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# An Example Employing Convexity in Functional Fixed Point Arguments

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**Abstract**

This paper illustrates a convexity technique to be employed with functional fixed point theorems in proving the existence of solutions to boundary value problems.

**Key words:** Multiple fixed-point theorems, convexity, Leggett-Williams, expansion, compression.

**AMS Subject Classification:** 47H110, 34B15

## 1 Introduction

Functional arguments to prove the existence of solutions to boundary value problems originated with the work of Leggett-Williams [13] when the authors utilized concave functionals in conjunction with the norm. The five functionals fixed point theorem [3] generalized the Leggett-Williams fixed point theorem by providing flexibility in choosing a convex functional instead of using the norm when proving the existence of triple positive solutions. Axiomatic index theory has been used to create and extend many functional fixed point theorems. Functional fixed point theorems (including [1, 7, 8, 14]) can be traced back to Leggett and Williams [13] when they presented criteria which guaranteed the existence of a fixed point for a completely continuous map that did not require the operator to be invariant with regards to the concave functional boundary of a functional wedge. Avery, Henderson, and O'Regan [5], in a dual of the Leggett-Williams fixed point theorem, gave conditions which guaranteed the existence of a fixed point for a completely continuous map that did not require the operator to be invariant with regards to the convex functional boundary of a functional wedge.

Although there is a lot more flexibility that can be utilized by using the functional fixed point theorems when proving existence of solutions to boundary value problems, bounding the nonlinear term on intervals by constants as was done in arguments utilizing Krasnosel'skii's fixed point theorem [11] and the original Leggett-Williams [13] fixed point theorem in the past is still the most prevalant technique. In this paper we provide an example of an alternative technique using a functional involving the derivative as well as bounding the nonlinearity by functions defined in terms of the nonlinearity. We provide an example that is the sum of an increasing and decreasing function which is not well suited to being bounded either above or below on intervals by constants due to the concavity of the nonlinearity.

## 2 Preliminaries

In this section we will state the definitions that are used in the remainder of the paper.

**Definition 2.1** *Let $E$ be a real Banach space. A nonempty closed convex set $P \subset E$ is called a* cone *if for all $x \in P$ and $\lambda \geq 0$, $\lambda x \in P$ and if $x, -x \in P$ then $x = 0$.*

Every cone $P \subset E$ induces an ordering in $E$ given by $x \leq y$ if and only if $y - x \in P$.

**Definition 2.2** *An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.*

**Definition 2.3** *A map $\alpha$ is said to be a nonnegative continuous concave functional on a cone $P$ of a real Banach space $E$ if $\alpha : P \to [0, \infty)$ is continuous and*

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

*for all $x, y \in P$ and $t \in [0, 1]$. Similarly we say the map $\beta$ is a nonnegative continuous convex functional on a cone $P$ of a real Banach space $E$ if $\beta : P \to [0, \infty)$ is continuous and*

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

*for all $x, y \in P$ and $t \in [0, 1]$.*

Let $\psi$ and $\delta$ be nonnegative continuous functionals on $P$; then, for positive real numbers $a$ and $b$, we define the following sets:

$$P(\psi, b) = \{x \in P : \psi(x) < b\} \text{ and } P(\psi, \delta, a, b) = \{x \in P : a < \psi(x) \text{ and } \delta(x) < b\}.$$

**Theorem 2.4** *Let $P$ be a cone in a real Banach space $E$, $\alpha$ and $\psi$ be nonnegative continuous concave functionals on $P$, $\delta$ and $\beta$ be nonnegative continuous convex functionals on $P$, and $T : P \to P$ be a completely continuous operator. If there exist nonnegative numbers $a$, $b$, $c$ and $d$ such that*

$(A1)$ $\{x \in P : a < \alpha(x)$ *and* $\delta(x) < b\} \neq \emptyset$*;*

$(A2)$ *if $x \in P$ with $\delta(x) = b$ and $\alpha(x) \geq a$, then $\delta(Tx) < b$;*

$(A3)$ *if $x \in P$ with $\delta(x) = b$ and $\alpha(Tx) < a$, then $\delta(Tx) < b$;*

$(A4)$ $\{x \in P : c < \psi(x)$ *and* $\beta(x) < d\} \neq \emptyset$*;*

$(A5)$ *if $x \in P$ with $\psi(x) = c$ and $\beta(x) \leq d$, then $\psi(Tx) > c$;*

$(A6)$ *if $x \in P$ with $\psi(x) = c$ and $\beta(Tx) > d$, then $\psi(Tx) > c$;*

*and if*

$(H1)$ $\overline{P(\delta, b)} \subset P(\psi, c)$ *with $P(\psi, c)$ being bounded, then $T$ has a fixed point $x^*$ in $P(\delta, \psi, b, c)$;*

$(H2)$ $\overline{P(\psi, c)} \subset P(\delta, b)$ *with $P(\delta, b)$ being bounded, then $T$ has a fixed point $x^*$ in $P(\psi, \delta, c, b)$.*

We say that $T$ is LW-inward with respect to $P(\alpha, \delta, a, b)$ if the conditions $(A1)$, $(A2)$, and $(A3)$ are satisfied and $P(\delta, b)$ is bounded. Similarly we say that $T$ is LW-outward with respect to $P(\psi, \beta, c, d)$ if the conditions $(A4)$, $(A5)$, and $(A6)$ are satisfied and $P(\psi, c)$ is bounded.

## 3 Main Result - Using Convexity

In this section we will illustrate the key techniques for verifying the existence of a positive solution for a right focal boundary value problem using our main result. Right focal boundary value problems have received substantial study for many years. For an early paper, we mention the classical paper by Jackson [10], and for more recent studies, we cite [2] and [4], to name just a couple. For our purposes in this paper, under the expansion condition $(H1)$ we apply the properties of a Green's function, bound the nonlinearity by functions derived from the nonlinearity, and demonstrate how to utilize a functional defined in terms of the derivative to obtain the necessary inequalities to apply the compression and expansion fixed point theorem of functional type, Theorem 2.4. To proceed, consider the second-order nonlinear right focal boundary value problem

$$x''(t) + f(x(t)) = 0, \quad t \in (0, 1), \tag{3.1}$$

$$x(0) = 0 = x'(1), \tag{3.2}$$

where $f : \mathbb{R} \to [0, \infty)$ is continuous. If $x$ is continuous on $[0,1]$ and a fixed point of the operator $T$ defined by

$$Tx(t) := \int_0^1 G(t,s) f(x(s)) ds,$$

where

$$G(t,s) = \min\{t,s\}, \quad (t,s) \in [0,1] \times [0,1]$$

is the Green's function for the operator $L$ defined by

$$Lx(t) := -x'',$$

with right-focal boundary conditions

$$x(0) = 0 = x'(1),$$

then it is well known that $x$ is a solution of the boundary value problem (3.1), (3.2). Throughout this section of the paper we will use the facts that $G(t,s)$ is nonnegative, and for each fixed $s \in [0,1]$, the Green's function is nondecreasing in $t$, as well as a concavity property of the Green's function which is given by

$$\min_{s \in [0,1]} \frac{G(y,s)}{G(w,s)} \geq \frac{y}{w} \quad \text{for} \quad 0 \leq y < w \leq 1. \tag{3.3}$$

Define the cone $P \subset E = C^1[0,1]$ by

$$P := \{x \in E \ : \ x \text{ is nonnegative, nondecreasing, concave, and } x(0) = 0\},$$

and for $x \in P$, define the concave functionals $\alpha$ and $\psi$ on $P$ by

$$\psi(x) := \min_{t \in \left[\frac{1}{2}, 1\right]} x(t) = x\left(\frac{1}{2}\right) \ \text{ and } \ \alpha(x) := \min_{t \in \left[\frac{1}{4}, 1\right]} x(t) = x\left(\frac{1}{4}\right),$$

and define the convex functionals $\beta$ and $\delta$ on $P$ by

$$\delta(x) := \max_{t \in [0,1]} x(t) = x(1) \ \text{ and } \ \beta(x) := \max_{t \in \left[\frac{1}{2}, 1\right]} x'(t) = x'\left(\frac{1}{2}\right).$$

Thus if $x \in P$, then by the concavity of $x$ we have $x\left(\frac{1}{2}\right) \geq \left(\frac{1}{2}\right) x(1)$, since

$$\frac{x\left(\frac{1}{2}\right) - x(0)}{\frac{1}{2} - 0} \geq \frac{x(1) - x(0)}{1 - 0}.$$

Therefore, for all $x \in P$, we have

$$\left(\frac{1}{2}\right) \delta(x) \leq \psi(x). \tag{3.4}$$

In the following theorem, we demonstrate how to apply condition $(H1)$ of Theorem 2.4 to prove the existence of at least one positive solution to (3.1), (3.2).

**Theorem 3.1** *If $m$, $M$, $b$ and $c$ are positive real numbers with $b < c$, and $f : [0, 2c] \to [0, \infty)$ is a continuous function such that*

(*i*) $\int_{\frac{1}{4}}^{1} \left(s - \frac{1}{4}\right) g(s)\, ds < \frac{b}{2}$ *where* $g(s) := \max_{r \in [bs, b]} f(r)$,

(*ii*) $\int_0^{\frac{1}{4}} s\, h(s)\, ds < m$ *where* $h(s) := \max_{r \in [2bs, b]} f(r)$ *for* $s \in \left[0, \frac{1}{4}\right]$,

(*iii*) $\int_{\frac{1}{4}}^{1} s\, k(s)\, ds < b - m$ *where* $k(s) := \max_{r \in \left[\frac{b(2s+1)}{3}, b\right]} f(r)$ *for* $s \in \left[\frac{1}{4}, 1\right]$,

(*iv*) $f\left(\frac{3c}{2}\right) > 4c + M$ *where* $|f(x) - f(y)| \leq M$ *whenever* $x, y \in [c, 2c]$ *and* $|x - y| \leq \frac{c}{2}$,

*then the right focal problem* (3.1), (3.2) *has at least one positive solution* $x^* \in P(\delta, \psi, b, c)$.

*Proof:* Let $a = \frac{b}{2}$ and $d = 2c$. For $x \in P(\delta, \psi, b, c)$, if $t \in (0, 1)$, then by the properties of the Green's function

$$(Tx)''(t) = -f(x(t))$$

and

$$Tx(0) = 0 = (Tx)'(1).$$

By the Arzela-Ascoli Theorem it is a standard exercise to show that $T$ is a completely continuous operator using the properties of $G$ and $f$, and for each $x \in \overline{P(\delta, \psi, b, c)}$, $Tx \in P$. Therefore we have that

$$T : \overline{P(\beta, \alpha_\mu, b, c)} \to P$$

is continuous. Thus by Dugundji's Theorem, there is a continuous extension, which we will again denote by $T$, such that $T : P \to P$ (a proof can be found in [6]). We will now verify that properties $(i) - (v)$ imply that $T$ is LW-inward with respect to $P(\alpha, \delta, a, b)$ and that $T$ is LW-outward with respect to $P(\psi, \beta, c, d)$.

Claim 1: $T$ is LW-inward with respect to $P(\alpha, \delta, a, b)$.

Define the function $x_L \in P$ by

$$x_L(t) \equiv \frac{5a\sqrt{t}(2 - \sqrt{t})}{3}.$$

Then

$$\alpha(x_L) = x_L\left(\frac{1}{4}\right) = \frac{5a}{4} > a,$$

and

$$\delta(x_L) = x_L(1) = \frac{5a}{3} < 2a = b.$$

Therefore, we have that

$$\{x \in P : a < \alpha(x) \text{ and } \delta(x) < b\} \neq \emptyset,$$

and if $x \in \overline{P(\delta, b)}$, then

$$\|x\| = \delta(x) \leq b.$$

Thus $\overline{P(\delta, b)}$ is bounded as well.

Subclaim 1.1: $\delta(Tx) < b$ for all $x \in P$, with $\delta(x) = b$ and $\alpha(Tx) < a$.

Let $x \in P$ with $\delta(x) = b$ and $\alpha(Tx) < a$. By the concavity of $x$, we have $\frac{x(t)-x(0)}{t-0} \geq \frac{x(1)-x(0)}{1-0} = b$. Hence $x(t) \geq bt$ and since $x$ is increasing with $x(1) = b$, we have $bt \leq x(t) \leq b$, for all $t \in [0, 1]$. Thus

$$\begin{aligned}
\delta(Tx) &= \int_0^1 G(1,s)\ f(x(s))\ ds = \int_0^1 s\ f(x(s))\ ds \\
&= \int_0^{\frac{1}{4}} s\ f(x(s))\ ds + \int_{\frac{1}{4}}^1 s\ f(x(s))\ ds \\
&= \int_0^{\frac{1}{4}} s\ f(x(s))\ ds + \int_{\frac{1}{4}}^1 \left(\frac{1}{4}\right)\ f(x(s))\ ds + \int_{\frac{1}{4}}^1 \left(s - \frac{1}{4}\right)\ f(x(s))\ ds \\
&= \int_0^1 G\left(\frac{1}{4}, s\right)\ f(x(s))\ ds + \int_{\frac{1}{4}}^1 \left(s - \frac{1}{4}\right)\ f(x(s))\ ds \\
&= \alpha(Tx) + \int_{\frac{1}{4}}^1 \left(s - \frac{1}{4}\right)\ g(s)\ ds \\
&< a + a = b,
\end{aligned}$$

and we have verified that $\delta(Tx) < b$.

Subclaim 1.2: If $x \in P$ with $\delta(x) = b$ and $\alpha(x) \geq a$, then $\delta(Tx) < b$.

Let $x \in P$ with $\delta(x) = b$ and $\alpha(x) \geq a$. Thus by the concavity of $x$, and since $x$ is increasing with $x(1) = b$, we have

$$\frac{as}{\frac{1}{4}} \leq x(s) \leq b \ \text{ for } \ s \in \left[0, \frac{1}{4}\right],$$

and

$$\frac{a(s - \frac{1}{4})}{1 - \frac{1}{4}} + a \leq x(s) \leq b \ \text{ for } \ s \in \left[\frac{1}{4}, 1\right].$$

Thus

$$\begin{aligned}
\delta(Tx) &= \int_0^1 G(1,s)\ f(x(s))\ ds = \int_0^1 s\ f(x(s))\ ds \\
&= \int_0^{\frac{1}{4}} s\ f(x(s))\ ds + \int_{\frac{1}{4}}^1 s\ f(x(s))\ ds \\
&\leq \int_0^{\frac{1}{4}} s\ h(s)\ ds + \int_{\frac{1}{4}}^1 s\ k(s)\ ds \\
&< b,
\end{aligned}$$

and we have verified that $\delta(Tx) < b$.

Therefore, we have verified that $T$ is LW-inward with respect to $I(\alpha, \delta, a, b)$.

Claim 2: $T$ is LW-outward with respect to $O(\psi, \beta, c, d)$.

Define the function $x_O \in P$ by

$$x_O(t) \equiv \frac{3ct(2-t)}{2}.$$

Then

$$\psi(x_O) = x_O\left(\frac{1}{2}\right) = \frac{9c}{8} > c,$$

and

$$\beta(x_O) = x_O'(1) = \frac{3c}{2} < 2c = d.$$

Therefore, we have that

$$\{x \in P : c < \psi(x) \text{ and } \beta(x) < d\} \neq \emptyset,$$

and $P(\psi, c)$ is a bounded subset of the cone $P$, since if $x \in P(\psi, c)$, then by (3.4) we have that

$$\left(\frac{1}{2}\right)\|x\| = \left(\frac{1}{2}\right)\delta(x) \leq \psi(x),$$

and so

$$\|x\| =\leq 2\psi(x) \leq 2c.$$

Subclaim 2.1: $\psi(Tx) > c$ for all $x \in P$, with $\psi(x) = c$ and $\beta(x) \leq d$.

Let $x \in P$ with $\psi(x) = c$ and $\beta(x) \leq d = 2c$. Then by the concavity of $x$ for $t \in \left(\frac{1}{2}, 1\right]$, we have

$$\frac{x(t) - x\left(\frac{1}{2}\right)}{t - \frac{1}{2}} < x'\left(\frac{1}{2}\right) = d = 2c,$$

and since $x\left(\frac{1}{2}\right) = \psi(x) = c$, we have

$$c \leq x(t) \leq 2ct$$

for $t \in \left[\frac{1}{2}, 1\right]$. Thus for each $t \in \left[\frac{1}{2}, 1\right]$,

$$|x(t) - \frac{3c}{2}| \leq \frac{c}{2}.$$

Therefore,

$$\begin{aligned}
\psi(Tx) &= \int_0^1 G\left(\frac{1}{2}, s\right) f(x(s))\ ds \\
&\geq \int_{\frac{1}{2}}^1 \left(\frac{1}{2}\right) f(x(s))\ ds \\
&\geq \left(\frac{1}{2}\right) \int_{\frac{1}{2}}^1 f\left(\frac{3c}{2}\right) - M\ ds \\
&> \left(\frac{1}{2}\right) \left[\left(2c + \frac{M}{2}\right) - \frac{M}{2}\right] = c.
\end{aligned}$$

Subclaim 2.2: $\psi(Tx) > c$ for all $x \in P$, with $\psi(x) = c$ and $\beta(Tx) > d$.

Let $x \in P$ with $\beta(Tx) > d$ and $\psi(x) = c$. Thus

$$\beta(Tx) = (Tx)'\left(\frac{1}{2}\right) = \int_{\frac{1}{2}}^1 f(x(s))\ ds > d = 2c,$$

and

$$\begin{aligned}
\psi(Tx) &= \int_0^1 G(\frac{1}{2}, s)\ f(x(s))\ ds \\
&= \int_0^{\frac{1}{2}} s\ f(x(s))\ ds + \int_{\frac{1}{2}}^1 \left(\frac{1}{2}\right) f(x(s))\ ds \\
&\geq \left(\frac{1}{2}\right) \beta(Tx) = c.
\end{aligned}$$

Therefore we have verified that $T$ is LW-outward with respect to $O(\psi, \beta, c, d)$.

Define

$$x_{subset}(t) \equiv \int_0^1 (b+c)\ G(t,s) ds = \frac{(b+c)t(2-t)}{2}.$$

Then $x_{subset}(t) \in P(\delta, \psi, b, c)$, since

$$\psi(x_{subset}) = x_{subset}\left(\frac{1}{2}\right) = \frac{(b+c)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{2} = \frac{3(b+c)}{8} < c$$

and

$$\delta(x_{subset}) = x_{subset}(1) = \frac{b+c}{2} > b.$$

Consequently we have that $\{x \in P\ : b < \delta(x)\ \text{ and }\ \psi(x) < c\} \neq \emptyset$, and if $x \in \overline{P(\delta, b)}$, then

$$\psi(x) = x\left(\frac{1}{2}\right) \leq x(1) = \delta(x) \leq b < c.$$

Thus $\overline{P(\beta, b)} \subset P(\psi, c)$. Therefore, $(H1)$ of Theorem 2.4 has been satisfied; thus the operator $T$ has at least one fixed point $x^* \in P(\delta, \psi, b, c)$, which is a desired solution of (3.1), (3.2). $\square$

When we have that $f$ is the sum of an increasing and a decreasing function, then determining the maximum value of these monotonic functions on a given interval is trivial. In the next corollary we demonstrate that if more restrictive conditions defined using the monotonic functions are satisfied, then the result is even easier to verify and simpler to state.

**Corollary 3.2** *If $b$ and $c$ are positive real numbers with $b < c$, $M$ is a positive number, $f(x) = f_1(x) + f_2(x)$ is a continuous function with $f_1, f_2 : [0, 2c] \to [0, \infty)$, $f_1$ is decreasing, and $f_2$ is increasing such that*

*(i)* $\int_{\frac{1}{4}}^{1} \left(s - \frac{1}{4}\right) \left(f_1(bs) + f_2(b)\right) ds < \frac{b}{2}$,

*(ii)* $\int_0^{\frac{1}{4}} s \left(f_1(2bs) + f_2(b)\right) ds + \int_{\frac{1}{4}}^{1} s \left(f_1\left(\frac{b(2s+1)}{3}\right) + f_2(b)\right) ds < b$,

*(iii)* $f\left(\frac{3c}{2}\right) > 4c + M$ *where* $|f(x) - f(y)| \leq M$ *whenever* $x, y \in [c, 2c]$ *and* $|x - y| \leq \frac{c}{2}$,

*then the right focal problem* (3.1), (3.2) *has at least one positive solution* $x^* \in P(\delta, \psi, b, c)$.

The following example illustrates the preceding Corollary.
**Example:** Let

$$f(x) = e^{-x} + x^2.$$

Then the boundary value problem

$$x'' + f(x) = 0, \text{ for } t \in (0, 1),$$

with right focal boundary conditions

$$x(0) = 0 = x'(1),$$

has at least one positive solution $x^*$, which can be verified by Corollary 3.2 using $b = 1$, $c = 10$, and $M = 50$.

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