  by

$$\bar{y}(x) = \begin{cases} \phi_1(x), & \text{if } x \in J_1, \\ \left. \begin{array}{l} v(x), & \text{if } y(x) > v(x), \\ y(x), & \text{if } v(x) \geq y(x) \geq w(x), \\ w(x), & \text{if } y(x) < w(x), \end{array} \right\} & x \in I_1, \\ \left. \begin{array}{l} v(x), & \text{if } y(x) < v(x), \\ y(x), & \text{if } v(x) \leq y(x) \leq w(x), \\ w(x), & \text{if } y(x) > w(x), \end{array} \right\} & x \in I_2, \\ \phi_2(x), & \text{if } x \in J_2, \end{cases}$$

and define  $F(x, y(x), y(g(x))) = f(x, \bar{y}(x), \bar{y}(g(x)))$ . The function  $F$  is continuous and bounded on  $I \times \mathbf{R}^2$  and it follows from the Schauder Fixed Point Theorem that the problem

$$Ly = F(x, y(x), y(g(x))),$$

$$Ty = \begin{Bmatrix} c \\ \phi \end{Bmatrix}$$

has a solution  $y(x)$ . We now show that  $y(x)$  satisfies

$$(2.12) \quad w(x) \leq y(x) \leq v(x), \quad x \in I_1, \quad w(x) \geq y(x) \geq v(x), \quad x \in I_2,$$

and hence that  $y(x)$  is a solution of (2.1)–(2.2) which satisfies (2.9). Consider  $w(x) - y(x)$ . Using the definition of  $\bar{y}$ , we find that

$$Lw - Ly = f[x, w] + P(|v(x) - w(x)| + |v(g(x)) - w(g(x))|) - f(x, \bar{y}(x), \bar{y}(g(x))) \geq 0,$$

and

$$T(w - y) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Thus,  $w(x) \leq y(x)$ ,  $x \in I_1$ ,  $w(x) \geq y(x)$ ,  $x \in I_2$ . Similarly,  $v(x) \geq y(x)$ ,  $x \in I_1$ ,  $v(x) \leq y(x)$ ,  $x \in I_2$ . Hence,  $y(x)$  satisfies (2.12) and the proof is complete.

REMARKS. (a) The procedure developed here can be applied to additional kinds of boundary value problems, including  $k$ -focal problems with retarded argument, see [2]. We obtained an analogous result for  $k$ -focal problems for ordinary differential equations in an earlier paper [3]. The computations in the two-point  $k$ -focal case are simpler because the Green's function is of constant sign on the entire interval.

(b) If  $G = \max_{x \in I} |\int_I G(x, s) ds|$  and if  $2PG < 1$ , a contraction mapping argument may be used to prove the existence and uniqueness of a solution of (2.1)–(2.2). If, in fact,  $6PG < 1$ , then  $A_m(x)$ , defined by (2.8), tends to zero as  $m \rightarrow \infty$ . Thus,  $v(x) = w(x)$  is the unique solution of (2.1)–(2.2).

(c) If  $G$  is as in (b),  $2PG < 1$ , and  $|f(x, y, z)|$  is bounded by a constant  $B$  for all  $(x, y, z) \in I \times \mathbb{R}^2$ , the functions  $v_1(x)$  and  $w_1(x)$  can be chosen as

$$v_1(x) = \begin{cases} \phi_1(x), & x \in J_1, \\ \zeta_c(x) - \frac{B}{1-2PG} \int_I G(x, s) ds & \\ \phi_2(x), & x \in J_2, \end{cases}$$

$$w_1(x) = \begin{cases} \phi_1(x), & x \in J_1, \\ \zeta_c(x) + \frac{B}{1-2PG} \int_I G(x, s) ds & \\ \phi_2(x), & x \in J_2. \end{cases}$$

(d) The requirement that  $v_1(x)$  and  $w_1(x)$  satisfy the boundary conditions (2.2) can be relaxed somewhat. If  $v_1$  and  $w_1$  satisfy conditions analogous to the conditions (3.1)–(3.4) of Theorem 3.1 of [8], a modification of the iteration procedure leads to the conclusion of Theorem 2.1.

(e) As an example, consider the BVP

$$(2.13) \quad y''' = 1 - xy(x) + y(2x - 1),$$

$$y(x) \equiv -x, \quad x \in J_1 = [-1, 0],$$

$$(2.14) \quad y(0) = y(1) = y(2) = 0,$$

$$y(x) \equiv x - 2, \quad x \in J_2 = [2, 3].$$

For this problem,  $P = 2$ . Let  $w_1(x) = x(x - 1)(x - 2)$ ,  $v_1(x) = -w_1$ , for  $x \in I$ . Then it is easy to see that

$$Lv_1 - f[x, v_1] + A_1(x) = -6 - f[x, v_1] + A_1(x) \leq 0;$$

$$Lw_1 - f[x, w_1] - A_1(x) \geq 0, \quad x \in I.$$

Hence the problem (2.13)–(2.14) has a solution  $y(x)$  between  $v_1$  and  $w_1$ .

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