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Quasilinearization and Boundary Value Problems at Resonance

Kareem Alanazi

Meshal Alshammari

Paul W. Eloe

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Kareem Alanazi, Meshal Alshammari and Paul Eloe*

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Abstract: A quasilinearization algorithm is developed for boundary value problems at resonance. To do so, a standard monotonicity condition is assumed to obtain the uniqueness of solutions for the boundary value problem at resonance. Then the method of upper and lower solutions and the shift method are applied to obtain the existence of solutions. A quasilinearization algorithm is developed and sequences of approximate solutions are constructed, which converge monotonically and quadratically to the unique solution of the boundary value problem at resonance. Two examples are provided in which explicit upper and lower solutions are exhibited.

Keywords: Boundary value problem at resonance, shift method, upper and lower solutions, quasilinearization

MSC 2010: 34B15, 34A45, 47H05

1 Introduction

The method of quasilinearization, introduced by Bellman [4, 5] in the 1960s, offers a numerical method to approximate solutions of nonlinear problems with sequences of solutions of linear problems. Under suitable hypotheses, the sequences of approximate solutions converge monotonically and quadratically.

The method has been particularly useful in the study of boundary value problems for ordinary differential equations and we cite a number of its applications here [1, 2, 9, 10, 12, 16–19, 21]. Although it appears that the interest in the application has waned in recent years, we apply a new application of the method of quasilinearization to a two-point boundary value problem for an ordinary differential equation at resonance. For the problem considered, we first obtain the uniqueness and existence of a solution using well-known methods, and obtain the monotone and quadratic convergence of a sequence of approximate solutions. The application is interesting since the uniqueness of solutions implies the monotone convergence of approximate solutions and the shift argument [11] implies the quadratic rate of convergence.

In Section 2, we first employ the method of upper and lower solutions and, under suitable hypotheses, obtain the uniqueness of solutions of a two-point boundary value problem at resonance for a second order ordinary differential equation. Then we apply the shift argument and obtain the existence of that unique solution. In Section 3, we construct a sequence of upper and lower solutions that converge monotonically to the unique solution, and in Section 4, we employ the shift argument and obtain a quadratic rate of convergence. We close, in Section 5, with two examples in which upper and lower solutions are explicitly exhibited.

***Corresponding author: Paul Eloe,** Department of Mathematics, University of Dayton, Dayton, OH 45469-2316, USA, e-mail: peloe1@udayton.edu. https://orcid.org/0000-0002-6590-9931
Kareem Alanazi, Department of Mathematics, College of Science and Arts, Aljouf University, El-Qurayat, Saudi Arabia, e-mail: sunshine-w@hotmail.com
Meshal Alshammari, Department of Mathematics, Aljouf College of Technology, Al-Jawf, Saudi Arabia, e-mail: mishaaal2009@hotmail.com

The employment of the shift argument is interesting. In this paper, the shift argument produces an invertible boundary value problem that produces a Green's function that is negative and, in particular, the usual maximum principle is valid. The hypotheses and inequalities produced here agree with those found in numerous applications of quasilinearization (see [1, 2, 9, 10, 12, 16–19, 21]). Moreover, it was shown in [3], and is shown again here by two examples, that with the shift argument implies nontrivial solutions to the linear problem at resonance provide excellent candidates as upper or lower solutions.

Recently, Al Mosa and Eloe [3] employed a shift argument So, that upper and lower solution methods coupled with monotone methods could be applied to a boundary value problem at resonance. In that application, the shift argument produced a Green's function of opposite sign, and applications associated with the maximum principle were not valid. Only recently the authors have discovered a method of upper and lower solutions in reverse order which in fact is pertinent to the study in [3]; see, for example, [6–8]. In [6–8], anti-maximum principles are obtained, and delicate iteration schemes are constructed to produce monotone convergence. In these arguments, a sequence of lower solutions converges in a monotonically decreasing fashion to a maximal solution and a sequence of upper solutions converges in a monotonically increasing fashion to a minimal solution. Since maximum principles apply in this article and the uniqueness of solutions is obtained, the methods here are completely different from those found in [3] or [6–8]. In [8], Cherpion et al. study both cases when the anti-maximum principle or the maximum principle applies. In the case when the maximum principle applies, a sequence of lower solutions converge in a monotonically increasing fashion to a minimal solution, and a sequence of upper solutions converge in a monotonically decreasing fashion to a maximal solution. Again, the uniqueness of solutions is a key feature in this study and in [6–8], sequences of iterates are constructed that converge to minimal or maximal solutions.

The application of the method of quasilinearization to boundary value problems at resonance is not new; see [22, 23]. The motivation and development here is different from that in [22] or [23], since the uniqueness of solutions is a key feature in this work, while the multiplicity of solutions is the key in [22] or [23].

2 Existence and uniqueness

Consider the second order boundary value problem for the differential equation

$$y''(t) = f(t, y(t)), \quad 0 \le t \le 1, \tag{2.1}$$

with homogeneous Neumann boundary condition

$$y'(0) = (0), \quad y'(1) = 0, \tag{2.2}$$

where $f\colon [0,1] \times R \to R$ is continuous. The boundary value problem (2.1), (2.2) is at resonance, since constant functions are solutions of the homogeneous problem $y'' = 0$ and satisfy the boundary conditions (2.2).

We begin with the assumption that $f_y > 0$ on $[0,1] \times \mathbb{R}$, and obtain the result on the uniqueness of solutions. The condition $f_y > 0$ is a standard assumption to imply the uniqueness of solutions of two-point boundary value problems for second order ordinary differential equations. In fact, in [13], Kiguradze used the hypothesis and a further hypothesis, $f(t, 0) = 0$, to obtain the uniqueness of solutions for a Neumann boundary value problem.

Theorem 2.1. *Assume $f\colon [0,1] \times R \to R$ is continuous, $\frac{\partial f}{\partial y} = f_y\colon [0,1] \times R \to R$ is continuous and $f_y > 0$ on $[0,1] \times \mathbb{R}$. Then the solutions of the boundary value problem* (2.1), (2.2) *are unique if they exist.*

Proof. Assume for the sake of contradiction that $y_1(t)$ and $y_2(t)$ denote two distinct solutions of the boundary value problem (2.1), (2.2). Assume, without loss of generality, that $y_1 - y_2$ has a positive maximum at $t_0 \in [0,1]$. (If this is not the case, then $y_2 - y_1$ has a positive maximum at some $t_0 \in [0,1]$.)

First, assume $t_0 \in (0,1)$. Then $(y_1 - y_2)''(t_0) \le 0$. However, y_1 and y_2 satisfy (2.1), and

$$(y_1 - y_2)''(t_0) = f(t_0, y_1(t_0)) - f(t_0, y_2(t_0)) > 0,$$

since f is increasing at y. Thus, $y_1 - y_2$ does not have a positive maximum at $t_0 \in (0,1)$.

Second, assume $t_0 = 0$ and recall $y_1'(0) = y_2'(0) = 0$. By Taylor's expansion, there exists $c \in (0, t)$ such that

$$\begin{aligned}(y_1 - y_2)(t) &= (y_1 - y_2)(0) + (y_1 - y_2)'(0)t + (y_1 - y_2)''(c)\frac{t^2}{2!}\\ &= (y_1 - y_2)(0) + (f(c, y_1(c)) - f(c, y_2(c)))\frac{t^2}{2!}\\ &> (y_1 - y_2)(0)\end{aligned}$$

for $0 < t$ sufficiently small. Thus, $y_1 - y_2$ does not have a positive maximum at $t_0 = 0$.

Third, assume $t_0 = 1$ and recall $y_1'(1) = y_2'(1) = 0$. By Taylor's expansion, there exists $c \in (t, 1)$ such that

$$\begin{aligned}(y_1 - y_2)(t) &= (y_1 - y_2)(1) + (y_1 - y_2)'(1)(t-1) + (y_1 - y_2)''(c)\frac{(t-1)^2}{2!}\\ &= (y_1 - y_2)(1) + (f(c, y_1(c)) - f(c, y_2(c)))\frac{(t-1)^2}{2!}\\ &> (y_1 - y_2)(1)\end{aligned}$$

for $0 < 1 - t$ sufficiently small. Thus, $y_1 - y_2$ does not have a positive maximum at $t_0 = 1$. □

Definition 2.2. We say $\alpha \in C^2[0, 1]$ is a lower solution of the BVP (2.1), (2.2) if $\alpha'(0) = 0$ and $\alpha'(1) = 0$, and

$$\alpha''(t) \geq f(t, \alpha(t)), \quad 0 \leq t \leq 1.$$

We say $\beta \in C^2[0, 1]$ is an upper solution of the BVP (2.1), (2.2) if $\beta'(0) = 0$ and $\beta'(1) = 0$ and

$$\beta''(t) \leq f(t, \beta(t)), \quad 0 \leq t \leq 1.$$

Theorem 2.3. *Assume $f: [0, 1] \times R \to R$ is continuous, $\frac{\partial f}{\partial y} = f_y: [0, 1] \times R \to R$ is continuous and $f_y > 0$ on $[0, 1] \times \mathbb{R}$. Assume α is a lower solution of the boundary value problem* (2.1), (2.2) *and β is an upper solution of the boundary value problem* (2.1), (2.2). *Then*

$$\alpha(t) \leq \beta(t), \quad 0 \leq t \leq 1.$$

Proof. The proof of this theorem is very similar to the proof of the uniqueness theorem, Theorem 2.1. Assume α is a lower solution and β is an upper solution of the BVP (2.1), (2.2). Assume for the sake of contradiction that $\alpha(t) \leq \beta(t)$ is false. Assume without loss of generality that $\alpha - \beta$ has a positive maximum at $t_0 \in [0, 1]$.

First, assume $t_0 \in (0, 1)$. Then $(\alpha - \beta)''(t_0) \leq 0$. However, α and β are, respectively, lower and upper solutions of (2.1), (2.2) and

$$(\alpha - \beta)''(t_0) \geq f(t_0, \alpha(t_0)) - f(t_0, \beta(t_0)) > 0,$$

since f is increasing in y. Thus, $\alpha - \beta$ does not have a positive maximum at $t_0 \in (0, 1)$.

Second, assume $t_0 = 0$ and recall $\alpha'(0) = \beta'(0) = 0$. By Taylor's expansion, there exists $c \in (0, t)$ such that

$$\begin{aligned}(\alpha - \beta)(t) &= (\alpha - \beta)(0) + (\alpha - \beta)'(0)t + (\alpha - \beta)''(c)\frac{t^2}{2!}\\ &\geq (\alpha - \beta)(0) + (f(c, \alpha(c)) - f(c, \beta(c)))\frac{t^2}{2!}\\ &> (\alpha - \beta)(0)\end{aligned}$$

for $0 < t$ sufficiently small. Thus, $\alpha - \beta$ does not have a positive maximum at $t_0 = 0$.

Third, assume $t_0 = 1$ and recall $\alpha'(1) = \beta'(1) = 0$. As in the proof of Theorem 2.1, Taylor's expansion produces a contradiction. □

Remark 2.4. Theorem 2.1 is an immediate corollary of Theorem 2.3, since the solution is both an upper and a lower solution.

To obtain the existence of solutions, we shall employ the method of upper and lower solutions and apply the shift argument [11]. In [14], Kiguradze and Lezhava have obtained the existence of solutions for a Neumann boundary value problem using only the method of upper and lower solutions, coupled with sequential compactness arguments. As later we will need the estimates related to a Green's function, we introduce the shift method here and employ the Schauder fixed point theorem using an argument modeled after the work [20] or [15].

Assume $K \neq 0$ and consider the shifted equation

$$y''(t) - K^2 y(t) = g(t, y(t)) = f(t, y(t)) - K^2 y(t), \quad 0 \le t \le 1. \tag{2.3}$$

The boundary value problem (2.1), (2.3) is not at resonance for any $K \neq 0$, and so we construct the corresponding Green's function in the form

$$G_K(t,s) = \frac{-1}{K\sinh(K)} \begin{cases} \cosh(K(s-1))\cosh(Kt), & 0 \le t \le s \le 1, \\ \cosh(Ks)\cosh(K(t-1)), & 0 \le s \le t \le 1. \end{cases} \tag{2.4}$$

Theorem 2.5. *Assume $f\colon [0,1] \times R \to R$ is continuous, $\frac{\partial f}{\partial y} = f_y\colon [0,1] \times R \to R$ is continuous and $f_y > 0$ on $[0,1] \times \mathbb{R}$. Assume α is a lower solution of the boundary value problem (2.1), (2.2) and β is an upper solution of the boundary value problem (2.1), (2.2). Then there exists a unique solution y of (2.1), (2.2) satisfying*

$$\alpha(t) \le y(t) \le \beta(t), \quad 0 \le t \le 1.$$

Proof. Let $K \neq 0$ and define the truncation of $g(t,y) = f(t,y) - K^2 y$ by

$$F(t, y(t)) = \begin{cases} f(t, \beta(t)) - K^2\beta(t) + \frac{y(t)-\beta(t)}{1+y(t)-\beta(t)}, & y(t) > \beta(t), \\ f(t, y(t)) - K^2 y(t), & \alpha(t) \le y(t) \le \beta(t), \\ f(t, \alpha(t)) - K^2\alpha(t) + \frac{y(t)-\alpha(t)}{1+\alpha(t)-y(t)}, & y(t) < \alpha(t). \end{cases}$$

Define an operator $T\colon C[0,1] \to C[0,1]$ by

$$Ty(t) = \int_0^1 G_K(t,s)F(s, y(s))\,ds,$$

where $G_K(t,s)$ is given by (2.4). Then $y \in C^2[0,1]$ is a solution of the boundary value problem

$$y''(t) - K^2 y(t) = F(t, y(t)), \quad 0 \le t \le 1, \tag{2.5}$$

with boundary conditions (2.2) if and only if $y \in C[0,1]$ and

$$y(t) = \int_0^1 G_K(t,s)F(s, y(s))\,ds, \quad 0 \le t \le 1.$$

Note that the truncation, F, is bounded and continuous on $[0,1] \times \mathbb{R}$. So, it is a straightforward application of the Schauder fixed point theorem to show that the boundary value problem (2.5), (2.2) has a solution. To see this, let

$$M = \sup\{|F(t,y)| : 0 \le t \le 1,\ y \in \mathbb{R}\}$$

and

$$G = \max_{0 \le t \le 1} \int_0^1 |G_K(t,s)|\,ds.$$

Then, if $y \in C[0,1]$,

$$\|Ty\| = \max_{0 \le t \le 1} \left| \int_0^1 G_K(t,s)F(s, y(s))\,ds \right| \le MG,$$

where $\|\cdot\|$ denotes the supremum norm on $[0,1]$.

Define

$$\mathcal{U} = \{y \in C[0,1] : \|y\| \leq MG\}.$$

Then $\mathcal{U}$ is a closed convex subset of $C[0,1]$ and $T\colon \mathcal{U} \to \mathcal{U}$. It can be shown that T on $C[0,1]$ is a completely continuous map, and so the Schauder fixed point theorem implies that there exists a fixed point $y \in \mathcal{U}$ of the operator T.

Let y denote a fixed point of T, and so y satisfies the boundary value problem

$$y''(t) - K^2 y(t) = F(t, y(t)), \quad 0 \leq t \leq 1, \qquad y'(0) = (0), \quad y'(1) = 0.$$

Now by showing that

$$\alpha(t) \leq y(t) \leq \beta(t), \quad 0 \leq t \leq 1,$$

it will follow that the fixed point y is a solution of the original boundary value problem (2.1), (2.2). We show $y(t) \leq \beta(t)$, $0 \leq t \leq 1$. Assume without loss of generality that $y - \beta$ has a positive maximum at $t_0 \in [0,1]$.

First, assume, $t_0 \in (0,1)$. Then $(y-\beta)''(t_0) \leq 0$. Since

$$y''(t_0) - K^2 y(t_0) = f(t, \beta(t)) - K^2\beta(t) + \frac{y(t) - \beta(t)}{1 + y(t) - \beta(t)},$$

and β is an upper solution of (2.1), (2.2), it follows that

$$\begin{aligned}(y-\beta)''(t_0) &\geq f(t_0, \beta(t_0)) + K^2(y(t_0) - \beta(t_0)) + \frac{y(t_0) - \beta(t_0)}{1 + y(t_0) - \beta(t_0)} - f(t_0, \beta(t_0)) \\ &= K^2(y(t_0) - \beta(t_0)) + \frac{y(t_0) - \beta(t_0)}{1 + y(t_0) - \beta(t_0)} > 0.\end{aligned}$$

Thus, $y - \beta$ does not have a positive maximum at $t_0 \in (0,1)$.

Second, assume $t_0 = 0$ and recall $y'(0) = \beta'(0) = 0$. By Taylor's expansion, there exists $c \in (0, t)$ such that

$$\begin{aligned}(y-\beta)(t) &= (y-\beta)(0) + (y-\beta)'(0)t + (y-\beta)''(c)\frac{t^2}{2!} \\ &= (y-\beta)(0) + (y-\beta)''(c)\frac{t^2}{2!} \\ &\geq (y-\beta)(0) + \left(f(c, \beta(c)) + K^2(y(c) - \beta(c)) + \frac{y(c) - \beta(c)}{1 + y(c) - \beta(c)} - f(c, \beta(c))\right)\frac{t^2}{2!} \\ &= (y-\beta)(0) + \left(K^2(y(c) - \beta(c)) + \frac{y(c) - \beta(c)}{1 + y(c) - \beta(c)}\right)\frac{t^2}{2!} \\ &> (y-\beta)(0)\end{aligned}$$

for $0 < t$ sufficiently small. Thus, $y - \beta$ does not have a positive maximum at $t_0 = 0$.

Similarly, $y - \beta$ does not have a positive maximum at $t_0 = 1$, and so

$$y(t) \leq \beta(t), \quad 0 \leq t \leq 1.$$

The argument for

$$\alpha(t) \leq y(t), \quad 0 \leq t \leq 1,$$

is completely analogous and the proof of existence is complete.

Since the hypotheses of Theorem 2.1 are assumed, the proof of existence and uniqueness is complete. □

3 The monotone method

In this section we add one more assumption on f to develop a monotone method. In the final section, we will add one further assumption on f, that f_{yy} exists and $f_{yy} \geq 0$, and show the monotone method will converge quadratically.

Theorem 3.1. *Assume $f : [0, 1] \times R \to R$ is continuous, $\frac{\partial f}{\partial y} = f_y : [0, 1] \times R \to R$ is continuous and $f_y > 0$ on $[0, 1] \times \mathbb{R}$. Assume in addition that f_{yy} exists and $f_{yy} \geq 0$. Assume α_0 is a lower solution of the boundary value problem (2.1), (2.2) and β_0 is an upper solution of the boundary value problem (2.1), (2.2). Then there exists a unique solution y of (2.1), (2.2) satisfying*

$$\alpha(t) \leq y(t) \leq \beta(t), \quad 0 \leq t \leq 1.$$

Moreover, there exist sequences $\{\alpha_n\}$ and $\{\beta_n\}$ of lower and upper solutions, respectively, of the boundary value problem (2.1), (2.2), each of which converges in $C[0, 1]$ to the unique solution y and satisfy

$$\alpha_n(t) \leq \alpha_{n+1}(t) \leq y(t) \leq \beta_{n+1}(t) \leq \beta_n(t), \quad 0 \leq t \leq 1.$$

Proof. Let α_0 and β_0 denote a lower and an upper solution of (2.1), (2.2), respectively. So, under the assumption that $f_y(t, y) > 0$ on $[0, 1] \times \mathbb{R}$,

$$\alpha_0(t) \leq \beta_0(t), \quad 0 \leq t \leq 1.$$

Define the function $h(t; \alpha_0, \beta_0)$ on $[0, 1]$ by

$$h(\alpha_0, \beta_0; t, y) = f(t, \alpha_0(t)) + f_y(t, \beta_0(t))(y - \alpha_0)(t)$$

and consider the BVP for the linear nonhomogeneous ordinary differential equation

$$y''(t) = h(\alpha_0, \beta_0; t, y(t)), \quad 0 \leq t \leq 1, \qquad y'(0) = 0, \quad y'(1) = 0. \tag{3.1}$$

Note that

$$h(\alpha_0, \beta_0; t, \alpha_0(t)) = f(t, \alpha_0(t)), \quad 0 \leq t \leq 1,$$

and so,

$$\alpha_0''(t) \geq f(t, \alpha_0(t)) = h(\alpha_0, \beta_0; t, \alpha_0(t)), \quad 0 \leq t \leq 1.$$

Moreover, since

$$f(t, \beta_0(t)) = f(t, \alpha_0(t)) + f_y(t, c(t))(\beta_0 - \alpha_0)(t)$$

for some $\alpha_0(t) \leq c(t) \leq \beta_0(t)$ and f_y is increasing in y for each $t \in [0, 1]$, we get

$$f(t, \alpha_0(t)) + f_y(t, c(t))(\beta_0 - \alpha_0)(t) \leq f(t, \alpha_0(t)) + f_y(t, \beta_0(t))(\beta_0 - \alpha_0)(t) = h(\alpha_0, \beta_0; t, \beta_0(t)), \quad 0 \leq t \leq 1.$$

Thus,

$$h(\alpha_0, \beta_0; t, \beta_0(t)) \geq f(t, \beta_0(t)) \geq \beta_0''(t), \quad 0 \leq t \leq 1.$$

In particular, α_0 and β_0 denote, respectively, a lower and an upper solution of (3.1), as well. Since h satisfies the hypotheses of Theorem 2.5, there exists a solution $\alpha_1(t)$ of (3.1) satisfying

$$\alpha_0(t) \leq \alpha_1(t) \leq \beta_0(t), \quad 0 \leq t \leq 1.$$

Note that there exists $\alpha_0(t) \leq c(t) \leq \alpha_1(t) \leq \beta_0(t)$ such that

$$f(t, \alpha_1(t)) - f(t, \alpha_0(t)) = f_y(t, c(t))(\alpha_1(t) - \alpha_0(t)) \leq f_y(t, \beta_0(t))(\alpha_1(t) - \alpha_0(t)),$$

and so

$$\alpha_1''(t) = h(\alpha_0, \beta_0; t, \alpha_1(t)) \geq f(t, \alpha_1(t)), \quad 0 \leq t \leq 1.$$

In particular, α_1 is a lower solution of (2.1), (2.2).

Now define the function $k(\beta_0; t, y)$ on $[0, 1]$ by

$$k(\beta_0; t, y) = f(t, \beta_0(t)) + f_y(t, \beta_0(t))(y - \beta_0)(t)$$

and consider the boundary value problem for the linear nonhomogeneous ordinary differential equation

$$y''(t) = k(\beta_0; t, y(t)), \quad 0 \leq t \leq 1, \qquad y'(0) = 0, \quad y'(1) = 0. \tag{3.2}$$

Note that

$$k(\beta_0; t, \beta_0(t)) = f(t, \beta_0(t)), \quad 0 \le t \le 1,$$

and

$$\beta_0''(t) \le f(t, \beta_0(t)) = k(\beta_0; t, \beta_0(t)), \quad 0 \le t \le 1.$$

Thus, β_0 is an upper solution of (3.2). Note that there exists $\alpha_0(t) \le c(t) \le \beta_0(t)$ such that

$$\begin{aligned} \alpha_0''(t) \ge f(t, \alpha_0(t)) &= f(t, \beta_0(t)) + f_y(t, c(t))(\alpha_0(t) - \beta_0(t)) \\ &\ge f(t, \beta_0(t)) + f_y(t, \beta_0(t))(\alpha_0(t) - \beta_0(t)) \\ &= k(\beta_0; t, \alpha_0(t)), \quad 0 \le t \le 1, \end{aligned}$$

and so α_0 is a lower solution of (3.2). Since k satisfies the hypotheses of Theorem 2.5, there exists a solution, $\beta_1(t)$, of (3.2) satisfying

$$\alpha_0(t) \le \beta_1(t) \le \beta_0(t), \quad 0 \le t \le 1.$$

An application of the mean value theorem again will give

$$k(\beta_0; t, \beta_1(t)) \le f(t, \beta_1(t)), \quad 0 \le t \le 1.$$

To see this, for some $\beta_1(t) \le c(t) \le \beta_0(t)$,

$$f(t, \beta_1(t)) = f(t, \beta_0(t)) + f_y(t, c(t))(\beta_1(t) - \beta_0(t)) \ge f(t, \beta_0(t)) + f_y(t, \beta_0(t))(\beta_1(t) - \beta_0(t)).$$

Thus,

$$\beta_1''(t) = k(\beta_0; t, \beta_1(t)) \le f(t, \beta_1(t)), \quad 0 \le t \le 1,$$

and β_1 is an upper solution of (2.1), (2.2).

Finally, apply Theorem 2.3 to obtain

$$\alpha_1(t) \le \beta_1(t), \quad 0 \le t \le 1;$$

in particular,

$$\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t), \quad 0 \le t \le 1.$$

Applying Theorem 2.5 with lower and upper solutions α_1 and β_1, respectively, and keeping in mind that the solution y obtained in Theorem 2.5 is unique, we have

$$\alpha_0(t) \le \alpha_1(t) \le y(t) \le \beta_1(t) \le \beta_0(t), \quad 0 \le t \le 1,$$

where $y(t)$ is the unique solution of the boundary value problem (2.1), (2.2).

Assume the sequences $\{\alpha_k\}_{k=1}^n$ and $\{\beta_k\}_{k=1}^n$ have been constructed inductively so that for each k,

$$\begin{aligned} h(\alpha_k, \beta_k; t, y) &= f(t, \alpha_k(t)) + f_y(t, \beta_k(t))(y - \alpha_k)(t), \\ k(\beta_k; t, y) &= f(t, \beta_k(t)) + f_y(t, \beta_k(t))(y - \beta_k)(t), \end{aligned}$$

α_k is the solution of the BVP

$$y''(t) = h(\alpha_{k-1}, \beta_{k-1}; t, y(t)), \quad 0 \le t \le 1, \qquad y'(0) = 0, \quad y'(1) = 0,$$

β_k is the solution of the BVP

$$y''(t) = k(\beta_{k-1}; t, y(t)), \quad 0 \le t \le 1, \qquad y'(0) = 0, \quad y'(1) = 0,$$

and

$$\alpha_{k-1}(t) \le \alpha_k(t) \le y(t) \le \beta_k(t) \le \beta_{k-1}(t), \quad 0 \le t \le 1,$$

$k = 0, \dots, n$, where α_k and β_k, $k = 1, \dots, n$, denote a lower solution and an upper solution, respectively, of (2.1), (2.2), and y is the unique solution of the boundary value problem (2.1), (2.2).

To finish the induction argument, consider the boundary value problem for the linear nonhomogeneous ordinary differential equation

$$y''(t) = h(\alpha_n, \beta_n; t, y(t)), \quad 0 \le t \le 1, \qquad y'(0) = 0, \quad y'(1) = 0. \tag{3.3}$$

Note that

$$h(\alpha_n, \beta_n; t, \alpha_n(t)) = f(t, \alpha_n(t)), \quad 0 \le t \le 1,$$

and

$$h(\alpha_n, \beta_n; t, \beta_n(t)) \ge f(t, \beta_n(t)), \quad 0 \le t \le 1.$$

So, α_n and β_n denote a lower and an upper solution of (3.3), respectively, as well. The function h satisfies the hypotheses of Theorem 2.5, and so there exists a solution $\alpha_{n+1}(t)$ of (3.3) satisfying

$$\alpha_n(t) \le \alpha_{n+1}(t) \le \beta_n(t), \quad 0 \le t \le 1.$$

Moreover,

$$\alpha''_{n+1}(t) = h(\alpha_n, \beta_n; t, \alpha_{n+1}(t)) \ge f(t, \alpha_{n+1}(t)), \quad 0 \le t \le 1,$$

and α_{n+1} is a lower solution of (2.1), (2.2).

Consider the boundary value problem for the linear nonhomogeneous ordinary differential equation:

$$y''(t) = k(\beta_n; t, y(t)), \quad 0 \le t \le 1, \qquad y'(0) = 0, \quad y'(1) = 0. \tag{3.4}$$

Note that

$$k(\beta_n; t, \alpha_n(t)) \ge f(t, \alpha_n(t)), \quad 0 \le t \le 1,$$

and

$$k(\beta_n; t, \beta_n(t)) = f(t, \beta_n(t)), \quad 0 \le t \le 1.$$

So, α_n and β_n denote a lower and an upper solution of (3.4), respectively, as well. The function k satisfies the hypotheses of Theorem 2.5, and so there exists a solution, $\beta_{n+1}(t)$, of (3.4) satisfying

$$\alpha_n(t) \le \beta_{n+1}(t) \le \beta_n(t), \quad 0 \le t \le 1.$$

Moreover,

$$\beta''_{n+1}(t) = k(\beta_n; t, \beta_{n+1}(t)) \le f(t, \beta_{n+1}(t)), \quad 0 \le t \le 1,$$

and β_{n+1} is an upper solution of (2.1), (2.2).

Finally, apply Theorem 2.3 to obtain

$$\alpha_n(t) \le \alpha_{n+1}(t) \le \beta_{n+1}(t) \le \beta_n(t), \quad 0 \le t \le 1,$$

and Theorem 2.5 to obtain

$$\alpha_n(t) \le \alpha_{n+1}(t) \le \beta_{n+1}(t) \le \beta_n(t), \quad 0 \le t \le 1.$$

To complete the proof, $\{\alpha_n\}$ and $\{\beta_n\}$ are monotone sequences of continuous functions bounded above or below, respectively, on a compact domain. So, by Dini's theorem, each converges uniformly to $\alpha(t)$ and $\beta(t)$, respectively, on $[0, 1]$. Thus,

$$k(\beta_n; t, \beta_{n+1}) = f(t, \beta_n(t)) + f_y(t, \beta_0(t))(\beta_{n+1} - \beta_n)(t) \to f(t, \beta)$$

as $n \to \infty$. So, β is the unique solution of (2.1), (2.2). Moreover,

$$h(\alpha_n, \beta_n; t, \alpha_{n+1}) = f(t, \alpha_n(t)) + f_y(t, \beta_n(t))(\alpha_{n+1} - \alpha_n)(t) \to f(t, \alpha)$$

as $n \to \infty$, and so α is also the unique solution of (2.1), (2.2). □

4 Quadratic convergence

We now provide an estimate on the error bound. For each n, define the error e_n as follows:

$$e_n(t) = \beta_n(t) - \alpha_n(t), \quad 0 \le t \le 1.$$

So, $0 \le e_n(t)$ for $0 \le t \le 1$. Denote by $\|e_n\|$ the error bound

$$\|e_n\| = \max_{0\le t\le 1} |e_n(t)|.$$

In order to show quadratic convergence, we assume the hypotheses of Theorem 3.1, employ the iteration scheme of Section 3.1 and include one final additional assumption that

$$f_y(t,y) \ge K^2, \quad \alpha_0(t) \le y \le \beta_0(t), \quad 0 \le t \le 1.$$

Theorem 4.1. *Assume $f: [0,1] \times R \to R$ is continuous, $\frac{\partial f}{\partial y} = f_y: [0,1] \times R \to R$ is continuous and $f_y > 0$ on $[0,1] \times \mathbb{R}$. Assume in addition that $f_{yy} \ge 0$. Assume α_0 is a lower solution of the boundary value problem* (2.1), (2.2) *and β_0 is an upper solution of the boundary value problem* (2.1), (2.2). *Assume further that*

$$f_y(t,y) \ge K^2, \quad \alpha_0(t) \le y \le \beta_0(t), \quad 0 \le t \le 1.$$

Then there exist sequences $\{\alpha_n\}$ and $\{\beta_n\}$, lower and upper solutions, respectively, of the boundary value problem (2.1), (2.2), *each of which converges quadratically in $C[0,1]$ to the unique solution y and satisfy*

$$\alpha_n(t) \le \alpha_{n+1}(t) \le y(t) \le \beta_{n+1}(t) \le \beta_n(t), \quad 0 \le t \le 1.$$

Proof. Employ the construction in the proof of Theorem 3.1 and recall

$$\alpha''_{n+1}(t) = h(\alpha_n, \beta_n; t, \alpha_{n+1}(t)) = f(t, \alpha_n(t)) + f_y(t, \beta_n(t))(\alpha_{n+1}(t) - \alpha_n(t))$$
$$\beta''_{n+1}(t) = k(\beta_n; t, \beta_{n+1}(t)) = f(t, \beta_n(t)) + f_y(t, \beta_n(t))(\beta_{n+1}(t) - \beta_n(t)).$$

Then

$$\begin{aligned} e''_n(t) = \beta''_{n+1}(t) - \alpha''_{n+1}(t) &= [f(t,\beta_n(t)) - f(t,\alpha_n(t))] + f_y(t,\beta_n(t))[\beta_{n+1}(t) - \beta_n(t) - \alpha_{n+1}(t) + \alpha_n(t)] \\ &= [f(t,\beta_n(t)) - f(t,\alpha_n(t))] + f_y(t,\beta_n(t))[e_{n+1}(t) - e_n(t)]. \end{aligned}$$

By the mean value theorem, there exists $\alpha_n(t) < c_n(t) < \beta_n(t)$ such that

$$f(t,\beta_n(t)) - f(t,\alpha_n(t)) = f_y(t, c_n(t))e_n(t).$$

Thus,

$$\begin{aligned} e''_{n+1}(t) &= f_y(t,c_n(t))e_n(t) + f_y(t,\beta_n(t))e_{n+1}(t) - f_y(t,\beta_n(t))e_n(t) \\ &= f_y(t,\beta_n(t))e_{n+1}(t) + [f_y(t,c_n(t)) - f_y(t,\beta_n(t))]e_n(t). \end{aligned}$$

Using the mean value theorem again for $f_y(t,c_n(t)) - f_y(t,\beta_n(t))$, there exists

$$c_n(t) < \hat{c_n}(t) < \beta_n(t)$$

such that

$$f_y(t,c_n(t)) - f_y(t,\beta_n(t)) = f_{yy}(t,\hat{c_n}(t))(c_n(t) - \beta_n(t)).$$

Then

$$e''_{n+1}(t) = f_y(t,\beta_n(t))e_{n+1}(t) + f_{yy}(t,\hat{c_n}(t))(c_n(t) - \beta_n(t))e_n(t).$$

Apply the shift argument, assume $K \ne 0$ and

$$e''_{n+1}(t) - K^2 e_{n+1}(t) = (f_y(t,\beta_n(t) - K^2)e_{n+1}(t) + f_{yy}(t,\hat{c_n}(t))(c_n(t) - \beta_n(t))e_n(t).$$

Note that e_{n+1} satisfies the boundary conditions (2.2) and employ the Green function (2.4) and

$$\begin{aligned} 0 \le e_{n+1}(t) &= \int_0^1 G_K(t,s)(f_y(s,\beta_n(s)) - K^2)e_{n+1}(s) + \int_0^1 G_K(t,s) f_{yy}(s,\hat{c_n}(s))(c_n(s)-\beta_n(s))e_n(s)\,ds \\ &\le \int_0^1 |G_K(t,s)| f_{yy}(s,\hat{c_n}(s))(\beta_n(s)-c_n(s))e_n(s)\,ds \\ &\le \int_0^1 |G_K(t,s)| f_{yy}(s,\hat{c_n}(s))e_n^2(s)\,ds, \end{aligned} \tag{4.1}$$

since $G_K(t,s) < 0$ on $(0,1)\times(0,1)$ and $f_y(t,\beta_n(t)) - K^2 \ge 0$ for $0 \le t \le 1$ implies

$$G_K(t,s)(f_y(s,\beta_n(s)) - K^2)e_{n+1}(s)) \le 0, \quad 0 \le s \le 1.$$

Let

$$M = \max\{|f_{yy}(t,y)|,\ \alpha_0(t) \le y \le \beta_0(t),\ 0 \le t \le 1\}$$

and

$$G = \max_{0\le t\le 1} \int_0^1 |G_K(t,s)|\,ds.$$

Then, from (4.1),

$$|e_{n+1}(t)| \le \int_0^1 |G_K(t,s)| f_{yy}(s,\hat{c_n}(s))e_n^2(s)\,ds \le M \int_0^1 |G_K(t,s)|\,ds \|e_n\|^2 \le MG\|e_n\|^2,$$

and the rate of convergence is quadratic. □

5 Two examples

The method of upper and lower solutions is only as good as one's ability to exhibit the existence of upper and lower solutions. Unfortunately, we do not have a general algorithm to do this; however, it was shown in [3] that constant solutions, the nontrivial solutions of the homogeneous boundary value problem at resonance provide excellent candidates as upper or lower solutions.

Theorem 5.1. *Assume $f\colon [0,1]\times R \to R$ is continuous, $\frac{\partial f}{\partial y} = f_y\colon [0,1]\times R \to R$ is continuous and $f_y > 0$ on $[0,1]\times \mathbb{R}$. Assume in addition that $f_{yy} \ge 0$. Assume α_0 is a lower solution of the boundary value problem* (2.1), (2.2) *and β_0 is an upper solution of the boundary value problem* (2.1), (2.2). *Assume further that*

$$f_y(t,y) \ge K^2$$

if $\alpha_0(t) \le y \le \beta_0(t)$, $0 \le t \le 1$. Also assume there exist $\sigma \in C[0,1]$ and a nondecreasing function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(t,y) - K^2 y| \leqslant \sigma(t)\psi(|y|), \quad (t,y) \in [0,1]\times\mathbb{R},$$

and there exists $M > 0$ such that

$$\frac{K^2 M}{\|\sigma\|_0 \psi(M)} > 1.$$

Then there exist sequences $\{\alpha_n\}$ and $\{\beta_n\}$ of lower and upper solutions, respectively, of the boundary value problem (2.1), (2.2), *each of which converges quadratically in $C[0,1]$ to the unique solution y and satisfies*

$$\alpha_n(t) \le \alpha_{n+1}(t) \le y(t) \le \beta_{n+1}(t) \le \beta_n(t), \quad 0 \le t \le 1.$$

Proof. To exhibit β_0, an upper solution, set

$$\beta_0 = M.$$

Then

$$\beta_0''(t) - K^2\beta_0 = -K^2 M \leq -\|\sigma\|_0 \psi(M) \leq f(t, \beta_0) - K^2\beta_0.$$

To exhibit α_0, a lower solution, set

$$w_0 = -M. \qquad \square$$

In a similar way, if the growth condition $|f(t,y) - K^2 y| \leqslant \sigma(t)\psi(|y|)$ is replaced by a boundedness condition, there exists $M > 0$ such that

$$|f(t,y) - K^2 y| \leqslant M, \quad (t,y) \in [0,1] \times \mathbb{R},$$

then upper and lower solutions are readily exhibited. Set $\beta_0 = \frac{M}{K^2}$ and $\alpha_0 = -\beta_0$.

Theorem 5.2. *Assume $f\colon [0,1] \times R \to R$ is continuous, $\frac{\partial f}{\partial y} = f_y\colon [0,1] \times R \to R$ is continuous and $f_y > 0$ on $[0,1] \times \mathbb{R}$. Assume in addition that $f_{yy} \geq 0$. Assume α_0 is a lower solution of the boundary value problem* (2.1), (2.2) *and β_0 is an upper solution of the boundary value problem* (2.1), (2.2). *Assume further that*

$$f_y(t,y) \geq K^2$$

if $\alpha_0(t) \leq y \leq \beta_0(t)$, $0 \leq t \leq 1$. In addition, there exists $M > 0$ such that

$$|f(t,y) - K^2 y| \leqslant M, \quad (t,y) \in [0,1] \times \mathbb{R}.$$

Then there exist sequences $\{\alpha_n\}$ and $\{\beta_n\}$ of lower and upper solutions, respectively, of the boundary value problem (2.1), (2.2), *each of which converges quadratically in $C[0,1]$ to the unique solution y and satisfies*

$$\alpha_n(t) \leq \alpha_{n+1}(t) \leq y(t) \leq \beta_{n+1}(t) \leq \beta_n(t), \quad 0 \leq t \leq 1.$$

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