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CONJUGATE TYPE BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS

P. W. ELOE AND L. J. GRIMM

Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

1. Introduction and preliminaries. Two-point boundary value problems (BVP's) for delay differential equations have been studied extensively, beginning with the work of G. A. Kamenskii, S. B. Norkin and others (see [5], [7]) which was motivated by variational problems and problems in oscillation theory. L. J. Grimm and K. Schmitt [4] and Ju. I. Kovač and L. I. Savčenko [6] employed solutions of various differential inequalities for the study of two-point problems with retarded argument. In this paper, we show how a bilateral iteration procedure can be developed to yield existence and inclusion theorems for multipoint boundary value problems of conjugate type for nonlinear functional-differential equations.

Let $n > 1$, $I = [a, b]$ be a real compact interval, let $a = x_1 < x_2 < \dots < x_k = b$, let $p_1(x), p_2(x), \dots, p_n(x)$ be continuous on I , and define the linear differential operator L by

$$(1.1) \quad Ly = y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_n(x)y.$$

A Ju. Levin (see Coppel [1]) has obtained the following result which will play a central role in our work.

THEOREM 1.1. *Let L and I be as above, and suppose that L is disconjugate on I . Then the Green's function $G(x, s)$ for the k -point conjugate type boundary value problem*

$$(1.2) \quad Ly = 0,$$

$$(1.3) \quad y^{(i)}(x_j) = 0, \quad i = 0, \dots, n_j - 1, \quad j = 1, \dots, k,$$

where $\sum_{j=1}^k n_j = n$, satisfies the inequality

$$(1.4) \quad G(x, s)(x - x_1)^{n_1}(x - x_2)^{n_2} \dots (x - x_k)^{n_k} \geq 0, \quad x_1 < s < x_k.$$

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2. Multipoint problems. Let I be as above, with L defined by (1.1) and disconjugate on I ; let $f: I \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g: I \rightarrow \mathbb{R}$ be continuous, and let $c_{ij}, i = 0, \dots, n_j - 1, j = 1, \dots, k$, be real constants, where $\sum_{j=1}^k n_j = n$. Define $\alpha = \min(\min_{x \in I} g(x), a)$, $\beta = \max(\max_{x \in I} g(x), b)$, $J_1 = [\alpha, a]$, and $J_2 = [b, \beta]$.

Consider the conjugate type BVP

$$(2.1) \quad Ly(x) = f(x, y(x), y(g(x))),$$

$$(2.2) \quad \begin{aligned} y^{(i)}(x_j) &= c_{ij}, \quad 0 \leq i \leq n_j - 1, j = 1, \dots, k, \\ y(x) &\equiv \phi_\nu(x), \quad x \in J_\nu, \nu = 1, 2, \end{aligned}$$

where $\phi_\nu(x)$ is continuous on J_ν and $\phi_1(a) = c_{01}, \phi_2(b) = c_{0k}$. We shall denote (2.1) by

$$(2.3) \quad Ly = f[x, y],$$

and the boundary conditions (2.2) by

$$(2.4) \quad Ty = \begin{Bmatrix} c \\ \phi \end{Bmatrix}.$$

Assume that f satisfies the uniform Lipschitz condition

$$(2.5) \quad |f(x, y_1, z_1) - f(x, y_2, z_2)| \leq P(|y_1 - y_2| + |z_1 - z_2|)$$

for all $(x, y_1, z_1), (x, y_2, z_2)$ in $I \times \mathbb{R}^2$, where P is a constant. Suppose there exist functions $v_1(x)$ and $w_1(x)$ continuous on $J_1 \cup I \cup J_2$ and n times continuously differentiable on I , such that

$$Tv_1 = Tw_1 = \begin{Bmatrix} c \\ \phi \end{Bmatrix},$$

and such that, for $x \in I$,

$$(2.6) \quad \begin{aligned} Lv_1 - f[x, v_1] + A_1(x) &\leq 0, \\ Lw_1 - f[x, w_1] - A_1(x) &\geq 0, \end{aligned}$$

where $A_1(x) \equiv P(|v_1(x) - w_1(x)| + |v_1(g(x)) - w_1(g(x))|)$. Let $l_c(x)$ denote the unique solution of the problem $Lu = 0, u^{(i)}(x_j) = c_{ij}, i = 0, \dots, n_j - 1, j = 1, \dots, k$, and construct sequences $\{v_m(x)\}$ and $\{w_m(x)\}$ as follows:

$$(2.7) \quad \begin{aligned} v_{m+1}(x) &= \begin{cases} \phi_1(x), & x \in J_1, \\ l_c(x) + \int_I G(x, s)(f[s, v_m] - A_m(s))ds, & x \in I, \\ \phi_2(x), & x \in J_2; \end{cases} \\ w_{m+1}(x) &= \begin{cases} \phi_1(x), & x \in J_1, \\ l_c(x) + \int_I G(x, s)(f[s, w_m] + A_m(s))ds, & x \in I, \\ \phi_2(x), & x \in J_2, \end{cases} \end{aligned}$$

where

$$(2.8) \quad A_m(x) = P(|v_m(x) - w_m(x)| + |v_m(g(x)) - w_m(g(x))|), m \geq 1.$$

THEOREM 2.1. *Let L be given by (1.1) and be disconjugate on $I = [a, b]$. Let $f: I \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g: I \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and let f satisfy (2.5). Suppose there exist functions $v_1(x)$ and $w_1(x)$ which satisfy (2.4) and (2.6), and define the sequences $\{v_m(x)\}$ and $\{w_m(x)\}$ by (2.7). Then the BVP (2.1)–(2.2) has a solution $y(x)$ such that, for each $m \geq 1$,*

$$(2.9) \quad \begin{aligned} v_m(x) &\geq v_{m+1}(x) \geq y(x) \geq w_{m+1}(x) \geq w_m(x), \quad x \in I_1, \\ v_m(x) &\leq v_{m+1}(x) \leq y(x) \leq w_{m+1}(x) \leq w_m(x), \quad x \in I_2, \end{aligned}$$

where $I_1 = \{x \in I: G(x, s) \leq 0\}$ and $I_2 = \{x \in I: G(x, s) \geq 0\}$.

PROOF. Set $u_m(x) = v_m(x) - w_m(x)$, $m \geq 1$. By (2.6), $Lu_1 \leq 0$ for $x \in I$, and

$$Tu_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix};$$

thus, $u_1(x) = \int_I G(x, s)Lu_1(s)ds$ has sign opposite to that of $G(x, s)$ for $x \in I$. Similarly, for each $m > 1$,

$$u_{m+1}(x) = \int_I G(x, s)(f[s, v_m] - f[s, w_m] - 2A_m(s))ds,$$

for each $x \in I$. Noting that $f[x, v_m] - f[x, w_m] - 2A_m(x) \leq 0$ for $x \in I$, it follows that, for each $m \geq 1$,

$$(2.10) \quad v_m(x) \geq w_m(x), \quad x \in I_1; \quad v_m(x) \leq w_m(x), \quad x \in I_2.$$

We now show the monotonicity of the sequences $\{v_m(x)\}$ and $\{w_m(x)\}$ on I_1 and on I_2 . From (2.6), note that $L(v_1 - v_2) \leq 0$ and $L(w_1 - w_2) \geq 0$ for $x \in I$. Since

$$T(v_1 - v_2) = T(w_1 - w_2) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

$(v_1 - v_2)(x) \geq 0 \geq (w_1 - w_2)(x)$, $x \in I_1$ and $(v_1 - v_2)(x) \leq 0 \leq (w_1 - w_2)(x)$, $x \in I_2$. For each $m \geq 2$,

$$\begin{aligned} L(v_m - v_{m+1}) &= f[x, v_{m-1}] - f[x, v_m] - A_{m-1}(x) + A_m(x) \\ &= \begin{cases} f[x, v_{m-1}] - f[x, v_m] - P(v_{m-1}(x) - v_m(x)) \\ \quad + P(w_{m-1}(x) - w_m(x)) - P|v_{m-1}(g(x)) - w_{m-1}(g(x))| \\ \quad + P|v_m(g(x)) - w_m(g(x))|, \quad x \in I_1; \\ f[x, v_{m-1}] - f[x, v_m] + P(v_{m-1}(x) - v_m(x)) \\ \quad - P(w_{m-1}(x) - w_m(x)) - P|v_{m-1}(g(x)) - w_{m-1}(g(x))| \\ \quad + P|v_m(g(x)) - w_m(g(x))|, \quad x \in I_2. \end{cases} \end{aligned}$$

(2.11)

$$\begin{aligned}
 L(w_m - w_{m+1}) &= f[x, w_{m-1}] - f[x, w_m] + A_{m-1}(x) - A_m(x) \\
 &= \begin{cases} f[x, w_{m-1}] - f[x, w_m] - P(w_{m-1}(x) - w_m(x)) \\ \quad + P(v_{m-1}(x) - v_m(x)) + P|v_{m-1}(g(x)) - w_{m-1}(g(x))| \\ \quad - P|v_m(g(x)) - w_m(g(x))|, x \in I_1; \\ f[x, w_{m-1}] - f[x, w_m] + P(w_{m-1}(x) - w_m(x)) \\ \quad - P(v_{m-1}(x) - v_m(x)) + P|v_{m-1}(g(x)) - w_{m-1}(g(x))| \\ \quad - P|v_m(g(x)) - w_m(g(x))|, x \in I_2. \end{cases}
 \end{aligned}$$

Assume now, as induction hypothesis, that for $m > 1$,

$$(v_{m-1} - v_m)(x) \geq 0 \geq (w_{m-1} - w_m)(x), x \in I_1,$$

$$(v_{m-1} - v_m)(x) \leq 0 \leq (w_{m-1} - w_m)(x), x \in I_2.$$

Consider $Lv_m - Lv_{m+1}$, for $x \in I$. Suppose first that $x \in I_1$. From (2.11), it follows that

$$\begin{aligned}
 Lv_m - Lv_{m+1} &\leq P|v_{m-1}(g(x)) - v_m(g(x))| + P(w_{m-1}(x) - w_m(x)) \\
 &\quad - P|v_{m-1}(g(x)) - w_{m-1}(g(x))| + P|v_m(g(x)) - w_m(g(x))|.
 \end{aligned}$$

If $g(x)$ is in J_1 or J_2 , then

$$Lv_m - Lv_{m+1} \leq P(w_{m-1}(x) - w_m(x)) \leq 0.$$

If $g(x)$ is in I_1 , then

$$Lv_m - Lv_{m+1} \leq P(w_{m-1}(x) - w_m(x)) + P(w_{m-1}(g(x)) - w_m(g(x))) \leq 0.$$

If $g(x)$ is in I_2 , then

$$Lv_m - Lv_{m+1} \leq P(w_{m-1}(x) - w_m(x)) + P(w_m(g(x)) - w_{m-1}(g(x))) \leq 0.$$

Thus, for $x \in I_1$, $Lv_m - Lv_{m+1} \leq 0$. Similarly, for $x \in I_2$, $Lv_m - Lv_{m+1} \leq 0$. Since

$$T(v_m - v_{m+1}) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

$v_m - v_{m+1} \geq 0$, $x \in I_1$ and $v_m - v_{m+1} \leq 0$, $x \in I_2$. Analogously, we find that $Lw_m - Lw_{m+1} \geq 0$ on I and that $w_m - w_{m+1} \leq 0$, $x \in I_1$ and $w_m - w_{m+1} \geq 0$, $x \in I_2$. Hence,

$$v_m(x) \geq v_{m+1}(x) \geq w_{m+1}(x) \geq w_m(x), x \in I_1,$$

$$v_m(x) \leq v_{m+1}(x) \leq w_{m+1}(x) \leq w_m(x), x \in I_2, m \geq 1.$$

It remains to show that there is a solution $y(x)$ of (2.1)–(2.2) which satisfies (2.9). Note that, on I_1 , I_2 , J_1 and J_2 , the sequences $\{v_m(x)\}$ and $\{w_m(x)\}$ are monotonic, bounded, and equicontinuous. By Ascoli's

theorem, they have uniform limits $v(x)$ and $w(x)$ with $v(x) \geq w(x), x \in I_1, v(x) \leq w(x), x \in I_2$, and $v(x) \equiv w(x) \equiv \phi_\nu(x)$ on $J_\nu, \nu = 1, 2$. It follows from (2.7) that, for $x \in I$,

$$Lv(x) = f[x, v] - A(x),$$

$$Lw(x) = f[x, w] + A(x),$$

where $A(x) = P(|v(x) - w(x)| + |v(g(x)) - w(g(x))|)$, and that

$$Tv = Tw = \begin{Bmatrix} c \\ \phi \end{Bmatrix}.$$

Now, for each function $y(x) \in C(J_1 \cup I \cup J_2)$, define \bar{y} by

$$\bar{y}(x) = \begin{cases} \phi_1(x), & \text{if } x \in J_1, \\ \left. \begin{array}{l} v(x), & \text{if } y(x) > v(x), \\ y(x), & \text{if } v(x) \geq y(x) \geq w(x), \\ w(x), & \text{if } y(x) < w(x), \end{array} \right\} & x \in I_1, \\ \left. \begin{array}{l} v(x), & \text{if } y(x) < v(x), \\ y(x), & \text{if } v(x) \leq y(x) \leq w(x), \\ w(x), & \text{if } y(x) > w(x), \end{array} \right\} & x \in I_2, \\ \phi_2(x), & \text{if } x \in J_2, \end{cases}$$

and define $F(x, y(x), y(g(x))) = f(x, \bar{y}(x), \bar{y}(g(x)))$. The function F is continuous and bounded on $I \times \mathbb{R}^2$ and it follows from the Schauder Fixed Point Theorem that the problem

$$Ly = F(x, y(x), y(g(x))),$$

$$Ty = \begin{Bmatrix} c \\ \phi \end{Bmatrix}$$

has a solution $y(x)$. We now show that $y(x)$ satisfies

$$(2.12) \quad w(x) \leq y(x) \leq v(x), \quad x \in I_1, \quad w(x) \geq y(x) \geq v(x), \quad x \in I_2,$$

and hence that $y(x)$ is a solution of (2.1)–(2.2) which satisfies (2.9). Consider $w(x) - y(x)$. Using the definition of \bar{y} , we find that

$$Lw - Ly = f[x, w] + P(|v(x) - w(x)| + |v(g(x)) - w(g(x))|) - f(x, \bar{y}(x), \bar{y}(g(x))) \geq 0,$$

and

$$T(w - y) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Thus, $w(x) \leq y(x)$, $x \in I_1$, $w(x) \geq y(x)$, $x \in I_2$. Similarly, $v(x) \geq y(x)$, $x \in I_1$, $v(x) \leq y(x)$, $x \in I_2$. Hence, $y(x)$ satisfies (2.12) and the proof is complete.

REMARKS. (a) The procedure developed here can be applied to additional kinds of boundary value problems, including k -focal problems with retarded argument, see [2]. We obtained an analogous result for k -focal problems for ordinary differential equations in an earlier paper [3]. The computations in the two-point k -focal case are simpler because the Green's function is of constant sign on the entire interval.

(b) If $G = \max_{x \in I} |\int_I G(x, s) ds|$ and if $2PG < 1$, a contraction mapping argument may be used to prove the existence and uniqueness of a solution of (2.1)–(2.2). If, in fact, $6PG < 1$, then $A_m(x)$, defined by (2.8), tends to zero as $m \rightarrow \infty$. Thus, $v(x) = w(x)$ is the unique solution of (2.1)–(2.2).

(c) If G is as in (b), $2PG < 1$, and $|f(x, y, z)|$ is bounded by a constant B for all $(x, y, z) \in I \times \mathbb{R}^2$, the functions $v_1(x)$ and $w_1(x)$ can be chosen as

$$v_1(x) = \begin{cases} \phi_1(x), & x \in J_1, \\ \zeta_c(x) - \frac{B}{1-2PG} \int_I G(x, s) ds & \\ \phi_2(x), & x \in J_2, \end{cases}$$

$$w_1(x) = \begin{cases} \phi_1(x), & x \in J_1, \\ \zeta_c(x) + \frac{B}{1-2PG} \int_I G(x, s) ds & \\ \phi_2(x), & x \in J_2. \end{cases}$$

(d) The requirement that $v_1(x)$ and $w_1(x)$ satisfy the boundary conditions (2.2) can be relaxed somewhat. If v_1 and w_1 satisfy conditions analogous to the conditions (3.1)–(3.4) of Theorem 3.1 of [8], a modification of the iteration procedure leads to the conclusion of Theorem 2.1.

(e) As an example, consider the BVP

$$(2.13) \quad y''' = 1 - xy(x) + y(2x - 1),$$

$$y(x) \equiv -x, \quad x \in J_1 = [-1, 0],$$

$$(2.14) \quad y(0) = y(1) = y(2) = 0,$$

$$y(x) \equiv x - 2, \quad x \in J_2 = [2, 3].$$

For this problem, $P = 2$. Let $w_1(x) = x(x - 1)(x - 2)$, $v_1(x) = -w_1$, for $x \in I$. Then it is easy to see that

$$Lv_1 - f[x, v_1] + A_1(x) = -6 - f[x, v_1] + A_1(x) \leq 0;$$

$$Lw_1 - f[x, w_1] - A_1(x) \geq 0, \quad x \in I.$$

Hence the problem (2.13)–(2.14) has a solution $y(x)$ between v_1 and w_1 .

REFERENCES

1. W. A. Coppel, *Disconjugacy*, Lecture Notes in Mathematics vol. 220, Springer, Berlin, 1971.
2. P. W. Eloe, *Solutions of differential inequalities and existence theorems for multipoint boundary value problems*, Dissertation, University of Missouri-Rolla, 1980.
3. ——— and L. J. Grimm, *Monotone iteration and Green's functions for boundary value problems*, Proc. Amer. Math. Soc. **78** (1980), 533–538.
4. L. J. Grimm and K. Schmitt, *Boundary value problems for differential equations with deviating arguments*, Aequationes Math. **3** (1969), 321–322; **4** (1970), 176–190.
5. G. A. Kamenskii and A. D. Myškis, *Variational and boundary value problems for differential equations with deviating argument*, Proc. Equadiff IV, Lecture Notes in Mathematics vol. 703, 1979, 179–188.
6. Ju. I. Kovač and L. I. Savčenko, *A boundary value problem for a nonlinear system of differential equations with retarded argument*, Ukrain. Mat. Ž. **22** (1970), 12–21.
7. S. B. Norkin, *Second order differential equations with retarded argument*, Providence: American Mathematical Society translations of Mathematical Monographs, vol. 31, 1972.
8. K. Schmitt, *Boundary value problems and comparison theorems for ordinary differential equations*, SIAM J. Appl. Math. **26** (1974), 670–678.

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