A Synthesis of finite difference methods and the jump process arising in the pricing of Contingent Claim

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A synthesis of finite difference method and jump process

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Abstract

It is demonstrated that approximation of the solution of the Black-Scholes partial differential equation by using a finite difference method is equivalent to approximating the diffusion process by a jump process and therefore the finite difference approximation is a type of numerical integration. In particular, we establish that the explicit finite difference approximation is equivalent to approximating the diffusion process by a jump process, initially introduced by Cox and Ross, while the implicit finite difference approximation amounts to approximating the diffusion process by a more general type of jump process. This work has been introduced by Brennan and Schwartz, The Journal of Financial and Quantitative Analysis, [13] (1978).

Introduction

In mathematics, finite-difference methods are numerical methods for approximating the solutions to differential equations using finite difference equations to approximate derivatives. The Black-Scholes model is a mathematical model of a financial market containing certain derivative investment instruments.

In 1973, Black and Scholes published the paper "The Pricing of Option and Corporate Liabilities" and the Black-Scholes Model was first introduced.

\[
\frac{1}{2} \sigma^2 S^2 H_{ss} + rS H_s + H_t + H = 0
\]

We use the log transform of Black-Scholes equation. We define \( y=\ln S \), \( W(y,t)=H(s,t) \) and get the log transform of Black-Scholes partial differential equation

\[
\frac{1}{2} \sigma^2 W_{yy} + (r - \frac{1}{2} \sigma^2) W_y + W_t - rW = 0
\]

Explicit Finite Difference Method

\[
W(y,t)=W(y,h,0)=W_{i,j}
\]
\[
W_y=(W_{i+1,j} - W_{i-1,j})/2h
\]
\[
W_{yy}=(W_{i+1,j+1} - 2W_{i,j+1} + W_{i-1,j+1})/h^2
\]
\[
W_t=(W_{i,j+1} - W_{i,j})/kh
\]

We make the substitution so that the corresponding difference equation is

\[
W_{ij}(1+rk)=aW_{i-1,j+1}+bW_{ij}+cW_{i+1,j+1}
\]

where

\[
a=[\frac{1}{2} \sigma^2/h^2 - \frac{1}{2}(r - \frac{1}{2} \sigma^2)/h]k
\]
\[
b=1 - \frac{\sigma^2}{h^2}k
\]
\[
c=[\frac{1}{2} \sigma^2/h^2 + \frac{1}{2}(r - \frac{1}{2} \sigma^2)/h]k
\]

If we choose

\[
h \leq \sigma^2/[(r - \frac{1}{2} \sigma^2)^2] \quad \text{and} \quad k \leq \sigma^2/(r - \frac{1}{2} \sigma^2)^2
\]

We can consider \( a, b, c \) are probabilities and the log transform of the stock price follow the jump process.

\[
E[dy] = h[p + p] = (r - \frac{1}{2} \sigma^2)k
\]
\[
V(dy) = \sigma^2 k - (r - \frac{1}{2} \sigma^2)^2 k^2
\]

According to the mean and variance of the jump process. We can write

\[
dy = (r - \frac{1}{2} \sigma^2)dt + \sigma dz
\]

The jump process approximate the diffusion process.

Implicit Finite Difference Method

\[
W(y,t)=W(y,h,0)=W_{i,j}
\]
\[
W_y=(W_{i+1,j} - W_{i-1,j})/2h
\]
\[
W_{yy}=(W_{i+1,j+1} - 2W_{i,j+1} + W_{i-1,j+1})/h^2
\]
\[
W_t=(W_{i,j+1} - W_{i,j})/kh
\]

So that the corresponding equation is

\[
W_{ij+1}(1-rk)=aW_{i-1,j}+bW_{ij}+cW_{i+1,j}
\]

Where

\[
a=[-\frac{1}{2} \sigma^2/h^2 + \frac{1}{2}(r - \frac{1}{2} \sigma^2)/h]k
\]
\[
b=1 + \frac{\sigma^2}{h^2}k
\]
\[
c=[-\frac{1}{2} \sigma^2/h^2 - \frac{1}{2}(r - \frac{1}{2} \sigma^2)/h]k
\]

This system of equations may be written in matrix form as \( AW=f \). We solve the system using Gaussian elimination.

\[
W_{ij} = (1 - rk) \sum_{n=-\infty}^{\infty} P_n \ast W_{i+n,j+1}
\]

\[\approx \frac{1}{1+rk} \sum_{n=-\infty}^{\infty} P_n \ast W_{i+n,j+1}\]

The expected value of the claim at the next instant is obtained by assuming that \( y \), the logarithm of the stock price follows the jump process. We obtain the mean and the variance of \( dy \).

\[
E[dy] = (r - \frac{1}{2} \sigma^2)k
\]
\[
V(dy) = \sigma^2 k - (r - \frac{1}{2} \sigma^2)^2 k^2
\]
\[
dy = (r - \frac{1}{2} \sigma^2)dt + \sigma dz
\]

The jump process approximate the diffusion process.