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Faster Numerical Method for the Black-Sholes Equation of American Option

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Abstract

A new numerical algorithm is developed to produce a numerical solution of a boundary value problem for the Black-Scholes partial differential equation on a certain region that includes a free boundary. In this algorithm, an artificial boundary is introduced and a method to find the free boundary is developed.

$$LC^* \equiv -\frac{\partial C^*}{\partial \tau} + S^2\frac{\partial^2 C^*}{\partial S^2} + (r^* - D^*)S\frac{\partial C^*}{\partial S} - r^*C^* = 0,$$
$$0 < S < S_f^*(\tau), \quad 0 < \tau < \tau^*,$$
$$C^*(S,0) = h(S), \quad 0 \le S \le S_f^*(0),$$
$$C^*(S_f^*(\tau),\tau) = h(S_f^*(\tau)), \quad \frac{\partial C^*}{\partial S}(S_f^*(\tau),\tau) = 1, \quad 0 \le \tau \le \tau^*,$$
$$C^*(S,\tau) \to 0 \text{ as } S \to 0.$$

Introduction/Motivation

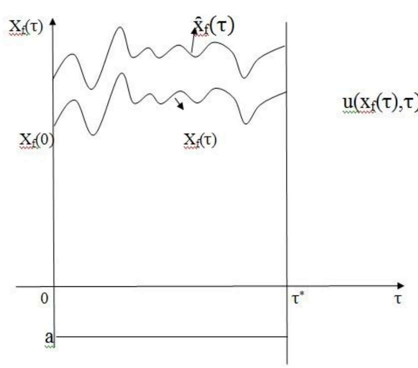
The algorithm is to solve the partial differential equation of the Black-Sholes model. Based on the initial boundary conditions, The method add a artificial boundary to contribute the problem. Using finite different method, the algorithm build up a matrix to calculate the option values. This method gives much more accurate solutions more smaller errors and much less CPU time.

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad a < x < x_f(\tau), \quad 0 < \tau \le \tau^*,$$
$$u(x,0) = g(x,0), \quad a < x < x_f(0),$$
$$u(x_f(\tau),\tau) = g(x_f(\tau),\tau), \quad 0 < \tau \le \tau^*,$$
$$e^{(\alpha-1)x_f(\tau)+\beta\tau}\left[\frac{\partial u(x_f(\tau),\tau)}{\partial x} + \alpha u(x_f(\tau),\tau)\right] = 1, \quad 0 \le \tau \le \tau^*,$$
$$\left.\frac{\partial u}{\partial x}\right|_{x=a} = \frac{1}{\sqrt{\pi}}\int_0^\tau \frac{\partial u(a,\lambda)}{\partial \lambda}\frac{d\lambda}{\sqrt{t-\lambda}}.$$

Algorithm

- Set up a linear system by using Crank-Nicoslon finite difference method.
- Solving the boundary problem by assuming an artificial boundary to find the option values by time.
- To evaluate the value of artificial boundary.
- Theorems give us the statement that in the limited domain u(x, τ) > g(x,τ), and if the free boundary moved above until crossing the boundary, out of the domain, then the value will satisfy u(x, τ) < g(x,τ),

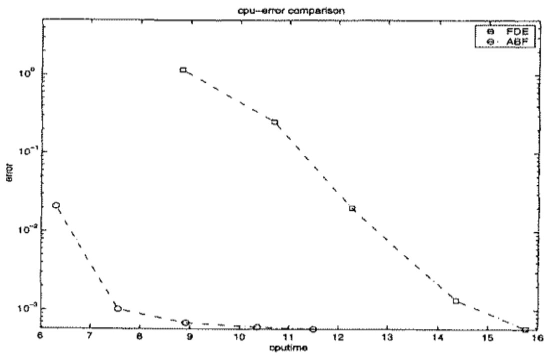
$$g(x,\tau) = e^{-\alpha x - \beta\tau}\max(e^x - 1, 0).$$



$$u(x_f(\tau),\tau) = g(x_f(\tau),\tau)$$

Conclusion

The artificial boundary condition gives the more accurate relations of the solutions at the boundary. Using the finite difference method, we obtain in a small truncated domain in more efficient way.



(H. Han and X.Wu, A Fast Numerical Method for the Method for the Black-Scholes Equation of American Option, SIAM J. Numer. Anal., 41 (2003), pp. 2081-2095.)

Further Work

Applying this method in a regime-switching model and introducing the entries $q_{i,j}$ of the matrix Q is the factor regime-switching situations. The problem will be rewritten as below:

$$\frac{\partial C(S,t,i)}{\partial t} + \frac{1}{2}\sigma(i)^2S^2\frac{\partial^2 C(S,t,i)}{\partial S^2} + (r(i) - D_0)S\frac{\partial C(S,t,i)}{\partial S} + \sum_{j\neq i} q_{ij}[C(S,t,j) - C(S,t,i)] - r(i)C = 0$$