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INITIAL VALUE PROBLEMS FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Let $n \geq 1$ denote an integer and let $n-1 < \alpha \leq n$. We consider an initial value problem for a nonlinear Caputo fractional differential equation of order α and obtain results analogous to well known results for initial value problems for ordinary differential equations. These results include Picard's existence and uniqueness theorem, Peano's existence theorem, extendibility of solutions to the right, maximal intervals of existence, a Kamke type convergence theorem, and the continuous dependence of solutions on parameters. The nonlinear term is assumed to depend on higher order derivatives and solutions are obtained in the space of $n-1$ times continuously differentiable functions.

1. INTRODUCTION

The purpose of this article is to develop some of the fundamental qualitative results for initial value problems for nonlinear fractional differential equations of Caputo type. We are particularly interested that the nonlinearity depend on higher order derivatives. Fractional differential equations have been of interest to many researchers in recent years and there are many authoritative accounts of fractional calculus and fractional differential equations [1, 9, 10, 15, 16, 17]; in these and other accounts, analytic solution methods and appropriate special functions are developed and computation and numerical methods are addressed. It is also common that Picard type existence and uniqueness theorems, or Peano type existence theorems are stated and proved for an initial value problem for a nonlinear fractional differential equation.

The study of the qualitative theory of fractional equations is currently receiving considerable attention. We fail to give an exhaustive bibliography and provide the following references. Diethelm, Ford and co-authors ([2, 3, 4, 5, 6], for example) have studied fractional differential equations qualitatively in their extensive work to develop the applications of fractional differential equations to numerical analysis. The works of Lakshmikantham and Vatsala [11, 12, 13] for example, are frequently referenced in today's work. More recently, there has been rapid growth in the study

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of existence and uniqueness results, continuous dependence results and extendibility results for initial value problems for fractional differential equations; see [14, 21, 22], for example. We also cite some interesting recent work of Tisdell [18, 19, 20].

In the search of existing work, we conclude that efforts have focused on properties of solutions from a space of continuous functions regardless of the order of the fractional equation. The purpose of this article is to develop some fundamental properties of solutions of initial value problems analogous to the study of initial value problems for higher order ordinary differential equations; in particular, if $n \geq 1$ is an integer, $n-1 < \alpha \leq n$, and the order of the fractional equation is α, we obtain properties of solutions of initial value problems that are from a space of $n-1$ times continuously differentiable functions. We ask that the nonlinearities depend on classical order derivatives since polynomials span the solution space of the corresponding linear homogeneous Caputo fractional differential equation. It would be natural to ask that the nonlinear term depend on fractional derivatives as well. We do not address that case in this work.

In what follows, a Caputo derivative of a function and several lemmas and theorems related to Picard type existence and uniqueness theorems and Peano type existence theorems for initial value problems are obtained. Then a method to extend solutions of initial value problems to the right is developed and an important analogue of Hartman's Corollary 2.1 [8, page 11] is obtained. With this corollary, solutions of initial value problems on maximal intervals of existence and a version of the Kamke convergence theorem [8, Theorem 3.2] for solutions of initial value problems of Caputo fractional differential equations are obtained. The article closes with two corollaries of Kamke's theorem giving sufficient conditions for the continuous dependence of solutions of initial values. The style of presentation somewhat follows that of Hartman [8], but it more accurately follows a development for ordinary differential equations that can be found in [7].

2. Initial Value Problems

We begin with the definitions of the Riemann-Liouville fractional integral and the Caputo fractional derivative.

Definition 2.1. Let $0 < \alpha$. For $x^* \in \mathbb{R}$, the α-th Riemann-Liouville fractional integral of a function, y, is defined by

$$I_{x^*}^{\alpha} y(x) = \frac{1}{\Gamma(\alpha)} \int_{x^*}^{x} (x-s)^{\alpha-1} y(s) ds, \quad x^* \leq x,$$

provided the right-hand side exists. For $\alpha = 0$, define $I_{x^*}^{\alpha}$ to be the identity map. Moreover, let n denote a positive integer and assume $n-1 < \alpha \leq n$. The Riemann-Liouville fractional derivative of order α is defined as

$$D_{x^*}^{\alpha} y(x) = D^n I_{x^*}^{n-\alpha} y(x)$$

where D^n denotes the classical nth order derivative, if the right-hand side exists. If a function y is such that

$$D_{x^*}^{\alpha} \Big(y(x) - \sum_{i=0}^{n-1} y^{(i-1)}(x^*) \frac{(x-x^*)^i}{i!} \Big)$$

exists, then the Caputo fractional derivative of order α of the function y is defined by

$$D^{\alpha}_{*x^*}y(x) = D^{\alpha}_{x^*}\Big(y(x) - \sum_{i=0}^{n-1} y^{(i-1)}(x^*)\frac{(x-x^*)^i}{i!}\Big).$$

Again for $\alpha = 0$, define $D^{\alpha}_{*x^*}$ to be the identity map.

Let $n \geq 1$ denote an integer and let $n-1 < \alpha \leq n$. Let $a < b$ and initially assume that $f : (a,b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous. Let $x^* \in (a,b)$. Let $y_i \in \mathbb{R}$, $i = 1, \dots, n$. Consider the initial value problem

$$D^{\alpha}_{*x^*}y(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x)), \quad a < x^* < x < b, \tag{2.1}$$

$$y^{(i-1)}(x^*) = y_i, \quad i = 1, \dots, n. \tag{2.2}$$

We shall introduce some norm notations and then begin with a preliminary lemma.

For $y \in C[c,d]$, define $||y||_{0,[c,d]} = \max_{c \leq x \leq d}|y(x)|$ and if $k \geq 1$ is an integer, for $y \in C^k[c,d]$ define

$$||y||_{k,[c,d]} = \max\{||y||_{0,[c,d]}, ||y'||_{0,[c,d]}, \dots, ||y^k||_{0,[c,d]}\}.$$

Lemma 2.1. *Let $n \geq 1$ denote an integer and let $n-1 < \alpha \leq n$. Let $g : (a,b) \to \mathbb{R}$, and assume g is $n-1$ times continuously differentiable on any compact subset of (a,b). Assume $K > 0$ and $h^* > 0$. Let $x^* \in (a,b)$. Define $G \subset \mathbb{R}^{n+1}$ by*

$$G = \{(x, v_1, \dots, v_n) : x^* \leq x \leq x^* + h^*, |v_i - g^{(i-1)}(x)| \leq K, i = 1, \dots, n\}. \tag{2.3}$$

Assume $f : [x^*, x^* + h^*] \times \mathbb{R}^n \to \mathbb{R}$ is continuous. Let

$$M \geq \sup_{(x,v_1,\dots,v_n)\in G} |f(x, v_1, \dots, v_n)|$$

and assume $M > 0$. Set

$$h = \min_{j=0,1,\dots,n-1}\{(\frac{K\Gamma(\alpha+1-j)}{M})^{\frac{1}{\alpha-j}}\}. \tag{2.4}$$

Then there exists $y \in C^{n-1}[x^*, x^* + h]$ satisfying the integral relation

$$y(x) = g(x) + \int_{x^*}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), y'(s), \dots, y^{(n-1)}(s))ds, \quad x^* \leq x \leq x^* + h. \tag{2.5}$$

Proof. Define $\mathcal{A} : C^{n-1}[x^*, x^* + h] \to C^{n-1}[x^*, x^* + h]$ by

$$\mathcal{A}y(x) = g(x) + \int_{x^*}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), \dots, y^{(n-1)}(s))ds.$$

To see that $\mathcal{A}y \in C^{n-1}[x^*, x^* + h]$, assume $x^* \leq x_1 < x_2 \leq x^* + h$. Let $y \in C^{n-1}[x^*, x^* + h]$ and let

$$\hat{M} = \sup_{x^* \leq x \leq x^*+h} |f(x, y(x), \dots, y^{(n-1)}(x))|.$$

For each $i = 1, \ldots, n$,

$$\begin{aligned}|(\mathcal{A}y)^{(i-1)}(x_2) - (\mathcal{A}y)^{(i-1)}(x_1)| &\leq |g^{(i-1)}(x_2) - g^{(i-1)}(x_1)| \\ &+ \frac{\hat{M}}{\Gamma(\alpha - (i-1))}\Big(\int_{x^*}^{x_1} |((x_2 - s)^{\alpha-1-(i-1)} - (x_1 - s)^{\alpha-1-(i-1)})| ds \\ &+ \int_{x_1}^{x_2} (x_2 - s)^{\alpha-1-(i-1)} ds\Big).\end{aligned}$$

For $i < n$,

$$|((x_2 - s)^{\alpha-1-(i-1)} - (x_1 - s)^{\alpha-1-(i-1)})| = ((x_2 - s)^{\alpha-1-(i-1)} - (x_1 - s)^{\alpha-1-(i-1)})$$

and for $i = n$,

$$|((x_2 - s)^{\alpha-1-(i-1)} - (x_1 - s)^{\alpha-1-(i-1)})| = ((x_1 - s)^{\alpha-1-(i-1)} - (x_2 - s)^{\alpha-1-(i-1)}).$$

Thus, for $i < n$,

$$\begin{aligned}&\frac{\hat{M}}{\Gamma(\alpha - (i-1))} \int_{x^*}^{x_1} |((x_2 - s)^{\alpha-1-(i-1)} - (x_1 - s)^{\alpha-1-(i-1)})| ds \\ &\qquad = \frac{\hat{M}}{\Gamma(\alpha + 1 - (i-1))}\Big((x_2 - x^*)^{\alpha-(i-1)} \\ &\qquad - (x_1 - x^*)^{\alpha-(i-1)} - (x_2 - x_1)^{\alpha-(i-1)}\Big)\end{aligned}$$

and

$$\begin{aligned}|(\mathcal{A}y)^{(i-1)}(x_2) - (\mathcal{A}y)^{(i-1)}(x_1)| &\leq |g^{(i-1)}(x_2) - g^{(i-1)}(x_1)| \\ &+ \frac{\hat{M}}{\Gamma(\alpha + 1 - (i-1))}\Big((x_2 - x^*)^{\alpha-(i-1)} \\ &- (x_1 - x^*)^{\alpha-(i-1)} - (x_2 - x_1)^{\alpha-(i-1)}\Big) \\ &+ \frac{\hat{M}}{\Gamma(\alpha + 1 - (i-1))}(x_2 - x_1)^{\alpha-(i-1)} \\ &= |g^{(i-1)}(x_2) - g^{(i-1)}(x_1)| \\ &+ \frac{\hat{M}}{\Gamma(\alpha + 1 - (i-1))}\Big((x_2 - x^*)^{\alpha-(i-1)} - (x_1 - x^*)^{\alpha-(i-1)}\Big).\end{aligned} \tag{2.6}$$

For $i = n$,

$$\begin{aligned}&\frac{\hat{M}}{\Gamma(\alpha - (n-1))} \int_{x^*}^{x_1} |((x_2 - s)^{\alpha-1-(n-1)} - (x_1 - s)^{\alpha-1-(n-1)})| ds \\ &\qquad = \frac{\hat{M}}{\Gamma(\alpha + 1 - (n-1))}\Big((x_1 - x^*)^{\alpha-(n-1)} \\ &\qquad - (x_2 - x^*)^{\alpha-(n-1)} + (x_2 - x_1)^{\alpha-(n-1)}\Big) \\ &\qquad \leq (x_2 - x_1)^{\alpha-(n-1)}\end{aligned}$$

and

$$|(\mathcal{A}y)^{(n-1)}(x_2) - (\mathcal{A}y)^{(n-1)}(x_1)| \le |g^{(n-1)}(x_2) - g^{(n-1)}(x_1)| \\ + \frac{2\hat{M}}{\Gamma(\alpha+1-(n-1))}(x_2-x_1)^{\alpha-(n-1)}. \tag{2.7}$$

For $i = 1, \ldots, n-1$, (2.6) implies $(\mathcal{A}y)^{(i-1)}$ is uniformly continuous on $[x^*, x^*+h]$ and for $i = n$, (2.7) implies $(\mathcal{A}y)^{(i-1)}$ is uniformly continuous on $[x^*, x^*+h]$.

Now set

$$\mathcal{U} = \{y \in C^{n-1}[x^*, x^*+h] : ||y - g||_{n-1,[x^*,x^*+h]} \le K\}.$$

The proof is complete once we show that $\mathcal{A} : \mathcal{U} \to \mathcal{U}$, that $\mathcal{A}(\mathcal{U})$ is uniformly bounded and that each $\mathcal{A}^{(i-1)}(\mathcal{U}), i = 1, \ldots, n$, is equicontinuous. Then an application of the Schauder fixed point theorem gives the existence of y satisfying the integral relation (2.5) and the lemma is proved. To see that $\mathcal{A} : \mathcal{U} \to \mathcal{U}$, let $y \in \mathcal{U}$. To apply (2.4), set $j = i-1$. For $x^* \le x \le x^*+h$,

$$|(\mathcal{A}y)^{(j)}(x) - g^{(j)}(x)| \le \frac{M}{\Gamma(\alpha+1-j)}(x-x^*)^{\alpha-j}, \quad j = 0, \ldots, n-1,$$

and so,

$$||\mathcal{A}y - g||_{n-1,[x^*,x^*+h]} \le K.$$

Thus, $\mathcal{A} : \mathcal{U} \to \mathcal{U}$ which also implies $\mathcal{A}(\mathcal{U})$ is uniformly bounded. For the equicontinuity, (2.6) and (2.7) remain valid if $\hat{M}$ is replaced by M. Thus, if $i = 1, \ldots, n-1$, and $y \in \mathcal{U}$,

$$|(\mathcal{A}y)^{(i-1)}(x_2) - (\mathcal{A}y)^{(i-1)}(x_1)| \le |g^{(i-1)}(x_2) - g^{(i-1)}(x_1)| \\ + \frac{M}{\Gamma(\alpha+1-(i-1))}\Big((x_2-x^*)^{\alpha-(i-1)} - (x_1-x^*)^{\alpha-(i-1)}\Big),$$

implying $\mathcal{A}^{(i-1)}(\mathcal{U}), i = 1, \ldots, n-1$, is equicontinuous, and for $i = n$, $y \in \mathcal{U}$,

$$|(\mathcal{A}y)^{(n-1)}(x_2) - (\mathcal{A}y)^{(n-1)}(x_1)| \le |g^{(n-1)}(x_2) - g^{(n-1)}(x_1)| \\ + \frac{2M}{\Gamma(\alpha+1-(n-1))}(x_2-x_1)^{\alpha-(n-1)}$$

implying $\mathcal{A}^{(n-1)}(\mathcal{U})$ is equicontinuous. □

Lemma 2.2. *Let $n \ge 1$ denote an integer and let $n-1 < \alpha \le n$. Let $g : (a,b) \to \mathbb{R}$, and assume g is $n-1$ times continuously differentiable on any compact subset of (a,b). Assume $K > 0$ and $h^* > 0$. Let $x^* \in (a,b)$. Define $G \subset \mathbb{R}^{n+1}$ by (2.3). Assume $f : [x^*, x^*+h^*] \times \mathbb{R}^n \to \mathbb{R}$ is continuous. Let $M \ge \sup_{(x,v_1,\ldots,v_n)\in G} |f(x,v_1,\ldots,v_n)|$ and assume $M > 0$. Set*

$$h = \min\{h^*, \min_{j=0,1,\ldots,n-1}(\frac{K\Gamma(\alpha+1-j)}{M})^{\frac{1}{\alpha-j}}\}.$$

In addition, assume $f : [x^*, x^*+h^*] \times \mathbb{R}^n \to \mathbb{R}$ satisfies a Lipschitz condition

$$|f(x,v_1,\ldots,v_n) - f(x,w_1,\ldots,w_n)| \le L \max_{i=1,\ldots,n} |v_i - w_i|$$

for some $L > 0$. Then there exists a unique $y \in C^{n-1}[x^*, x^*+h]$ satisfying the integral relation (2.5).

We omit the proof as it employs the standard method of successive approximations employed in Picard type uniqueness results. A rigorous proof in the case $f(x,y)$ is independent of $y',\ldots,y^{(n-1)}$ is found in [1, page 93].

The next lemma provides the specific fixed point operator for the initial value problem (2.1), (2.2). Diethelm [1, Lemma 6.2] proved Lemma 2.3 in the case $f(x,y)$ is independent of higher order derivatives and obtained a fixed point operator to obtain solutions $y \in C[x_1, x_1+h]$.

Lemma 2.3. *Let $n \geq 1$ denote an integer and let $n-1 < \alpha \leq n$. Let $y_1,\ldots,y_n \in \mathbb{R}$, $K>0$, $h^*>0$. Let $x_1 \in (a,b)$. Define*

$$g(x) = \sum_{k=0}^{n-1} y_{k+1} \frac{(x-x_1)^k}{k!} \tag{2.8}$$

and define

$$G = \{(x, v_1,\ldots,v_n) : x_1 \leq x \leq x_1+h^*, |v_i - g^{(i-1)}(x)| \leq K, i=1,\ldots,n\}. \tag{2.9}$$

Assume $f : G \to \mathbb{R}^n$ is continuous. Let

$$M \geq \sup_{(x,v_1,\ldots,v_n)\in G} |f(x,v_1,\ldots,v_n)|$$

and assume $M>0$. Set

$$h = \min\{h^*, \min_{j=0,1,\ldots,n-1} \left(\frac{K\Gamma(\alpha+1-j)}{M}\right)^{\frac{1}{\alpha-j}}\}.$$

Then $y \in C^{n-1}[x_1,x_1+h]$ is a solution of the initial value problem (2.1), (2.2) on $[x_1,x_1+h]$ if, and only if, $y \in C^{n-1}[x_1,x_1+h]$ and

$$y(x) = g(x) + \int_{x_1}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), \ldots y^{(n-1)}(s))ds, \quad x_1 \leq x \leq x_1+h. \tag{2.10}$$

Proof. Consider the initial value problem (2.2) for the linear nonhomogeneous fractional differential equation

$$D^{\alpha}_{*x_1} y(x) = F(x), \quad a < x_1 < x < b,$$

where $F \in C[x_1,x_1+h]$. Then y is a solution of the nonhomogeneous initial value problem, if and only if,

$$y(x) = g(x) + \int_{x_1}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds, \quad x_1 \leq x \leq x_1+h.$$

Now consider the initial value problem (2.1), (2.2). If $y \in C^{n-1}[x_1,x_1+h]$ is a solution of the initial value problem (2.1), (2.2), set $F(x) = f(x, y(x), \ldots y^{(n-1)}(x))$. Then $F \in C[x_1,x_1+h]$ and so, y satisfies (2.10). Conversely, if $y \in C^{n-1}[x_1,x_1+h]$ and y satisfies (2.10) then the continuity of f [1, Theorem 3.7] is sufficient to give that $D^{\alpha}_{*x^*} I^{\alpha}_{x^*} f = f$. Thus, if $y \in C^{n-1}[x_1,x_1+h]$ and y satisfies (2.10), then y is a solution of the initial value problem (2.1), (2.2). □

We do point out that the assumption $y \in C^{n-1}[x_1,x_1+h]$ is only required to imply $F \in C[x_1,x_1+h]$. Thus we state and prove a lemma characterizing the differentiability of

$$y(x) = g(x) + \int_{x_1}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds, \quad x_1 \leq x \leq x_1+h,$$

where F is only assumed to be continuous.

Lemma 2.4. *Let $n \geq 1$ denote an integer and let $n-1 < \alpha \leq n$. Let $y_1, \ldots, y_n \in \mathbb{R}$, $K > 0$, $h^* > 0$. Let $x_1 \in (a,b)$. Define*

$$g(x) = \sum_{k=0}^{n-1} y_{k+1} \frac{(x-x_1)^k}{k!}.$$

Let $h > 0$ and assume $F : [x_1, x_1+h] \to \mathbb{R}$ is continuous. Then,

$$y(x) = g(x) + \int_{x_1}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} F(s)ds, \quad x_1 \leq x \leq x_1 + h,$$

implies that $y \in C^{n-1}[x_1, x_1+h]$.

Proof. Let $M > 0$ be such that $|F(x)| \leq M$ for $x_1 \leq x \leq x_1 + h$. Calculations similar to those provide in (2.6) and (2.7) give the following estimates. If $i = 1, \ldots, n-1$, and $x_1 \leq x_2 \leq x_3$, then

$$|y^{(i-1)}(x_3) - y^{(i-1)}(x_2)| \leq |g^{(i-1)}(x_3) - g^{(i-1)}(x_2)| + \frac{M}{\Gamma(\alpha+1-(i-1))} \Big((x_3-x_1)^{\alpha-(i-1)} - (x_2-x_1)^{\alpha-(i-1)} \Big).$$

and if $i = n$ and $x_1 \leq x_2 \leq x_3$, then

$$|y^{(n-1)}(x_3) - y^{(n-1)}(x_2)| \leq |g^{(n-1)}(x_3) - g^{(n-1)}(x_2)| + \frac{2M}{\Gamma(\alpha+1-(i-1))} (x_3-x_2)^{\alpha-(i-1)}.$$

Thus, for $i = 1, \ldots, n$, $y^{(i-1)} \in C[x_1, x_1+h]$ and $y \in C^{n-1}[x_1, x_1+h]$. □

This next theorem is a fractional version of the Peano theorem; again Diethelm [1] proved a version of the Peano theorem in the case $f(x,y)$ is independent of $y', \ldots, y^{(n-1)}$.

Theorem 2.1. *Let $n \geq 1$ denote an integer and let $n-1 < \alpha \leq n$. Let $y_1, \ldots, y_n \in \mathbb{R}$, $K > 0$, $h^* > 0$. Let $x_1 \in (a,b)$. Define g by (2.8) and define G by (2.9). Assume $f : G \to \mathbb{R}$ is continuous. Let $M \geq \sup_{(x,v_1,\ldots,v_n)\in G} |f(x, v_1, \ldots, v_n)|$ and assume $M > 0$. Set*

$$h = \min\{h^*, \min_{j=0,1,\ldots,n-1} \big(\frac{K\Gamma(\alpha+1-j)}{M}\big)^{\frac{1}{\alpha-j}}\}.$$

Then there exists a function $y \in C^{n-1}[x_1, x_1+h]$ that is a solution of the initial value problem (2.1), (2.2) on $[x_1, x_1+h]$.

Proof. Apply Lemma 2.3 and then apply Lemma 2.1 in the specific case where g is given by (2.8). □

We state without proof the corresponding Picard type uniqueness result.

Theorem 2.2. *Let $n \geq 1$ denote an integer and let $n-1 < \alpha \leq n$. Let $y_1, \ldots, y_n \in \mathbb{R}$, $K > 0$, $h^* > 0$. Let $x_1 \in (a,b)$. Define g by (2.8) and define G by (2.9). Assume $f : G \to \mathbb{R}$ is continuous. Let $M \geq \sup_{(x,v_1,\ldots,v_n)\in G} |f(x, v_1, \ldots, v_n)|$ and assume $M > 0$. Set*

$$h = \min\{h^*, \min_{j=0,1,\ldots,n-1} \big(\frac{K\Gamma(\alpha+1-j)}{M}\big)^{\frac{1}{\alpha-j}}\}.$$

In addition, assume $f : [x^*, x^*+h^*] \times \mathbb{R}^n \to \mathbb{R}$ satisfies a Lipschitz condition

$$|f(x, v_1, \ldots, v_n) - f(x, w_1, \ldots, w_n)| \leq L \max_{i=1,\ldots,n} |v_i - w_i|$$

for some $L > 0$. Then there exists a unique $y \in C^{n-1}[x_1, x_1 + h]$ that is a solution of the initial value problem (2.1), (2.2) on $[x_1, x_1 + h]$.

We now obtain an important corollary which an analogue of Hartman's Corollary 2.1 [8, page 11]. In the proof, we extend to the right a solution of an initial value problem for the fractional differential equation (2.1).

Corollary 2.1. *Let $E \subset (a, b)\times\mathbb{R}^n$, E open, connected and convex, and let $f : E \to \mathbb{R}$ be continuous. Let $\mathcal{K} \subset E$ be compact. Let $a < x_1 < x_2 < b$. Let $z \in C^{n-1}[x_1, x_2]$ denote a solution of the initial value problem (2.1), (2.2) for $x^* = x_1$ on $[x_1, x_2]$. If $(x_2, z(x_2), \dots, z^{(n-1)}(x_2)) \in \mathcal{K}$, then there exists $\delta_{\mathcal{K}} > 0$ such that there is a solution $y \in C^{n-1}[x_1, x_2]$ of the initial value problem (2.1), (2.2) on $[x_1, x_2 + \delta_{\mathcal{K}}]$ and if $x_1 \le x \le x_2$ then $y(x) = z(x)$.*

Proof. Let $z \in C^{n-1}[x_1, x_2]$ be a solution of the initial value problem (2.1), (2.2) on $[x_1, x_2]$ and define for $x \ge x_2$,

$$\bar{g}(x) = g(x) + \int_{x_1}^{x_2} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, z(s), \dots, z^{(n-1)}(s))ds \tag{2.11}$$

where $g(x)$ is given by (2.8).

We first construct a viable $\delta_{\mathcal{K}} > 0$.

For $(x, v_1, \dots, v_n) = (x, v) \in \mathbb{R}^{n+1}, (\hat{x}, \hat{v}_1, \dots, \hat{v}_n) = (\hat{x}, \hat{v}) \in \mathbb{R}^{n+1}$, define

$$\operatorname{dis}(\mathcal{K}, \partial E) = \inf_{(x,v)\in\mathcal{K},(\hat{x},\hat{v})\in\partial E}\Big(d((x, v), (\hat{x}, \hat{v}))\Big)$$

where

$$d((x, v), (\hat{x}, \hat{v})) = |x - \hat{x}| + \max_{i=1,\dots,n}\{|v_i - \hat{v}_i|\}.$$

Set $\eta = 1$ if $\operatorname{dis}(\mathcal{K}, \partial E) = +\infty$ and set $\eta = \frac{1}{2}\operatorname{dis}(\mathcal{K}, \partial E)$ if $\operatorname{dis}(\mathcal{K}, \partial E) < +\infty$. Define

$$\overline{E_1} = \{(x, v_1, \dots, v_n) \in E : d((x, v_1, \dots, v_n), \mathcal{K}) \le \eta\}.$$

Then $\mathcal{K} \subset \overline{E_1} \subset E$ and $\overline{E_1}$ is compact. Set $M \ge \sup_{(x,v_1,\dots,v_n)\in\overline{E_1}} |f|$, and assume $M > 0$.

Define

$$G = \{(x, v) \in \mathbb{R}^{n+1} : 0 \le x - x_2 \le \frac{\eta}{2}, |v_i - \bar{g}^{(i-1)}(x)| \le \frac{\eta}{2}\}.$$

If $(s, v_1, \dots, v_n) \in G$ then

$$d\big((s, v_1, \dots, v_n), (x_2, z(x_2), z'(x_2), \dots z^{(n-1)}(x_2))\big) \le \eta.$$

Since $(x_2, z(x_2), z'(x_2), \dots z^{(n-1)}(x_2)) \in \mathcal{K}$, then

$$d((s, v_1, \dots, v_n), \mathcal{K}) \le \eta \text{ and } (s, v_1, \dots, v_n) \in \overline{E}_1;$$

in particular, $G \subset \overline{E_1} \subset E$. So appeal to Lemma 2.1 with $x^* = x_2$ and $h^* = K = \frac{\eta}{2}$ and set

$$\delta_{\mathcal{K}} = \min\{\frac{\eta}{2}, \min_{j=0,1,\dots n-1}\big(\frac{\eta\Gamma(\alpha + 1 - j)}{2M}\big)^{\frac{1}{\alpha-j}}\}. \tag{2.12}$$

With the construction of $\delta_{\mathcal{K}} > 0$, for $x_1 \le x \le x_2 + \delta_{\mathcal{K}}$ we seek $y \in C^{n-1}[x_1, x_2 + \delta_{\mathcal{K}}]$ satisfying $y(x) = z(x)$ if $x_1 \le x \le x_2$ and

$$y(x) = g(x) + \int_{x_1}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), \dots, y^{(n-1)}(s))ds, \quad x_1 \le x \le x_2 + \delta_{\mathcal{K}}$$

where $g(x)$ is given by (2.8). If $x \in [x_2, x_2+\delta_{\mathcal{K}}]$ relabel $\bar{g}$ by

$$\bar{g} = g_{\mathcal{K}}(x) = g(x) + \int_{x_1}^{x_2} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, z(s), \dots, z^{(n-1)}(s))ds.$$

Define an operator $\mathcal{A}_{\mathcal{K}}$ on $C^{n-1}[x_2, x_2+\delta_{\mathcal{K}}]$ by

$$\begin{aligned} \mathcal{A}_{\mathcal{K}} y(x) &= g_{\mathcal{K}}(x) \\ &+ \int_{x_2}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), \dots, y^{(n-1)}(s))ds, \quad x_2 \le x \le x_2+\delta_{\mathcal{K}}. \end{aligned} \tag{2.13}$$

With the construction of $\delta_{\mathcal{K}}$ in (2.12), Lemma 2.1 applies and the operator $\mathcal{A}_{\mathcal{K}}$ has a fixed point $z_2 \in C^{n-1}[x_2, x_2+\delta_{\mathcal{K}}]$. Define

$$y(x) = \begin{cases} z(x), & x_1 \le x \le x_2, \\ \\ z_2(x), & x_2 < x \le x_2+\delta_{\mathcal{K}}. \end{cases} \tag{2.14}$$

By construction, $y(x) = z(x)$, for $x_1 \le x \le x_2$ and by construction

$$y \in C^{n-1}[x_1, x_2+\delta_{\mathcal{K}}].$$

For $x_1 \le x \le x_2$,

$$\begin{aligned} y(x) &= g(x) + \int_{x_1}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, z(s), \dots, z^{(n-1)}(s))ds \\ &= g(x) + \int_{x_1}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), \dots, y^{(n-1)}(s))ds. \end{aligned}$$

For $x_2 \le x \le x_2+\delta_{\mathcal{K}}$,

$$\begin{aligned} y(x) &= g_{\mathcal{K}}(x) + \int_{x_2}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), \dots, y^{(n-1)}(s))ds \\ &= g(x) + \int_{x_1}^{x_2} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, z(s), \dots, z^{(n-1)}(s))ds \\ &\quad + \int_{x_2}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), \dots, y^{(n-1)}(s))ds \\ &= g(x) + \int_{x_1}^{x_2} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), \dots, y^{(n-1)}(s))ds \\ &\quad + \int_{x_2}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), \dots, y^{(n-1)}(s))ds \\ &= g(x) + \int_{x_1}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), \dots, y^{(n-1)}(s))ds. \end{aligned}$$

Thus, the function y given by (2.14) satisfies

$$y(x) = \int_{x_1}^{x_2+\delta_{\mathcal{K}}} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), \dots, y^{(n-1)}(s))ds, \quad x_1 \le x \le x_2+\delta_{\mathcal{K}},$$

and extends the solution of (2.1), (2.2) on $[x_1, x_2]$ to $[x_1, x_2+\delta_{\mathcal{K}}]$. $\square$

It is important to note that $\delta_{\mathcal{K}}$, given by (2.12), depends only on η and so $\delta_{\mathcal{K}}$ depends only on $\mathcal{K}$.

We prove one more preliminary theorem prior to proving the continuation theorem.

Theorem 2.3. *Assume $E \subset \mathbb{R} \times \mathbb{R}^n$, E open, connected and convex, $f : E \to \mathbb{R}$ continuous. Assume y is a solution of the initial value problem* (2.1), (2.2) *for $x^* = x_1$ on $[x_1, \bar{x})$ such that if $x_1 \le x < \bar{x}$, then $(x, y(x), y'(x), \dots, y^{(n-1)}(x)) \in E$. Let $\{x_k\}$ denote a monotone sequence converging to $\bar{x}$ from below. Assume each limit, $\lim_{k\to\infty} y^{(i-1)}(x_k) = \bar{y}_i$ exists, $i = 1, \dots, n$. If there exist $0 < \gamma \le \bar{x} - x_1$, $\beta > 0$, and $M > 0$ such that $|f(x, v_1, \dots, v_m)| \le M$ on*

$$E \cap \{(x, v_1, \dots, v_n) : \bar{x} - \gamma \le x \le \bar{x}, \max_{i=1,\dots,n} |v_i - \bar{y}_i| \le \beta\}$$

then $\lim_{x\to\bar{x}^-} y^{(i-1)}(x) = \bar{y}_i$ exists for each $i = 1, \dots, n$. Moreover, if f is continuous on $E \cup \{(\bar{x}, \bar{y}_1, \dots, \bar{y}_n)\}$ extend $y(x)$ to $[x_1, \bar{x}]$ by $y^{(i-1)}(\bar{x}) = \bar{y}_i$, $i = 1, \dots, n$, and the extension of y is a solution of (2.1), (2.2) *on $[x_1, \bar{x}]$.*

Proof. Define

$$G = \{(x, v_1, \dots, v_n) : 0 < \bar{x} - x \le \gamma, \max_{i=1,\dots,n} |v_j - \bar{y}_i| \le \beta\}.$$

We first show that for k sufficiently large, if $x_k < x < \bar{x}$, then

$$(x, y(x), y'(x), \dots, y^{(n-1)}(x)) \in G.$$

For the sake of contradiction, suppose there exists a sequence $\{s_k\}$ converging to $\bar{x}$ from below, and $\max_{i=1,\dots,n} |y^{(i-1)}(s_k) - \bar{y}_i| > \beta$ for each k. Find K such that if $k \ge K$ then $\max_{i=1,\dots,n} |y^{(i-1)}(x_k) - \bar{y}_i| \le \frac{\beta}{2}$. Find subsequences $\{s_{k_l}\}$, $\{x_{k_l}\}$ and an integer $i \in \{0, 1, \dots, n-1\}$ such that

$$|y^{(i-1)}(s_{k_l}) - \bar{y}_i| > \beta, \text{ and } \max_{i=1,\dots,n} |y^{(i-1)}(x_{k_l}) - \bar{y}_i| \le \frac{\beta}{2},$$

for each l. Relabel $\{s_{k_l}\}$ by $\{s_k\}$ and $\{x_{k_l}\}$ by $\{x_k\}$ respectively. By continuity, there exists $\hat{x}_k \in (x_k, \bar{x})$ such that

$$\max_{i=1,\dots,n} |y^{(i-1)}(\hat{x}_k) - \bar{y}_i| = \beta \text{ and } \max_{i=1,\dots,n} |y^{(i-1)}(x) - \bar{y}_i| < \beta$$

for $x_k \le x < \hat{x}_k$. Then,

$$\frac{\beta}{2} \le \max_{i=1,\dots,n} |y^{(i-1)}(\hat{x}_k) - y^{(i-1)}(x_k)|. \tag{2.15}$$

Assume $i_0 = \max_{i=1,\dots,n} |y^{(i-1)}(\hat{x}_k) - y^{(i-1)}(x_k)|$. If $i_0 < n$, then (again, with calculations similar to those to produce (2.6))

$$\begin{aligned} |y^{(i_0-1)}(\hat{x}_k) - y^{(i_0-1)}(x_k)| &\le |g^{(i_0-1)}(\hat{x}_k) - g^{(i_0-1)}(x_k)| \\ &\quad + \frac{M}{\Gamma(\alpha + 1 - (i-1))} \Big((\hat{x}_k - x_1)^{\alpha-(i-1)} - (x_k - x_1)^{\alpha-(i-1)} \Big) \end{aligned}$$

where g is given by (2.8). By continuity, the right hand side vanishes as $x_k \to \bar{x}$. Choose $\gamma > 0$ sufficiently small so the right hand side is less than $\frac{\beta}{2}$ which contradicts (2.15). Thus, for k sufficiently large, $i_0 = n$. This contradicts (2.15) as well since calculations similar to those to produce (2.6) give

$$\begin{aligned} |y^{(n-1)}(\hat{x}_k) - y^{(n-1)}(x_k)| &\le |g^{(n-1)}(\hat{x}_k) - g^{(n-1)}(x_k)| \\ &\quad + \frac{2M}{\Gamma(\alpha + 1 - (n-1))} (\hat{x}_k - x_k)^{\alpha-(n-1)}. \end{aligned}$$

Thus, for k sufficiently large, if $x_k < x < \bar{x}$, then $(x, y(x), \ldots, y^{(n-1)}(x)) \in G$.

Now if $x_k < \hat{x}_1 < \hat{x}_2 < \bar{x}$,

$$\max_{i=1,\ldots,n-1} |y^{(i-1)}(\hat{x}_2) - y^{(i-1)}(\hat{x}_1)| \le \max_{i=1,\ldots,n-1} \Bigg(|g^{(i-1)}(\hat{x}_2) - g^{(i-1)}(\hat{x}_1)| + \frac{M}{\Gamma(\alpha+1-(i-1))} \Big((\hat{x}_2 - x_1)^{\alpha-(i-1)} - (\hat{x}_1 - x_1)^{\alpha-(i-1)} \Big) \Bigg)$$

and

$$|y^{(n-1)}(\hat{x}_2) - y^{(n-1)}(\hat{x}_1)| \le |g^{(n-1)}(\hat{x}_2) - g^{(n-1)}(\hat{x}_1) + \frac{2M}{\Gamma(\alpha+1-(i-1))} (\hat{x}_2 - \hat{x}_1)^{\alpha-(i-1)}.$$

By the Cauchy criterion, $\lim_{x\to\bar{x}} y^{(i-1)}(x)$ exists and equals $\bar{y}_i$, $i = 1, \ldots, n$.

Extend $y(x)$ to $[x_1, \bar{x}]$ by $y^{(i-1)}(\bar{x}) = \bar{y}_i$, $i = 1, \ldots, n$. Then $y \in C^{n-1}[x_1, \bar{x}]$ and by the continuity of f

$$y(x) = g(x) + \int_{x_1}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s), y'(s), \ldots, y^{(n-1)}(s)) ds, \quad x_1 \le x \le \bar{x};$$

in particular, the extension is a solution of (2.1), (2.2) on $[x_1, \bar{x}]$. □

We now give a precise definition of a right maximal interval of existence. Let $I = [x_1, b), I = [x_1, b]$ or $I = [x_1, \infty)$.

Definition 2.2. Let $y(x)$ be a solution of (2.1) on I. Then the interval I is said to be a right maximal interval of existence for $y(x)$ if there is no extension of y to the right that is a solution of (2.1) on the extended interval.

Definition 2.3. Assume $E \subset \mathbb{R} \times \mathbb{R}^n$, E open, connected and convex, $f : E \to \mathbb{R}$ continuous and let $y(x)$ be a solution of (2.1) on an interval $[x_1, b)$. If $b < \infty$, we say $y(x)$ approaches ∂E as $x \to b^-$ and write $y(x) \to \partial E$ as $x \to b^-$, if for any compact set $\mathcal{K} \subset E$, there exists $x_{\mathcal{K}} \in [x_1, b)$ such that $(x, y(x), y'(x), \ldots, y^{(n-1)}(x)) \notin \mathcal{K}$, for $x_{\mathcal{K}} < x < b$. If $I = [x_1, \infty)$, we say $y(x) \to \partial E$ as $x \to \infty$.

We are now in a position to state and prove a continuation theorem on a right maximal interval of existence for solutions of initial value problems for fractional differential equations.

Theorem 2.4. *Assume $E \subset \mathbb{R}\times\mathbb{R}^n$, E open, connected and convex, and $f : E \to \mathbb{R}$ continuous. Let I denote the interval $I = [x_1, b), I = [x_1, b]$ or $I = [x_1, \infty)$ and let $y(x)$ be a solution of (2.1) on I. Assume $\{(x, y(x), \ldots, y^{(n-1)}(x)) : x \in I\} \subset E$. Then $y(x)$ can be extended as a solution of (2.1) on a right maximal interval, $[x_1, \omega)$, and $y(x) \to \partial E$ as $x \to \omega^-$ (or as $x \to \infty$).*

Proof. Let $\{E_m\}_{m=1}^{\infty}$ denote a sequence of nonempty open, connected and convex subsets of E such that $\overline{E_m}$ is compact, $\overline{E_m} \subset E_{m+1}$ for each m and $E = \cup_{m=1}^{\infty} E_m$.

Let b denote the right endpoint of I. If $b = \infty$, then I is right maximal and $y(x) \to \partial E$ as $x \to \infty$.

Assume $b < \infty$ and assume $I = [x_1, b)$. There are two cases to consider:

i) There exists an increasing sequence $\{x_k\}$, $x_k \uparrow b$, and a set E_m such that $(x_k, y(x_k), y'(x_k), \ldots y^{(n-1)}(x_k)) \in \overline{E_m}$ for each $k \ge 1$;

ii) For each $m \ge 1$, there exists $x_k \in I$ such that $(x, y(x), y'(x), \ldots y^{(n-1)}(x)) \notin \overline{E_m}$ for $x_k < x < b$.

If case ii) happens, we argue that $I = [x_1, b)$ is maximal and $y(x) \to \partial E$ as $x \to b^-$. So, assume case ii) and assume for the sake of contradiction that $I = [x_1, b)$ is not maximal. Extend the solution y to $\overline{I} = [x_1, b]$ which can be done since I is not maximal. The compact set

$$\{(x, y(x), y'(x), \dots y^{(n-1)}(x)) : x_1 \leq x \leq b\} \subset \cup_{m=1}^{\infty} E_m,$$

which is a countable open cover. Select a finite subcover, $E_{m_1} \subset \cdots \subset E_{m_l}$, such that

$$\{(x, y(x), y'(x), \dots y^{(n-1)}(x)) : x_1 \leq x \leq b\} \subset E_{m_l} \subset \overline{E_{m_l}}.$$

This contradicts the assumption of case ii), so under the assumption of case ii), $I = [x_1, b)$ is maximal and $y(x) \to \partial E$ as $x \to b^-$.

We complete the proof by assuming case i) holds. Let $x_k \uparrow b$ and find E_m such that $(x_k, y(x_k), y'(x_k), \dots, y^{(n-1)}(x_k)) \in \overline{E_m}$ for each $k \geq 1$. Due to the compactness of $\overline{E_m}$ there exists a convergent subsequence $(x_{k_l}, y(x_{k_l}), \dots, y^{(n-1)}(x_{k_l}))$ and assume

$$(x_{k_l}, y(x_{k_l}), y'(x_{k_l}), \dots, y^{(n-1)}(x_{k_l})) \to (b, \overline{y_0}, \overline{y_1}, \dots, \overline{y_{n-1}}) \in \overline{E_m}.$$

Apply Theorem 2.3 and extend the solution $y(x)$ to $[x_1, b]$. (In particular, if case i) holds, we can assume that $I = [x_1, b]$.)

So $(b, y(b), y'(b), \dots, y^{(n-1)}(b)) \in \overline{E_m}$. Apply Corollary 2.1 and there exists $\delta_m > 0$ such that $y(x)$ can be extended to $[x_1, b + \delta_m]$. If $(b + \delta_m, y(b + \delta_m), y'(b + \delta_m), \dots, y^{(n-1)}(b + \delta_m)) \in \overline{E_m}$, apply Corollary 2.1 again to extend $y(x)$ to $[x_1, b + 2\delta_m]$. (This can be done since δ_m given by Corollary 2.1 depends only on $\overline{E_m}$.) Continue, extending y in increments of δ_m. $\overline{E_m}$ is compact so this process can be repeated only finitely many times. Find $j_m \geq 1$ such that

$$(b+(j_m-1)\delta_m, y(b+(j_m-1)\delta_m), y'(b+(j_m-1)\delta_m), \dots, y^{(n-1)}(b+(j_m-1)\delta_m)) \in \overline{E_m}$$

and

$$(b + j_m\delta_m, y(b + j_m\delta_m), y'(b + j_m\delta_m), \dots, y^{(n-1)}(b + j_m\delta_m)) \notin \overline{E_m}.$$

Let $b_1 = b + j_m\delta_m$. Since Corollary 2.1 has been applied at $b + (j_m - 1)\delta_m$, then $(b_1, y(b_1), y'(b_1), \dots, y^{(n-1)}(b_1)) \in E = \cup_{m=1}^{\infty} E_m$. So, there exists $m_1 > m$ such that $(b_1, y(b_1), y'(b_1), \dots, y^{(n-1)}(b_1)) \in \overline{E_{m_1}}$. Repeat the construction of the preceding paragraph and find $\delta_{m_1} > 0$ and $j_{m_1} \geq 1$ such that

$$(b + (j_{m_1} - 1)\delta_{m_1}, y(b + (j_{m_1} - 1)\delta_{m_1}), \dots, y^{(n-1)}(b + (j_{m_1} - 1)\delta_{m_1})) \in \overline{E_{m_1}}$$

and

$$(b + j_{m_1}\delta_{m_1}, y(b + j_{m_1}\delta_{m_1}), \dots, y^{(n-1)}(b + j_{m_1}\delta_{m_1})) \notin \overline{E_{m_1}}.$$

Set $b_2 = b_1 + j_{m_1}\delta_{m_1}$.

At each step, $(b_l, y(b_l), y'(b_l), \dots, y^{(n-1)}(b_l)) \in \overline{E_{m_l}} \subset E$, and so the process is not terminated in a finite number of steps. Construct an infinite sequence, $b < b_1 < \cdots < b_l < \cdots$ and an infinite sequence of integers, $m < m_1 < \cdots < m_l < \cdots$ such that $y(x)$ is extended to the closed interval, $[x_1, b_j]$, for all $j \geq 1$. Note that for $j \geq 2$,

$$(b_1, y(b_1), \dots, y^{(n-1)}(b_1)) \notin \overline{E_m}, \quad (b_j, y(b_j), \dots, y^{(n-1)}(b_j)) \notin \overline{E_{m_{j-1}}}. \tag{2.16}$$

Define $\omega = \sup_{j\geq 1}\{b_j\}$. Then $y(x)$ has been extended to $[x_1, \omega)$.

We close by showing $[x_1, \omega)$ is right maximal. If $\sup_{j\geq 1}\{b_j\} = \infty$ then $[x_1, \omega)$ is right maximal. If $\omega < \infty$, assume for the sake of contradiction that $y(x)$ is extended to $[x_1, \omega]$. Then, the compact set,

$$\{(x, y(x), y'(x), \ldots, y^{(n-1)}(x)) : b \leq x \leq \omega\} \subset \cup_{m=1}^{\infty} E_m,$$

which is a countable open covering. Select a finite subcover, and since $E_m \subset E_{m+1}$ for each m, find E_M such that

$$\{(x, y(x), y'(x), \ldots, y^{(n-1)}(x)) : b \leq x \leq \omega\} \subset E_M.$$

This contradicts (2.16) since there exists $m_{j-1} > M$ such that $\overline{E_M} \subset \overline{E_{m_{j-1}}}$. Thus,

$$(b_j, y(b_j), y'(b_j), \ldots, y^{(n-1)}(b_j)) \in \overline{E_M} \subset \overline{E_{m_{j-1}}}$$

and

$$(b_j, y(b_j), y'(b_j), \ldots, y^{(n-1)}(b_j)) \notin \overline{E_{m_{j-1}}}$$

by (2.16), giving the contradiction. Thus, $[x_1, \omega)$ is right maximal for the solution y, and $y(x) \to \partial E$ as $x \to \omega^-$.

□

Corollary 2.2. *Assume $x_1 < b$ and assume $f : [x_1, b] \times \mathbb{R}^n$ is continuous. Then there exists a solution y of (2.1), (2.2) and y exists on $I = [x_1, b]$ or y exists on a right maximal interval, $I = [x_1, \omega)$ where $x_1 < \omega \leq b$ and $||y||_{n-1,[x_1,x]} \to \infty$ as $x \to \omega^-$.*

To prove the corollary, define $F : [x_1, \infty) \times \mathbb{R}^n$ by

$$F(x, w_1, \ldots, w_n) = \begin{cases} f(x, w_1, \ldots, w_n), & x_1 \leq x \leq b, \\ \\ f(b, w_1, \ldots, w_n), & b \leq x. \end{cases}$$

F is continuous $[x_1, \infty) \times \mathbb{R}^n$ and so one applies Theorem 2.4 to $y(x)$ a solution of

$$D^{\alpha}_{*x_1} y(x) = F(x, y(x), y'(x), \ldots, y^{(n-1)}(x)), \quad x_1 \leq x.$$

We now state and prove a version of the Kamke convergence theorem for the initial value problem (2.1), (2.2).

Theorem 2.5. *Assume $E \subset \mathbb{R} \times \mathbb{R}^n$, E open, connected and convex and let $f_k : E \to \mathbb{R}$ denote a sequence of continuous functions that converge uniformly to a function f on every compact subset of E. Assume $(x^*, y_1, \ldots, y_n) \in E$. For each $k \geq 1$, assume $(x_k^*, y_k^1, \ldots, y_k^n) \in E$ and consider an initial value problem*

$$D^{\alpha}_{*x_k^*} y(x) = f_k(x, y(x), y'(x), \ldots, y^{(n-1)}(x)), \quad a < x_k^* < x < \omega_k, \tag{2.17}$$

$$y^{(i-1)}(x_k^*) = y_k^i, \quad i = 1, \ldots, n. \tag{2.18}$$

Let $y_k(x)$ denote the solution of (2.17), (2.18) on a maximal right interval $I_k = (x_k^*, \omega_k)$. Further, assume x_k^* is an increasing sequence and $x_k^* \uparrow x^*$ and

$$(x_k^*, y_k^1, \ldots, y_k^n) \to (x^*, y_1, \ldots, y_n) \text{ as } k \to \infty.$$

Then there exists a solution $y(x)$ of the initial value problem (2.1), (2.2) on a right maximal interval $I = [x^*, \omega)$ and there exists a subsequence $\{y_{k_l}\}$ of $\{y_k\}$ such that for each compact subset $J \subset [x^*, \omega)$, $||y_{k_l} - y||_{[n-1,J]} \to 0$ as $l \to \infty$.

Proof. Let $\{E_m\}_{m=1}^{\infty}$ denote a sequence of nonempty open, connected and convex subsets of E such that $\overline{E_m}$ is compact, $\overline{E_m} \subset E_{m+1}$ for each m, and $E = \cup_{m=1}^{\infty} E_m$. For each $m \geq 1$ define

$$\eta_m = \text{ dis } (\overline{E_m}, E \setminus E_{m+1})$$

and note $\eta_m > 0$. Define

$$E_m^* = \{(x, v_1, \ldots, v_n) \in E : \text{ dis } ((x, v_1, \ldots, v_n), \overline{E_m}) \leq \frac{\eta_m}{2}\}.$$

Then E_m^* is compact and $\overline{E_m} \subset E_m^* \subset E_{m+1}$. Since the sequence $\{f_k\}$ converges uniformly to f on E_m^*, a compact subset of E, there exists $M_m > 0$ such that $|f_k| \leq M_m$ on E_m^* for each $k \geq 1$. A modification of the proof of Corollary 2.1 provides that if

$$\delta_m = \min\{\frac{\eta_m}{4}, \min_{j=0,1,\ldots n-1} (\frac{\eta_m \Gamma(\alpha+1-j)}{4M_m})^{\frac{1}{\alpha-j}}\},$$

then, for each $k \geq 1$, $(\bar{x}, v_1, \ldots, v_n) \in E_m^*$, an initial value problem

$$D_{*\bar{x}}^{\alpha} y(x) = f_k(x, y(x), y'(x), \ldots, y^{(n-1)}(x)), \quad \bar{x} \leq x \leq \bar{x} + \delta_m,$$
$$y^{(i-1)}(\bar{x}) = v_i, \quad i = 1, \ldots, n,$$

has a solution that exists on $[\bar{x}, \bar{x} + \delta_m]$ (see (2.12)). Moreover, for $(\bar{x}, v_1, \ldots, v_n) \in E_m^*$, solutions of these initial value problems are bounded in norm in the space $C^{n-1}[\bar{x}, \bar{x} + \delta_m]$ since for each $i = 1, \ldots n$, $x \in [\bar{x}, \bar{x} + \delta_m]$,

$$|y^{(i-1)}(x)| \leq |y^{(i-1)}(x) - g_v^{(i-1)}(x)| + |g_v^{(i-1)}(x)| \leq \frac{\eta_m}{4} + ||g_v||_{n-1,[\bar{x},\bar{x}+\delta_m]}$$

where $g_v(x) = \sum_{i=0}^{n-1} v_{i+1} \frac{(x-\bar{x})^i}{i!}$.

Since $(x^*, y_1, \ldots, y_n) \in E$, there exists $m_1 \geq 1$ such that $(x^*, y_1, \ldots, y_n) \in E_{m_1}$. Then for K_1 sufficiently large, if $k \geq K_1$, then $(x_k^*, y_k^1, \ldots, y_k^n) \in E_{m_1}$ and $|x_k^* - x^*| \leq \frac{\delta_{m_1}}{2}$. We shall provide calculations to show $\{y_k^{(i-1)}\}_{k=K_1}^{\infty}$, $i = 1, \ldots, n$, is equicontinuous on $[x^*, x^* + \frac{\delta_{m_1}}{2}]$. Write

$$y_k(x) = g_k(x) + \int_{x_k^*}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y_k(s), \ldots, y_k^{(n-1)}(s)) ds,$$

where $g_k(x) = \sum_{j=0}^{n-1} y_k^{j+1} \frac{(x-x^*)^j}{j!}$. Let M_1 denote a uniform bound on the family $\{|f_k|\}$ on E_{m_1}. Since $x^* - \frac{\delta_{m_1}}{2} \leq x_k^*$, calculations similar to those used to obtain (2.6) and (2.7) give the following. For $x^* \leq x_1 < x_2 \leq x^* + \frac{\delta_{m_1}}{2}$, if $i < n$, then

$$\begin{aligned} |y_k^{(i-1)}(x_2) - y_k^{(i-1)}(x_1)| &\leq |g_k^{(i-1)}(x_2) - g_k^{(i-1)}(x_1)| \\ &+ \frac{M_1}{\Gamma(\alpha+1-(i-1))} \Big(((x_2 - (x^* - \frac{\delta m_1}{2}))^{\alpha-(i-1)} \\ &- (x_1 - (x^* - \frac{\delta m_1}{2}))^{\alpha-(i-1)} \Big) \end{aligned}$$

and if $i = n$,

$$\begin{aligned} |y_k^{(n-1)}(x_2) - y_k^{(n-1)}(x_1)| &\leq |g_k^{(i-1)}(x_2) - g_k^{(i-1)}(x_1)| \\ &+ \frac{2M_1}{\Gamma(\alpha+1-(i-1))} (x_2 - x_1)^{\alpha-(i-1)}. \end{aligned}$$

Repeated applications of the Arzela-Ascoli theorem give the existence of a a subsequence $\{k_1(j)\}_{j=1}^{\infty} \subset \{k\}_{k=1}^{\infty}$ such that the subsequence $\{y_{k_1(j)}\}_{j=1}^{\infty}$ converges in the space $C^{n-1}[x^*, x^* + \frac{\delta_{m_1}}{2}]$. Call the limiting function y_0 and

$$||y_{k_1(j)} - y_0||_{n-1,[x^*,x^*+\frac{\delta_{m_1}}{2}]} \to 0 \text{ as } j \to \infty.$$

Note that y_0 is a solution of the initial value problem

$$D^{\alpha}_{*x^*} y(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x)), \quad a < x^* \le x \le x^* + \frac{\delta m_1}{2}, \tag{2.19}$$

$$y^{(i-1)}(x^*) = y_i, \quad i = 1, \dots, n,$$

since for $x \in [x^*, x^* + \frac{\delta m_1}{2}]$, $y_{k_1(j)}(x) =$

$$\sum_{i=0}^{n-1} y^{i+1}_{k_1(j)} \frac{(x - x^*_k)^i}{i!} + \int_{x^*_{k_l}}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y_{k_1(j)}(s), \dots, (y_{k_1(j)})^{(n-1)}(s))ds$$

$$\to \sum_{k=0}^{n-1} y_{i+1} \frac{(x - x^*)^i}{i!} + \int_{x^*}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y_0(s), \dots, y_0^{(n-1)}(s))ds.$$

In particular,

$$y_0(x) = \sum_{k=0}^{n-1} y_{i+1} \frac{(x - x^*)^i}{i!} + \int_{x^*}^{x} \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y_0(s), \dots, y_0^{(n-1)}(s))ds$$

and y_0 is a solution of the initial value problem (2.19), (2.2) by Theorem 2.10.

For simplicity in exposition, we shall write

$$(x, \bar{y}(x)) = (x, y(x), \dots, y^{(n-1)}(x))$$

through the remainder of this proof. If $(x^* + \frac{\delta m_1}{2}, \bar{y}_0(x^* + \frac{\delta m_1}{2})) \in E_{m_1}$, since $\frac{\delta m_1}{2}$ depends only on $\overline{E_{m_1}}$, the process can be repeated (using a similar construction to obtain (2.16)) to construct a further subsequence $\{y_{k_2(j)}\}_{j=1}^{\infty}$ of $\{y_{k_1(j)}\}_{j=1}^{\infty}$ such that, for each $i = 1, \dots, n$, $\{y^{(i-1)}_{k_2(j)}\}_{j=1}^{\infty}$ converges uniformly on $[x^*, x^* + 2(\frac{\delta m_1}{2})]$. In particular, $\{y_{k_2(j)}\}_{j=1}^{\infty}$ converges to a solution, y, of

$$D^{\alpha}_{*x^*} y(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x)), \quad a < x^* \le x \le x^* + 2\frac{\delta m_1}{2},$$

$$y^{(i-1)}(x^*) = y_i, \quad i = 1, \dots, n,$$

and $y(x) = y_0(x)$, $x^* \le x \le x^* + \frac{\delta m_1}{2}$. So, label the solution y on $[x^*, x^* + 2\frac{\delta m_1}{2}]$ by $y = y_0$.

If $(x^* + 2(\frac{\delta m_1}{2}), \bar{y}_0(x^* + 2(\frac{\delta m_1}{2}))) \in E_{m_1}$ continue the process. Since $\overline{E_{m_1}}$ is compact, there is a first integer $l_1 \ge 1$ such that $(x^* + l_1(\frac{\delta m_1}{2}), \bar{y}_0(x^* + l_1(\frac{\delta m_1}{2}))) \notin E_{m_1}$. Note that we have obtained corresponding subsequences $\{k_l(j)\}_{j=1}^{\infty}$, $l = 1, \dots, l_1$ satisfying $\{k_{l+1}(j)\}_{j=1}^{\infty} \subset \{k_l(j)\}_{j=1}^{\infty}$, $l = 1, \dots, l_1 - 1$ if $l_1 > 1$. Define $x_1^* = x^* + l_1(\frac{\delta m_1}{2})$ and assume that $(x_1^*, \bar{y}_0(x_1^*)) \in E_{m_2}$ where $E_{m_1} \subset E_{m_2} \subset E$. Apply the process that has just been performed on E_{m_1} to E_{m_2}.

Proceed inductively to obtain a strictly increasing sequence of endpoints, $\{x_p^*\}$, and a strictly increasing (with respect to set containment) sequence of sets $\{E_{m_p}\}$ with corresponding subsequences of integers $\{k_{m_p}(j)\}_{j=1}^{\infty}$, satisfying

$$\{k_{m_{p+1}}(j)\}_{j=1}^{\infty} \subset \{k_{m_p}(j)\}_{j=1}^{\infty}$$

such that for each $i = 1, \ldots, n$, $y^{(i-1)}_{k_p(j)}$ converges uniformly to $y^{(i-1)}_0$ on $[x^*, x^*_p]$.

Set $\omega = \sup_{p\geq 1}\{x^*_p\}$. A standard diagonalization process, sometimes referred to as the Cantor Selection Theorem [8, pages 3 and 4], implies that there exists a subsequence $\{y_{k_l}\}$ of $\{y_k\}$ such that for each compact subset $J \subset [x^*, \omega)$, $||y_{k_l} - y_0||_{[n-1,J]} \to 0$ as $l \to \infty$. Finally, $(x^*_p, \bar{y}_0(x^*_p)) \notin \overline{E_{m_p}}$ for each p which implies that $[x^*, \omega)$ is right maximal for y_0. □

We close the article with two corollaries of the Kamke convergence theorem, Theorem 2.5, which give sufficient conditions for the continuous dependence of solutions on initial conditions.

Corollary 2.3. *Assume $E \subset \mathbb{R} \times \mathbb{R}^n$, E open, connected and convex and let $f_k : E \to \mathbb{R}$ denote a sequence of continuous functions that converge uniformly to a function f on every compact subset of E. Assume $(x^*, y_1, \ldots, y_n) \in E$ and for each $k \geq 1$, assume $(x^*_k, y^1_k, \ldots, y^n_k) \in E$. For each $k \geq 1$, consider an initial value problem* (2.17), (2.18) *and let $y_k(x)$ denote the solution of* (2.17), (2.18) *on a right maximal interval I_k. Further, assume x^*_k is an increasing sequence and $x^*_k \uparrow x^{*-}$ and $(x^*_k, y^1_k, \ldots, y^n_k) \to (x^*, y_1, \ldots, y_n)$ as $k \to \infty$. Assume the solution, y of* (2.1), (2.2)*, whose existence is given in Theorem 2.5, is unique with maximal interval of existence $[x^*, \omega)$. Let $[c, d] \subset [x^*, \omega)$. Then there exists K such that if $k \geq K$ then $[c, d] \subset I_k$, and*

$$||y_k(x) - y(x)||_{n-1,[c,d]} \to 0 \textit{ as } k \to \infty.$$

Proof. First assume there does not exist K such that if $k \geq K$ then $[c, d] \subset I_k$. Construct a subsequence $\{y_{k_l}\}$ such that $[c, d] \not\subseteq I_{k_l}$ for each l. Apply the process in the proof of Theorem 2.5 with $\{y_{k_l}(x)\}$ as the original sequence. Then since $[c, d] \subset [x^*, \omega)$ this subsequence has a further subsequence $\{y_{k_{l_j}}\}$ such that

$$||y_{k_{l_j}}(x) - y(x)||_{n-1,[c,d]} \to 0 \text{ as } j \to \infty.$$

This implies $[c, d] \subset I_{k_{l_j}}$ which contradicts $[c, d] \not\subseteq I_{k_l}$ for each l. Thus, K exists.

Second, assume $k \geq K$ and assume for the sake of contradiction that

$$||y_k(x) - y(x)||_{n-1,[c,d]} \nrightarrow 0 \text{ as } k \to \infty.$$

Find $\epsilon > 0$ and a subsequence $\{y_{k_l}\}$ of $\{y_k\}^{\infty}_{k=K}$ such that

$$||y_{k_l}(x) - y(x)||_{n-1,[c,d]} \geq \epsilon$$

for all k_l. Apply the process in the proof of Theorem 2.5 with $\{y_{k_l}(x)\}$ as the original sequence. Then there will exist a further subsequence $\{y_{k_{l_j}}(x)\}$ and a solution $\bar{y}_0$ of of (2.1), (2.2) such that

$$||y_{k_{l_j}}(x) - \bar{y}_0(x)||_{n-1,[c,d]} \to 0 \text{ as } j \to \infty.$$

This violates the uniqueness of the solution, y. □

Corollary 2.4. *Assume $E \subset \mathbb{R} \times \mathbb{R}^n$, E open, connected and convex and let $f : E \to \mathbb{R}$ be continuous. Assume that solutions for initial value problems for* (2.1) *with initial conditions in E are unique. Given any $(x^*, z_1, \ldots, z_n) \in E$, let $y(x; x^*, z)$ denote the solution of the initial value problem* (2.1) *on $[x^*, \omega^*_z)$, its right maximal interval of existence, with initial conditions, $y^{(i-1)}(x^*) = z_i$, $i = 1, \ldots, n$. Then for each $\epsilon > 0$ and each compact $[c, d] \subset [x^*, \omega^*_z)$ there exists $\delta > 0$ such that if $(x^*, v_1, \ldots, v_n) \in E$ and $\max_{i=1,\ldots n} |v_i - z_i| < \delta$ then $[c, d] \subset [x^*, \omega^*_v)$, the* *right maximal interval of existence of the solution* $y(x;x^*,v)$*, and* $||y(x;x^*,v) - y(x;x^*,z)||_{n-1,[c,d]} < \epsilon$.

Proof. Assume there exists $(x^*, z_1, \dots, z_n) \in E$, $[a,b] \subset [x^*, \omega_z^*)$ and $\epsilon > 0$ such that no such $\delta > 0$ exists. Construct a decreasing sequence $\delta_k > 0$ converging to zero such that for each δ_k, $y_k(x;x^*,v_k)$ (the solution of (2.1) satisfying $y_k^{(i-1)}(x^*) = v_i$, $i = 1,\dots,n$ where $(x^*, v_{k1}, \dots, v_{kn}) \in E$ and $\max_{i=1,\dots n} |v_{ki} - z_i| < \delta_k$) is such that

$$\max_{i=1,\dots,n} |y_k^{(i-1)}(x;x^*,v_k) - y^{(i-1)}(x;x^*,z)| \geq \epsilon$$

for some $x \in [c,d]$. Apply the process in the proof of the Kamke theorem to this sequence and employ the uniqueness of the solution y to obtain a contradiction; so the continuous dependence of solutions on initial conditions is proved. □

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