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POSITIVE SOLUTIONS AND J-FOCAL POINTS FOR TWO POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. Cone theory is applied to a class of two point boundary value problems for ordinary differential equations. Criteria for the existence of extremal points are obtained. These criteria are in terms of the existence of nontrivial solutions that lie in a cone, and in terms of the spectral radius of an associated compact linear operator.

1. Introduction. Let $n > 1$ be a positive integer, and let $\alpha < \beta$, $k \in \{1, \dots, n-1\}$, and $j \in \{0, \dots, k\}$ be given. Let $p_i \in C[\alpha, \beta]$, for $i = 0, \dots, j$, and consider the linear ordinary differential equation,

$$y^{(n)} = \sum_{i=0}^{j} p_i(x) y^{(i)}, \qquad \alpha \le x \le \beta. \tag{1.1}$$

We shall be concerned with extremal point properties of (1.1) with respect to the family of two point boundary conditions,

$$\begin{aligned} y^{(i)}(\alpha) &= 0, \qquad i = 0, \dots, k-1, \\ y^{(i)}(b) &= 0, \qquad i = j, \dots, n-k+j-1, \end{aligned} \tag{1.2_b}$$

where $b \in (\alpha, \beta]$. Note that, if $j = 0$, then (1.2_b) represents conjugate boundary conditions, if $j = k$, then (1.2_b) represents right focal boundary conditions, and if $j \in \{1, \dots, k-1\}$, then (1.2_b) represents boundary conditions that are "between" conjugate and right focal boundary conditions.

Definition. $b_0 \in (\alpha, \beta]$ is the *j-focal point of* (1.1) *corresponding to* (1.2_b) if, and only if,

$$b_0 = \inf\{b > \alpha \mid (1.1), (1.2_b) \text{ has a nontrivial solution}\}.$$

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Note that, if b_0 exists, then $b_0 > \alpha$ and there exists a nontrivial solution of the boundary value problem (1.1), (1.2_b).

Our purpose in this paper is to obtain, under certain sign conditions on the coefficients, p_i, $i = 0, \dots, j$, relationships among the existence of a j-focal point, b_0 of (1.1) corresponding to (1.2_b), the existence of a nontrivial solution of the boundary value problem (1.1), (1.2_b), that is positive with respect to a cone, and properties of the spectral radii of associated linear, compact, integral maps. For example, if $p \in C[\alpha, \beta]$, then b_0 is the conjugate point of $y'' = p(x)y$ corresponding to $y(\alpha) = y(b) = 0$ if and only if there exists a nontrivial solution of $y'' = p(x)y$, $y(\alpha) = y(b_0) = 0$, that does not vanish on (α, b_0); see [**3, 15**].

Although this classical result can be established by elementary methods, the theory of cones in a Banach space has been employed to carry this result over to other families of boundary value problems. Schmitt and Smith [**18**] applied the theory of cones to second order, m-dimensional systems of two point conjugate boundary value problems; Hankerson and Henderson [**9**] extended the techniques of Schmitt and Smith [**18**] to problems of the form (1.1), (1.2_b), with $k = n-1$, $j = n-2$. Recently, Eloe, Hankerson, and Henderson [**7**] extended these techniques to apply to multipoint conjugate boundary value problems. A number of other authors have studied similar questions for two point right focal boundary value problems; see, for example, [**8, 11, 19**].

This particular paper is largely motivated by the work of Schmitt and Smith [**18**], whose techniques have been extended by Hankerson and Henderson [**9**], and more recently, by Eloe, Hankerson, and Henderson [**7**]. The key argument in each of these papers is that a mapping, which maps a linear, compact operator, depending on b, to its spectral radius, is strictly increasing as a function of b. In [**18, 9**], the arguments to establish the strict monotoneity are geometric and rely on the specific boundary conditions. The geometric arguments have not readily carried over to other families of boundary value problems.

In [**2**], Bates and Gustafson prove that the Green's functions for a family of multipoint boundary value problems satisfy specific sign properties with respect to boundary points. In [**7**], this observation is exploited to obtain the monotoneity employed by Schmitt and Smith [**18**] and Hankerson and Henderson [**9**]. In this paper, we shall carry the results of Bates and Gustafson over to Green's functions for the family of boundary value problems, $y^{(n)} = 0$ satisfying (1.2_b), and hence, obtain the monotoneity as in [**18, 9, 7**]. We point out that the techniques we present here are valid for $j \in \{0, \dots, k-1\}$ and are not valid for the case, $j = k$. However, the case $j = k$ (i.e., the right focal case), has been resolved by Tomastik [**19**].

In Section 2, in order that the paper be self-contained, we provide preliminary definitions and results from the theory of cones in a Banach space. In Section 3, we shall obtain the sign properties of Green's functions that we will employ in Section 4 to define appropriate cones in Banach spaces. In Section 4, we shall apply the results from Section 2 and obtain criteria for the existence of a j-focal point of (1.1) corresponding to (1.2_b).

2. Cone theoretic preliminaries. In this section, in order that the paper be self-contained, we shall provide definitions and results from the theory of cones in a Banach space. We refer the reader to Krasnosel'skii [**12**] and also to the works of Amman [**1**], Deimling [**4**], Krein and Rutman [**13**], Schmitt and Smith [**18**], and Zeidler [**20**] for accounts of the definitions and results stated here.

Let $\mathbf{B}$ be a real Banach space and let $\mathbf{P}$ be a nonempty, closed subset of $\mathbf{B}$. $\mathbf{P}$ is a *cone* provided: (i) $\delta u + \gamma v \in \mathbf{P}$ for all $u, v \in \mathbf{P}$ and all $\delta, \gamma \geq 0$, and (ii) if $u, -u \in \mathbf{P}$, then $u = 0$. A cone is *reproducing* if for each $x \in \mathbf{B}$, there exist $u, v \in \mathbf{P}$ such that $x = u - v$.

A Banach space, $\mathbf{B}$, is called a *partially ordered Banach space* provided there exists a partial ordering, $\leq$, on $\mathbf{B}$ which satisfies: (i) $u \leq v$, for $u, v \in \mathbf{B}$, implies $tu \leq tv$, for all $t \geq 0$ and for $t < 0$, $tu \geq tv$ and $tu \neq tv$, and (ii) $u_1 \leq v_1$, $u_2 \leq v_2$, for $u_1, u_2, v_1, v_2 \in \mathbf{B}$, implies that $u_1 + u_2 \leq v_1 + v_2$. Let $\mathbf{P} \subset \mathbf{B}$ be a cone and define $u \leq v$, for $u, v \in \mathbf{B}$, if and only if $v - u \in \mathbf{P}$. Then $\leq$ is a partial ordering on $\mathbf{B}$, and we shall say that $\leq$ is the partial ordering induced by $\mathbf{P}$. Moreover, $\mathbf{B}$ is a partially ordered Banach space with respect to the partial ordering induced by $\mathbf{P}$.

Let $N_1, N_2 : \mathbf{B} \to \mathbf{B}$ be bounded, linear operators. We shall say that $N_1 \leq N_2$ with respect to $\mathbf{P}$ provided $N_1 u \leq N_2 u$, for all $u \in \mathbf{P}$. If $N : \mathbf{B} \to \mathbf{B}$ is bounded and linear, we shall say that N is *positive* with respect to $\mathbf{P}$ if $N(\mathbf{P}) \subseteq \mathbf{P}$.

Remark. In this paper $\leq$ denotes partial orderings with respect to cones and the usual partial ordering on $\mathbf{R}$ induced by $\mathbf{R}^+$. The particular implied ordering will be clear by context.

If $\mathbf{B}$ is a real Banach space and $N : \mathbf{B} \to \mathbf{B}$ is a bounded, linear operator, we shall employ $r(N)$ to denote the spectral radius of N.

A proof of the following theorem is found in [**16**].

Theorem 2.1. *Let N_b, $\alpha \leq b \leq \beta$, be a family of compact, linear operators on a Banach space such that the mapping $b \mapsto N_b$ is continuous in the uniform operator topology. Then the mapping $b \mapsto r(N_b)$ is continuous.*

Proofs of the following three theorems can be found in [**1, 12**]. In each of the following theorems, assume that $\mathbf{P}$ is a reproducing cone and that $N, N_1, N_2 : \mathbf{B} \to \mathbf{B}$ are compact, linear, and positive with respect to $\mathbf{P}$.

Theorem 2.2. *Assume $r(N) > 0$. Then $r(N)$ is an eigenvalue of N, and there is a corresponding eigenvector in $\mathbf{P}$.*

Theorem 2.3. *If $N_1 \leq N_2$ with respect to $\mathbf{P}$, then $r(N_1) \leq r(N_2)$.*

Theorem 2.4. *Suppose there exists $\mu > 0$, $u \in \mathbf{B}$, $-u \notin \mathbf{P}$ such that $Nu \geq \mu u$. Then N has an eigenvector in $\mathbf{P}$ which corresponds to an eigenvalue, $\lambda \geq \mu$.*

3. Sign properties of Green's functions. In this section, we state in a sequence of corollaries, sign properties of Green's functions for the family of boundary value problems, $y^{(n)} = 0$ satisfying (1.2_b). These properties will be employed in Section 4 to define the cones in Banach spaces in which we shall then apply the results listed in Section 2. The following theorem is proved by Eloe [**5**].

Theorem 3.1. *Let $G(j, b; x, s)$ be the Green's function for the boundary value problem, $y^{(n)} = 0$, (1.2_b). Then $(\partial^j/\partial x^j)G(j, b; x, s)$*

is the Green's function for the conjugate boundary value problem,

$$y^{(n-j)} = 0,$$

$$y^{(i)}(\alpha) = 0, \qquad i = 0, \dots, k-j-1,$$
$$y^{(i)}(b) = 0, \qquad i = 0, \dots, n-k-1.$$

Corollaries 3.2–3.5 now follow immediately from Theorem 3.1 and known sign properties of Green's functions for conjugate type boundary value problems. For Corollaries 3.2 and 3.3, see [**3, 6, 14**]. For Corollaries 3.4 and 3.5, see [**2, 7, 17**].

Corollary 3.2. $(-1)^{n-k}(\partial^j/\partial x^j)G(j,b;x,s) > 0$ *on* $(\alpha,b)\times(\alpha,b)$.

Corollary 3.3. $(-1)^{n-k}(\partial^k/\partial x^k)G(j,b;\alpha,s) > 0$ *on* (α,b).

Corollary 3.4. $(-1)^{n-k}(\partial/\partial b)((\partial^j/\partial x^j)G(j,b;x,s)) > 0$ *on* $(\alpha,b)\times(\alpha,b)$.

Corollary 3.5. $(-1)^{n-k}(\partial/\partial b)((\partial^k/\partial x^k)G(j,b;\alpha,s)) > 0$ *on* (α,b).

We remark that in addition to the results of [**2, 7, 17**] concerning the existence and properties of $\partial G/\partial b$, it can be proved in analogy to a result in Hartman [**10**, p. 97] that $(\partial^2/\partial x\partial b)G = (\partial^2/\partial b\partial x)G$. This is also used in establishing Corollaries 3.4 and 3.5.

4. Properties of j-focal points. In this section we employ the inequalities provided in Section 3 to construct appropriate cones and then apply the results of Section 2. Throughout this section, $j\in\{0,\dots,k-1\}$ is fixed. Let

$$\mathbf{B} = \{y\in C^{(j)}[\alpha,\beta] \mid y^{(i)}(\alpha) = 0,\ i=0,\dots,j\},$$

with norm $||y|| = \max_{i=0,\dots,j}\{\sup_{\alpha\le x\le\beta}|y^{(i)}(x)|\}$. Let $\mathbf{P}\subset\mathbf{B}$ be the cone defined by

$$\mathbf{P} = \{y\in\mathbf{B} \mid (-1)^{n-k}y^{(j)}(x)\ge 0,\ \alpha\le x\le\beta\}.$$

We note first that $\mathbf{P}$ is a reproducing cone. To see this, let $(z(x))^+ = \max\{z(x), 0\}$ and $(z(x))^- = \max\{-z(x), 0\}$, for each $\alpha \le x \le \beta$. Then, if $y \in \mathbf{B}$, $y(x) = (y(x))^+ - (y(x))^-$, if $j = 0$, and

$$y(x) = \int_0^x \frac{(x-s)}{(j-1)!}^{j-1} (y^{(j)}(s))^+ \, ds - \int_0^x \frac{(x-s)}{(j-1)!}^{j-1} (y^{(j)}(s))^- \, ds,$$

if $1 \le j \le k-1$. As a consequence, when suitable compact, positive operators are defined on $\mathbf{B}$, Theorems 2.2–2.4 will be applicable. We note second that, if $y \in \mathbf{P}$, then

$$(-1)^{n-k} y^{(i)}(x) \ge 0, \qquad \alpha \le x \le \beta,\ i = 0, \dots, j.$$

Further, for each $b \in (\alpha, \beta]$, we define another Banach space

$$\mathbf{B}_b = \{y \in C^{(n-1)}[\alpha, b] \mid y^{(i)}(\alpha) = 0,\ i = 0, \dots, k-1\},$$

with norm $||y||_b = \max_{i=0,\dots,n-1}\{\sup_{\alpha \le x \le b} |y^{(i)}(x)|\}$, and we define the cone $\mathbf{P}_b \subset \mathbf{B}_b$ by

$$\mathbf{P}_b = \{y \in \mathbf{B}_b \mid (-1)^{n-k} y^{(j)}(x) \ge 0,\ \alpha \le x \le b\}.$$

Again, we note that, if $y \in \mathbf{P}_b$, then

$$(-1)^{n-k} y^{(i)}(x) \ge 0, \qquad \alpha \le x \le b,\ i = 0, \dots, j.$$

Moreover, although $\operatorname{int} \mathbf{P} = \varnothing$,

$$\operatorname{int} \mathbf{P}_b = \{y \in \mathbf{B}_b \mid (-1)^{n-k} y^{(j)}(x) > 0,\ \alpha < x \le b, \\ \text{and } (-1)^{n-k} y^{(k)}(\alpha) > 0\}.$$

For each $b \in (\alpha, \beta]$, we now consider the boundary value problem (1.1), (1.2_b); in addition, we assume hereafter that $p_i \in C[\alpha, \beta]$ and $(-1)^{n-k} p_i(x) \ge 0$, $\alpha \le x \le \beta$, for each $i = 0, \dots, j$, and that p_0 does not vanish identically on each compact subinterval of $[\alpha, \beta]$.

For each $b \in (\alpha, \beta]$, define a linear operator $N_b : \mathbf{B} \to \mathbf{B}$ by

$$N_b y(x) = \begin{cases} \int_a^b G(j,b;x,s)(\sum_{i=0}^j p_i(s) y^{(i)}(s))\, ds, & \alpha \le x \le b, \\ \sum_{l=0}^j \frac{(x-b)^l}{l!} \int_\alpha^b \frac{\partial^l}{\partial x^l} G(j,b;b,s)(\sum_{i=0}^j p_i(s) y^{(i)}(s))\, ds, & b \le x \le \beta. \end{cases}$$

Then

$$(N_b y)^{(j)}(x) = \begin{cases} \int_a^b \frac{\partial^j}{\partial x^j} G(j,b;x,s)(\sum_{i=0}^j p_i(s) y^{(i)}(s))\, ds, & \alpha \le x \le b, \\ 0, & b \le x \le \beta. \end{cases}$$

Observe that, if $y \in \mathbf{P}$, then $p_i(s) y^{(i)}(s) \ge 0$ for $\alpha \le s \le \beta$, $i = 0, \dots, j$, and so by Corollary 3.2, $N_b(\mathbf{P}) \subseteq \mathbf{P}$. In the discussion that follows, we shall also restrict the operator N_b to $\mathbf{B}_b$; that is, define $N_b : \mathbf{B}_b \to \mathbf{B}_b$ by

$$N_b y(x) = \int_\alpha^b G(j,b;x,s) \left(\sum_{i=0}^j p_i(s) y^{(i)}(s) \right) ds, \quad \alpha \le x \le b. \tag{4.2}$$

Throughout the remainder of the paper, we shall specify the domain, $\mathbf{B}$ or $\mathbf{B}_b$, when referring to the operator N_b, in order to avoid confusion. We observe that again by Corollary 3.2, $N_b(\mathbf{P}_b) \subseteq \mathbf{P}_b$.

Remark. Properties of $G(j,b;x,s)$ and the Arzela-Ascoli Theorem are readily employed to show that N_b, defined on $\mathbf{B}$ or $\mathbf{B}_b$, is a compact operator for each $b \in (\alpha, \beta]$. Moreover, note that $\lambda \neq 0$ for all eigenvalues of the boundary value problem,

$$y^{(n)} = \lambda \sum_{i=0}^j p_i(x) y^{(i)}(x) \quad \text{and} \quad (1.2_b), \qquad \alpha \le x \le b. \tag{4.3}$$

If y is an eigenvector corresponding to an eigenvalue λ of (4.3) on $[\alpha, \beta]$, then we can extend y to $[\alpha, \beta]$ by $y(x) = \sum_{l=0}^j ((x-b)^l / l!) y^{(l)}(b)$, for $b \le x \le \beta$. This extension, y, satisfies $y = \lambda N_b y$, where N_b is defined on $\mathbf{B}$. Conversely, if $y \in \mathbf{B}$ is an eigenvector for N_b corresponding to an eigenvalue, $\mu \neq 0$, then the restriction of y to $[\alpha, b]$ is a nontrivial solution of the boundary value problem, (4.3) corresponding to $\lambda = 1/\mu$.

Remark. In the following, $r(N_b)$ will be associated with the operator N_b as a map from $\mathbf{B}$ to $\mathbf{B}$.

Theorem 4.1. *For $\alpha < b \leq \beta$, $r(N_b)$ is strictly increasing as a function of b.*

Proof. We first argue that for each $\alpha < b \leq \beta$, $r(N_b) > 0$. In [**6**], Eloe and Henderson showed that there exists $\lambda > 0$ and $u \in \mathbf{P}_b \backslash \{0\}$ such that, for $x \in [\alpha, b]$, $N_b u(x) = \lambda u(x)$. Extend u to $[\alpha, \beta]$ by $u(x) = \sum_{l=0}^{j}((x-b)^l/l!)u^{(l)}(b)$, $b \leq x \leq \beta$, and it follows that, for $x \in [\alpha, \beta]$, $N_b u(x) = \lambda u(x)$. Thus, $r(N_b) \geq \lambda > 0$.

Now, let $\alpha < b_1 < b_2 \leq \beta$. Since $r(N_{b_1}) > 0$, it follows by Theorem 2.2 that there exists $u \in \mathbf{P}\backslash\{0\}$ such that $N_{b_1} u = r(N_{b_1})u$. Set $y_1 = N_{b_1} u = r(N_{b_1})u$ and $y_2 = N_{b_2} u$. Then, for $x \in [\alpha, b_1]$,

$$(y_2 - y_1)^{(j)}(x) = \int_\alpha^{b_1} \left[\frac{\partial^j}{\partial x^j} G(j, b_2; x, s) - \frac{\partial^j}{\partial x^j} G(j, b_1; x, s)\right] \cdot \left(\sum_{i=0}^{j} p_i(s) u^{(i)}(s)\right) ds.$$

Since $u \in \mathbf{P}\backslash\{0\}$ and p_0 does not vanish identically on compact subintervals of $[\alpha, \beta]$, it follows from Corollary 3.4 that $(-1)^{n-k}(y_2 - y_1)^{(j)}(x) > 0$, for $\alpha < x \leq b_1$. Moreover, by Corollary 3.5, $(-1)^{n-k}(y_2 - y_1)^{(k)}(\alpha) > 0$. In particular, the restriction of $y_2 - y_1$ to $[\alpha, b_1]$ is an element of $\operatorname{int} \mathbf{P}_{b_1}$. Thus, there exists $\delta > 0$ such that $y_2 - y_1 \geq \delta u$, where this inequality is with respect to the cone $\mathbf{P}_{b_1}$. Since $(y_1)^{(j)}(x) = 0$, for $b_1 \leq x \leq \beta$, and $y_2 \in \mathbf{P}$, it readily follows that $y_2 - y_1 \geq \delta u$, where the inequality is now with respect to the cone $\mathbf{P}$. Thus, $y_2 \geq y_1 + \delta u = (r(N_{b_1} + \delta)u$. It now follows from Theorem 2.4 that $r(N_{b_2}) \geq r(N_{b_1}) + \delta > r(N_{b_1})$. The proof is complete. □

We now state and prove the main result of the paper.

Theorem 4.2. *The following are equivalent:*

i) b_0 *is the* j*-focal point of* (1.1) *corresponding to* (1.2_b);

ii) *there exists a nontrivial solution,* y*, of the boundary value problem,* (1.1), (1.2_{b_0})*, such that* $y \in \mathbf{P}_{b_0}$;

iii) $r(N_{b_0}) = 1$.

Proof. That iii) implies ii) follows immediately from Theorem 2.2.

For ii) implies i), let $u \in \mathbf{P}_{b_0} \backslash \{0\}$ satisfy (1.1) and (1.2_b). In [**6**], Eloe and Henderson employed a cone

$$\overline{\mathbf{P}}_{b_0} \subseteq \overline{\mathbf{B}}_{b_0} \equiv \{y \in C^{(n-1)}[\alpha, b_0] \mid \ y \text{ satisfies } (1.2_b)\},$$

where

$$\overline{\mathbf{P}}_{b_0} \equiv \{y \in \overline{\mathbf{B}}_{b_0} \mid (-1)^{(n-k)} y^{(i)}(x) \geq 0, \ \alpha \leq x \leq b_0, \ i = 0, \dots, j\},$$

and showed that $u \in \operatorname{int} \overline{\mathbf{P}}_{b_0}$.

Now extend u to $[\alpha, \beta]$ by $u(x) = \sum_{l=0}^{j} ((x - b_0)^l / l!) u^{(l)}(b_0)$, for $b_0 \leq x \leq \beta$. Then $r(N_{b_0}) \geq 1$. If $r(N_{b_0}) = 1$, the proof is complete by Theorem 4.1 since, for $\alpha < b < b_0$, $r(N_b) < 1$ and the boundary value problem, (1.1), (1.2_b) has only the trivial solution.

Assume then that $r(N_{b_0}) > 1$. Let $v \in \mathbf{P} \backslash \{0\}$ be such that $N_{b_0} v = r(N_{b_0}) v$. Again, by the results of [**6**], we have $v \in \operatorname{int} \overline{\mathbf{P}}_{b_0}$. But, since $u \in \operatorname{int} \overline{\mathbf{P}}_{b_0}$, there exists $\delta > 0$ such that $u \geq \delta v$, where this inequality is with respect to the cone $\overline{\mathbf{P}}_{b_0}$. Note that $u \geq \delta v$ implies that $(-1)^{n-k} u^{(j)}(x) \geq (-1)^{n-k} \delta v^{(j)}(x)$, $\alpha \leq x \leq b_0$; in particular, $u \geq \delta v$, where this inequality is with respect to the cone $\mathbf{P}_{b_0}$. Finally, we recall that u has been extended, for $b_0 \leq x \leq \beta$, and that $v^{(j)}(x) = 0$, for $b_0 \leq x \leq \beta$. Thus, $u \geq \delta v$, where this inequality is now with respect to the cone $\mathbf{P}$. Assume that δ is maximal. Then,

$$u = N_{b_0} u \geq N_{b_0}(\delta v) = \delta N_{b_0} v = \delta r(N_{b_0}) v,$$

which contradicts the maximality of δ, if $r(N_{b_0}) > 1$. Hence, $r(N_{b_0}) = 1$, and in particular, b_0 is the j-focal point of (1.1) corresponding to (1.2_b).

For i) implies iii), it is clear that i) implies $r(N_{b_0}) \geq 1$. Moreover, we note that $\lim_{b \to \alpha} r(N_b) = 0$. Thus, if $r(N_{b_0}) > 1$, it follows from Theorem 2.1 and the Intermediate Value Theorem that, for some $\alpha < b < b_0$, there exists a nontrivial solution of the boundary value problem (1.1), (1.2_b). But this contradicts i). Consequently, $r(N_{b_0}) = 1$, and the proof of the theorem is complete. □

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