2017

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Communications
on Applied Nonlinear Analysis
Volume 24(2017), Number 4, 101 - 108

Positive Solutions of Boundary Value Problems for Ordinary Differential Equations with Dependence on Higher Order Derivatives

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Communicated by Ram U. Verma
(Received May 31, 2017; Accepted October 2, 2017)

Abstract

In this paper, the compression contraction fixed point theorem is applied to a right focal boundary value problem for a second order ordinary differential equation to provide sufficient conditions for existence of solutions in a cone. The nonlinear term is a function of three variables and singularities are allowed. A cone is defined in the Banach space $C^1[0,1]$ and concavity of derivatives of solutions plays a key role. A family of examples is provided in which explicit sufficient conditions are exhibited.

AMS Subject Classification: 34B18; 34B15; 34B27

Key words: Positive solutions, right focal boundary value problem, existence of positive solutions.

1 Introduction

In this short note we obtain sufficient conditions for existence of positive solutions of a right focal boundary value problem for a nonlinear second order ordinary differential equation. We employ the compression contraction fixed point theorem, often called the Krasnosel'skii-Guo [5, 13] fixed point theorem. This work follows the convention that the sufficient conditions can be expressed in terms of sublinear or superlinear behavior of the nonlinear term. Erbe and Wang [4] presented the original application of the method to ordinary differential equations; the method has been modified and applied to many types of boundary value problems for functional differential or difference equations. For example, see [1, 3, 6, 14] for ordinary differential equations, [7] for systems of ordinary differential equations, [9] for discrete equations and [8, 11] for fractional differential equations. The citations we provide are far from exhaustive and we refer the reader to the recent monograph [10] for a more extensive bibliography.

Beginning with Erbe and Wang [4], it is commonly assumed that the nonlinear term has the form $f(x, y(x))$ where x denotes the independent variable and y denotes the solution to be obtained. The primary contribution of this work is to consider a nonlinear term of the form $f(x, y(x), y'(x))$ where y' denotes the derivative of the solution to be obtained. In particular, the Banach space here will be a space of continuously differentiable functions which complicates the concepts of superlinear or sublinear behavior. There has been previous work along these lines, [2], and the applications to systems of equations [10] must address similar complications. However, the calculations produced here are somewhat different because we consider a boundary value problem in which the supremum norm of the derivative of a solution is the norm of the solution and so, the sufficient conditions align more naturally with the original asymptotic conditions found in [4].

In Section 2, we state the fixed point theorem, we define the boundary value problem, and then we apply the fixed point theorem to obtain sufficient conditions for the existence of a positive solution. In Section 3, we provide a family of examples to illustrate the results.

2 Existence of positive solutions

We recall the definition of a cone in a Banach space and state the Krasnosel'skii-Guo fixed point theorem.

Definition 2.1 *Let E be a real Banach space. A nonempty closed convex set $K \subset E$ is called a cone if it satisfies the following two conditions:*

1. *If $x \in K$, and $\lambda \geq 0$, then $\lambda x \in K$.*
2. *If $x \in K$, and $-x \in K$, then $x = 0$.*

Every cone $K \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in K$.

Theorem 2.1 *Let E be a Banach space equipped with the norm $\|\cdot\|$ and let $K \subset E$ be a cone. Assume Ω_1 and Ω_2 are open balls in E with $0 \in \Omega_1 \subseteq \overline{\Omega}_1 \subseteq \Omega_2$, and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

1. *$\|Au\| \leq \|u\|, u \in K \cap \partial\Omega_1$ and $\|Au\| \geq \|u\|, u \in K \cap \partial\Omega_2$, or*
2. *$\|Au\| \geq \|u\|, u \in K \cap \partial\Omega_1$ and $\|Au\| \leq \|u\|, u \in K \cap \partial\Omega_2$.*

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Assume $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. We shall consider a right focal boundary value problem for an ordinary differential equation,

$$y''(x) = f(x, y(x), y'(x)), \quad 0 < x < 1, \tag{2.1}$$

$$y(0) = 0, \quad y'(1) = 0. \tag{2.2}$$

The Green's function for this right focal problem is well-known and is given here.

$$G(x,s) = \begin{cases} -x, & 0 \le x < s \le 1, \\ -s, & 0 \le s < x \le 1, \end{cases} \tag{2.3}$$

and

$$\frac{\partial}{\partial x} G(x,s) = G_x(x,s) = \begin{cases} -1, & 0 \le x < s \le 1, \\ 0, & 0 \le s < x \le 1. \end{cases}$$

Of importance in this work is that

$$G(x,s) < 0, \quad (x,s) \in (0,1) \times (0,1), \quad \frac{\partial}{\partial x} G(x,s) \le 0, \quad (x,s) \in (0,1) \times (0,1).$$

In cases where $f(x, y(x))$ is independent of y', a sign condition on f coupled with the sign of the Green's function is used to show the concavity of solutions. That concavity of the solution is then employed to define an appropriate cone for the application of the Krasnosel'skii-Guo fixed point theorem. With this motivation, we impose conditions to imply the concavity of y' and use that concavity to define an appropriate cone for the boundary value problem (2.1), (2.2). To obtain concavity of y' for solutions of (2.1), (2.2) in $C^1[0,1]$, we impose conditions on f such that solutions, y, of (2.1), (2.2) will satisfy $y''(x)$ is decreasing; in particular, y' is concave.

Consider the Banach space $C^1[0,1]$ with $||y|| = \max\{||y||_0, ||y'||_0\}$ where

$$||y||_0 = \max_{0 \le x \le 1} |y(x)|.$$

Lemma 2.2 *Assume $y(x) \in C^1[0,1]$, $y(0) = 0$, and assume $|y'(x)|$ is decreasing as a function of x. Then $||y|| = |y'(0)|$.*

Proof: Since $|y'(x)|$ is decreasing as a function of x, it follows that $||y'||_0 = |y'(0)|$. Moreover, if $x \in (0,1]$, there exists $c \in (0,x)$ such that

$$|y(x)| = |y(x) - y(0)| = |y'(c)|x \le |y'(c)|.$$

This implies $||y||_0 \le ||y'||_0 = |y'(0)|$. □

Lemma 2.3 *Assume $y(x) \in C^1[0,1]$, $y(0) = 0$, $y'(x) > 0$, $0 \le x < 1$ and assume y' is decreasing as a function of x. Then for $\frac{1}{4} \le x \le \frac{3}{4}$, $y(x) \ge \frac{y'(\frac{3}{4})}{4}$.*

Proof: Let $x \in [\frac{1}{4}, \frac{3}{4}]$. Then

$$y(x) = \int_0^x y'(s)ds \ge y'(\frac{3}{4})x \ge \frac{y'(\frac{3}{4})}{4}.$$

□

Define the cone $K \subset C^1[0,1]$ by

$$K = \{y \in C^1[0,1] : y(0) = 0, y'(x) \geq 0, 0 \leq x \leq 1,$$
$$y'(x) \text{ is decreasing as a function of } x, \text{ and } \min_{x\in[0,\frac{3}{4}]} y'(x) \geq \frac{|y'(0)|}{4}\}.$$

Note that if $y \in K$ then $||y|| = |y'(0)|$.

Remark 1 *Note that if $y \in K$, apply Lemma 2.2 and the condition $\min_{x\in[0,\frac{3}{4}]} y'(x) \geq \frac{|y'(0)|}{4}$ to conclude that if $x \in [\frac{1}{4}, \frac{3}{4}]$ then*

$$\frac{||y||}{16} \leq y(x) \leq ||y||.$$

Define $\mathcal{A}$ on $C^1[0,1]$ by

$$\mathcal{A}(y)(x) = \int_0^1 G(x,s) f(s, y(s), y'(s)) ds \tag{2.4}$$

where G is given in (2.3).

Theorem 2.4 *Assume $f : [0,1] \times (0,\infty)^2 \to (-\infty, 0]$ is continuous. Assume further that f satisfies a monotonicity condition:*

$$\text{if } 0 \leq x_1 < \bar{x_1} \leq 1, 0 \leq y_1 \leq \bar{y_1}, y_2 \geq \bar{y_2} \geq 0, \text{ then } 0 \geq f(x_1, y_1, y_2) \geq f(\bar{x_1}, \bar{y_1}, \bar{y_2}). \tag{2.5}$$

Assume f satisfies the asymptotic properties

$$a) \limsup_{u_2 \to 0^+} \max_{x\in[0,1]} \max_{|u_1|\leq|u_2|} \frac{|f(x, u_1(x), u_2(x))|}{|u_2|} = 0,$$

$$b) \liminf_{u_2 \to +\infty} \min_{x\in[\frac{1}{4},\frac{3}{4}]} \min_{\frac{|u_2|}{16}\leq|u_1|\leq|u_2|} \frac{|f(x, u_1(x), u_2(x))|}{|u_2|} = \infty.$$

Then there exists $y \in K \setminus \{0\}$ such that y is a solution of the boundary value problem, (2.1), (2.2).

Proof: To see that $\mathcal{A} : K \to K$, let y be an element of the cone K. Then $(\mathcal{A}y)(0) = 0$ since $G(0,s) = 0$. Since $f : [0,1] \times (0,\infty)^2 \to (-\infty, 0]$ and $G_x \leq 0$ on $[0,1] \times [0,1]$, $(\mathcal{A}y)'(x) \geq 0$, $0 \leq x \leq 1$. To see that $(\mathcal{A}y)'(x) \geq 0$ is decreasing in x , we note that $(\mathcal{A}y)''(x) \leq 0$ since we have that

$$(Ay)''(x) = f(x, y(x), y'(x)) \leq 0.$$

Finally to show $\min_{x\in[0,\frac{3}{4}]} (\mathcal{A}y)'(x) \geq \frac{|(\mathcal{A}y)'(0)|}{4}$ we first remark (2.5) is imposed to imply that $(\mathcal{A}y)'(x)$ is concave. Note that if $0 \leq w_1 < w_2 \leq 1$ then $0 \leq y(w_1) \leq y(w_2)$ and $y'(w_1) \geq y'(w_2) \geq 0$. Thus,

$$(\mathcal{A}y)''(w_1) = f(w_1, y(w_1), y'(w_1)) \geq f(w_2, y(w_2), y'(w_2)) = (\mathcal{A}y)''(w_2)$$

and $\frac{d}{dx}((\mathcal{A}y)')$ is decreasing. Set $u(x) = (\mathcal{A}y)'(x)$ and set $v(x) = (\mathcal{A}y)'(0)(1-x)$. We show $u(x) \geq v(x)$, $0 \leq x \leq 1$. Assume for the sake of contradiction that $\alpha(x) = u(x) - v(x) \geq 0$, $0 \leq x \leq 1$ is false. Note $\alpha(0) = \alpha(1) = 0$, and assume there exists $x_0 \in (0,1)$ such that $0 > \alpha(x_0)$. Then there exist $0 \leq c_1 < x_0 < c_2 \leq 1$ such that $\alpha'(c_1) < 0$ and $\alpha'(c_2) > 0$. This is a contradiction since u' is decreasing implies $\alpha'(x) = u'(x) - (\mathcal{A}y)'(0)$ is decreasing. Thus, $u(x) \geq v(x)$, $0 \leq x \leq 1$.

Assume $0 \leq x \leq \frac{3}{4}$. Then

$$(\mathcal{A}y)'(x) \geq (\mathcal{A}y)'(\frac{3}{4}) \geq v(\frac{3}{4}) = \frac{|(\mathcal{A}y)'(0)|}{4}.$$

We are now in a position to employ the asymptotic behaviors in a) and b) to construct the open balls Ω_1 and Ω_2 for an application of Theorem 2.1. Employ condition a) and choose $H_1 > 0$ such that if $|u_2| \leq H_1$ then $|f(x, u_1, u_2)| \leq |u_2|$ for all $x \in [0,1]$, $|u_1| \leq |u_2|$. Define

$$\Omega_1 = \{y \in C^1[0,1] : ||y|| \leq H_1\}.$$

Let $y \in K \cap \partial\Omega_1$. Then

$$\begin{aligned}||\mathcal{A}y|| = |(\mathcal{A}y)'(0)| &= |\int_0^1 f(s, y(s), y'(s))ds| = \int_0^1 -f(s, y(s), y'(s))ds \\ &= \int_0^1 |f(s, y(s), y'(s))|ds \leq \int_0^1 y'(s)ds \leq y'(0) = ||y||\end{aligned}$$

and

$$||\mathcal{A}y|| \leq H_1 = ||y||, \quad y \in K \cap \partial\Omega_1.$$

Now employ condition b) and choose $\hat{H}_2 > H_1$ such that if $|u_2| \geq \hat{H}_2$ then $|f(x, u_1, u_2)| \geq 8|u_2|$ for all $x \in [\frac{1}{4}, \frac{3}{4}]$, $\frac{|u_2|}{16} \leq |u_1| \leq |u_2|$. Set $H_2 = 4\hat{H}_2$ and define

$$\Omega_2 = \{y \in C^1[0,1] : ||y|| \leq H_2\}.$$

Let $y \in K \cap \partial\Omega_2$ and note that $y'(\frac{3}{4}) \geq \hat{H}_2$. Then, applying Lemma 2.3 and Remark 1, we have

$$\begin{aligned}(\mathcal{A}y)'(\frac{1}{4}) &= |\int_{\frac{1}{4}}^1 f(s, y(s), y'(s))ds| = \int_{\frac{1}{4}}^1 |f(s, y(s), y'(s))|ds \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} |f(s, y(s), y'(s))|ds \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}} 8y'(s)ds \geq 4y'(\frac{3}{4}) \geq ||y|| = H_2.\end{aligned}$$

In particular, if $y \in K \cap \partial\Omega_2$, then

$$||\mathcal{A}y|| = (\mathcal{A}y)'(0) \geq (\mathcal{A}y)'(\frac{1}{4}) \geq ||y||.$$

Note that the continuity of f, the continuity of G on $[0,1]\times[0,1]$ and the continuity of G_x on triangles $0\le x\le s\le 1$ and $0\le s\le x\le 1$ are sufficient to show that $\mathcal{A}: K\cap(\overline{\Omega}_2\setminus\Omega_1)\to K$ is a completely continuous map. The hypotheses of Theorem 2.1 are satisfied and there exists a solution $y\in K\setminus\{0\}$ of the boundary value problem (2.1), (2.2). □

Theorem 2.5 *Assume $f:[0,1]\times(0,\infty)^2\to(-\infty,0]$ is continuous. Assume further that f satisfies a monotonicity condition (2.5). Assume f satisfies the asymptotic properties*

$$c)\liminf_{u_2\to 0^+}\min_{x\in[\frac{1}{4},\frac{3}{4}]}\min_{\frac{|u_2|}{16}\le|u_1|\le|u_2|}\frac{|f(x,u_1(x),u_2(x))|}{|u_2|}=\infty,$$

$$d)\limsup_{u_2\to+\infty}\max_{x\in[0,1]}\max_{|u_1|\le u_2}\frac{|f(x,u_1(x),u_2(x))|}{|u_2|}=0,$$

hold. Then there exists a solution $y\in K\setminus 0$ such that y is a solution of the boundary value problem, (2.1), (2.2).

Proof: Employ condition c) and choose $H_1>0$ such that if $|u_2|\le H_1$ then $|f(x,u_1,u_2)|\ge 8|u_2|$ for all $x\in[\frac{1}{4},\frac{3}{4}], |u_1|\le|u_2|$. Define

$$\Omega_1=\{y\in C^1[0,1]:||y||\le H_1\},$$

and assume $y\in K\cap\partial\Omega_1$. Then

$$(\mathcal{A}y)'(\frac{1}{4})=|\int_{\frac{1}{4}}^{1}f(s,y(s),y'(s))ds|\ge\int_{\frac{1}{4}}^{\frac{3}{4}}|f(s,y(s),y'(s))ds|\ge\int_{\frac{1}{4}}^{\frac{3}{4}}8y'(s)ds\ge 4y'(\frac{3}{4})\ge||y||.$$

Thus, for $y\in K\cap\partial\Omega_1$ if follows that

$$||\mathcal{A}y||=(\mathcal{A}y)'(0)\ge(\mathcal{A}y)'(\frac{1}{4})\ge||y||.$$

Now employ condition d), and consider two cases, f is a bounded function or f is an unbounded function.

First assume that f is a bounded function. Assume there exists $M>0$ such that $|f(x,u_1,u_2)|\le M$ for all $x\in[0,1]$, $\frac{u_2}{16}\le u_1\le u_2<\infty$. Set $H_2=\max\{2H_1,M\}$. If $y\in K$ and $|y'(0)|\ge H_2$, then

$$(\mathcal{A}y)'(0)=|\int_0^1 f(s,y(s),y'(s))ds|\le M\le H_2;$$

in particular, define

$$\Omega_2=\{y\in C^1[0,1]:||y||\le H_2\},$$

and assume $y\in K\cap\partial\Omega_2$. Then $||\mathcal{A}y||\le||y||$.

For the final case, assume the f is unbounded. Find $H_2 \geq 2H_1$ such that if $|u_2| \geq H_2$ then $|f(x, u_1, u_2)| \leq |u_2|$ for all $x \in [0,1]$, $\frac{u_2}{16} \leq u_1 \leq u_2 < \infty$. Define

$$\Omega_2 = \{y \in C^1[0,1] : ||y|| \leq H_2\}$$

and assume $y \in K \cap \partial\Omega_2$. Then

$$(\mathcal{A}y)'(0) = |\int_0^1 f(s, y(s), y'(s))ds| \leq \int_0^1 y'(s)ds \leq y'(0) = ||y|| = H_2;$$

in particular, $||\mathcal{A}y|| \leq ||y||$ for $y \in K \cap \partial\Omega_2$.

Again the hypotheses of Theorem 2.1 are satisfied and there exists a solution $y \in K \setminus \{0\}$ of the boundary value problem (2.1), (2.2).

□

3 Examples

Example 1 *Let* $f(x, u_1, u_2) = -g(u_1)h(u_2)$ *where* $g : [0,\infty) \to [0,\infty)$ *is an increasing continuous function and* $h : [0,\infty) \to [0,\infty)$ *is a decreasing continuous function. Then* f *satisfies (2.5). For a specific example, let* $f(x, u_1, u_2) = -u_1^{\alpha}(1 + \exp(-u_2))$. *If* $\alpha > 1$ *then the conditions of Theorem 2.4 are satisfied and if* $0 < \alpha < 1$ *then the conditions of Theorem 2.5 are satisfied.*

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