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QUASILINEARIZATION APPLIED TO BOUNDARY VALUE PROBLEMS AT RESONANCE FOR RIEMANN-LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS

ABSTRACT. The quasilinearization method is applied to a boundary value problem at resonance for a Riemann-Liouville fractional differential equation. Under suitable hypotheses, the method of upper and lower solutions is employed to establish uniqueness of solutions. A shift method, coupled with the method of upper and lower solutions, is applied to establish existence of solutions. The quasilinearization algorithm is then applied to obtain sequences of lower and upper solutions that converge monotonically and quadratically to the unique solution of the boundary value problem at resonance.

1. Introduction. The method of quasilinearization was introduced by Bellman and Kalba [7, 8]. The method, as constructed here, is rather remarkable as both existence and uniqueness of solutions is established and a bilateral monotone iteration scheme is produced to approximate solutions of nonlinear problems with solutions of linear problems. Under suitable hypotheses, the sequences of approximate solutions converge quadratically to the unique solution.

Applications of quasilinearization are extensive. We cite [18, 19, 20, 23] for applications to initial value problems for ordinary differential equations and we cite [1, 2, 10, 11, 16, 21] for applications to boundary value problems for ordinary differential equations. More recently, researchers have successfully applied the method to fractional differential equations; see [6, 25, 27] for applications to initial value problems for fractional differential equations and see [9, 15] for applications to boundary value problems for fractional differential equations.

Quasilinearization, coupled with a shift method, has been shown to apply to boundary value problems at resonance. In the case of ordinary differential equations, see, for example, [5, 24, 28]. The purpose of this study is to apply the quasilinearization method, coupled with a shift method, to a boundary value problem at resonance for a fractional differential equation of Riemann-Liouville type.

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Uniqueness of solutions is essential to the algorithm, and in a recent article, [3], those authors considered a problem at resonance for an ordinary differential equation in which a new argument to obtain uniqueness of solutions was produced. In this article, we consider an analogous boundary value problem for the fractional differential equation and in doing so, produce a new argument for uniqueness of solutions. We stress that uniqueness of solutions is essential in this work and so this work differs from that in [24] or [28] where multiplicity of solutions is the motivation.

In Section 2 we provide preliminary definitions and state analogues of the second derivative test for fractional derivatives obtained in [4] and in [26]. In Section 3, we introduce the two-point fractional boundary value problem at resonance that is studied in this work. The method of upper and lower solutions is employed to obtain uniqueness of solutions. A shift method is applied and a Green's function is constructed using the Laplace transform method. Existence of solutions is then obtained through an application of the Schauder fixed point theorem. In Section 4, we apply the quasilinearization algorithm and construct a sequences of upper solutions and lower solutions that converge monotonically and quadratically to the unique solution. In Section 5, we provide a short conclusion.

2. Preliminaries.

Definition 2.1. [17] Let $0 < \alpha$ and $a \in \mathbb{R}$. The α^{th} -order Riemann-Liouville fractional integral of a function y is defined by

$$I_a^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s) ds, \quad a \leq t, \quad (1)$$

provided the right-hand side exists. For $\alpha = 0$, define I_a^α to be the identity map. Moreover, let n denote a positive integer and assume $n-1 < \alpha \leq n$. The α^{th} -order Riemann-Liouville fractional derivative is defined as

$$D_a^\alpha y(t) = D^n I_a^{n-\alpha} y(t), \quad a \leq t, \quad (2)$$

where D^n denotes the classical n^{th} -order derivative, if the right-hand side exists.

Definition 2.2. [17] Let $m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. We denote by $C^m[0, 1]$ a space of functions y which are m times continuously differentiable on $[0, 1]$ with the norm

$$\|y\|_{C^m} = \sum_{k=0}^m \|y^{(k)}\|_C = \sum_{k=0}^m \max_{t \in [0,1]} |y^{(k)}(t)|.$$

In particular, for $m = 0$, $C^0[0, 1] = C[0, 1]$ is the space of continuous functions y on $[0, 1]$ with the norm

$$\|y\|_C = \max_{t \in [0,1]} |y(t)|.$$

The following two theorems are analogues of the second derivative test and are proved (for a global minimum value) in [4] and [26]. These are important results for applications of upper and lower solutions to fractional differential equations.

Theorem 2.3. [4] Assume $y \in C^2[0, 1]$ attains its maximum value at $t_0 \in (0, 1)$. Assume $1 < \alpha < 2$. Then

$$D_0^\alpha y(t_0) \leq -\frac{(\alpha-1)}{\Gamma(2-\alpha)} t_0^{-\alpha} y(t_0).$$

Moreover, if $y(t_0) > 0$, then $D_0^\alpha y(t_0) < 0$.

The condition $y \in C^2[0, 1]$ is a very strong condition for applications to Riemann-Liouville fractional differential equations and so the following result has been obtained to address this difficulty.

Theorem 2.4. [26] *Let $1 < \alpha < 2$. Assume that $y \in C(0, 1]$ satisfies the following conditions:*

- (i) $D_0^\alpha u \in C[0, 1]$;
- (ii) y attains its global maximum at $t_0 \in (0, 1)$.

Then,

$$D_0^\alpha y(t_0) \leq -\frac{(\alpha - 1)}{\Gamma(2 - \alpha)} t_0^{-\alpha} y(t_0).$$

Moreover, if $y(t_0) > 0$, then $D_0^\nu y(t_0) < 0$.

We state two more preliminary results that will be applied in Section 3.

Theorem 2.5. [26] *Let $0 < \nu < 1$. Assume that $y \in C(0, 1]$ satisfies the following conditions:*

- (i) $D_0^\nu u \in C[0, 1]$;
- (ii) y attains its global maximum at $t_0 \in (0, 1]$.

Then,

$$D_0^\nu y(t_0) \geq \frac{1}{\Gamma(1 - \nu)} t_0^{-\nu} y(t_0).$$

Moreover, if $y(t_0) > 0$, then $D_0^\nu y(t_0) > 0$.

Theorem 2.6. [12] *Let m denote a positive integer and assume $m - 1 < \alpha \leq m$. Assume $y \in C^m[0, 1]$ and $D_0^\alpha y \in C[0, 1]$. Then there exists $c \in (0, t)$ such that*

$$y(t) = \frac{I_0^{m-\alpha} y(0)}{\Gamma(\alpha - m + 1)} t^{\alpha-m} + \sum_{k=1}^{m-1} \frac{D_0^{\alpha-k} y(0)}{\Gamma(\alpha - k + 1)} t^{\alpha-k} + \frac{D_0^\alpha y(c)}{\Gamma(\alpha + 1)} t^\alpha.$$

3. Uniqueness of Solutions and Existence of Solutions. Let $1 < \alpha < 2$ and assume throughout that $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. We consider the two point boundary value problem for a Riemann-Liouville fractional differential equation,

$$D_0^\alpha y(t) = f(t, y(t)), \quad 0 \leq t \leq 1, \tag{3}$$

$$y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1). \tag{4}$$

The fractional boundary value problem (3) - (4) is at resonance because constant multiples of $t^{\alpha-1}$ satisfy the homogeneous boundary value problem

$$D_0^\alpha y(t) = 0, \quad 0 \leq t \leq 1,$$

$$y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1).$$

Throughout, we shall assume that f is increasing in the second component. In the case of second order ordinary differential equations, this monotone assumption, coupled with the second derivative test, is standard to obtain uniqueness of solutions.

Theorem 3.1. *Assume $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ and $\frac{\partial f}{\partial y} = f_y : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous and assume that $f_y > 0$ on $[0, 1] \times \mathbb{R}$. Then, the fractional boundary value problem (3) - (4) has a unique solution in $C[0, 1]$, if it exists.*

Proof. Assume for the sake of contradiction that y_1 and y_2 denote two distinct solutions of the boundary value problem (3) - (4) in $C[0, 1]$. Set $u = y_1 - y_2$. Then, $u \in C[0, 1]$, $D_0^\alpha u \in C[0, 1]$ and

$$u(0) = 0, \quad D_0^{\alpha-1}u(0) = D_0^{\alpha-1}u(1).$$

Without loss of generality assume that $u(t)$ has a positive maximum at $t_0 \in [0, 1]$.

First, assume $t_0 \in (0, 1)$. Then, $u(t_0) > 0$. Apply Theorem 2.4, and

$$D_0^\alpha u(t_0) < 0.$$

However, y_1 and y_2 each satisfy (3), and so

$$D_0^\alpha u(t_0) = f(t_0, y_1(t_0)) - f(t_0, y_2(t_0)) > 0,$$

since f is increasing in y . Thus, $u(t)$ does not have a positive maximum at $t_0 \in (0, 1)$. We shall refer to this argument as the usual contradiction.

Second, assume $t_0 = 0$. Since $u(0) = 0$, u does not have a positive maximum at 0.

Third, we assume $t_0 = 1$. Then, $u(1) > 0$. Apply Theorem 2.5, and

$$D_0^{\alpha-1}u(1) > 0.$$

Now apply Theorem 2.6 with $m = 1$ and $0 < \alpha \leq 1$ so that for each $t \in (0, 1]$ there exists $c \in (0, t)$ such that

$$u(t) = \frac{I_0^{2-\alpha}u(0)}{\Gamma(\alpha-1)}t^{\alpha-2} + \frac{D_0^{\alpha-1}u(c)}{\Gamma(\alpha)}t^{\alpha-1}.$$

Note that $u(0) = 0$ implies $\frac{I_0^{2-\alpha}u(0)}{\Gamma(\alpha-1)} = 0$ and $D_0^{\alpha-1}u(0) = D_0^{\alpha-1}u(1) > 0$. Thus, $u(t) > 0$, in a right neighborhood of 0.

Set $g(t) = I_0^{2-\alpha}u(t)$, $t \in [0, 1]$. Then, $g \in C^2[0, 1]$ since

$$g''(t) = D_0^\alpha u(t) = f(t, y_1(t)) - f(t, y_2(t)).$$

Moreover, $g(0) = 0$ and

$$g'(0) = D_0^{\alpha-1}u(0) = D_0^{\alpha-1}u(1) = g'(1) > 0.$$

We argue that g' does not change sign in $(0, 1)$. For the sake of contradiction, assume $g'(t)$ changes sign at $\tau \in (0, 1)$ and assume $g'(t) > 0$ for $0 \leq t < \tau$. Then, $g'(\tau) = 0$ and $g''(\tau) \leq 0$. If $y_1(\tau) > y_2(\tau)$, then

$$g''(\tau) = D_0^\alpha u(\tau) = f(\tau, y_1(\tau)) - f(\tau, y_2(\tau)) > 0, \quad (5)$$

which contradicts $g''(\tau) \leq 0$. If $y_1(\tau) \leq y_2(\tau)$, then $u(\tau) \leq 0$. Since $u(0) = 0$, $u(t) > 0$ in a right neighborhood of 0 and $u(\tau) \leq 0$, implies $u(t)$ has a positive global maximum at $t_1 \in (0, \tau)$, which produces the usual contradiction with Theorem 2.4 applied on $(0, \tau]$. Thus, g' does not change sign and $g' > 0$ on $[0, 1]$. Therefore, g is an increasing function on $[0, 1]$. Since $g(0) = 0$, this implies $g > 0$ on $(0, 1]$.

Since $g(t) = I_0^{2-\alpha}u(t) > 0$ on $(0, 1]$ and $g'(t) = D_0^{\alpha-1}u(t) > 0$ on $[0, 1]$, it follows from Theorem 2.6 (applied with $m = 1, 0 < \alpha \leq 1$) that for each $t \in (0, 1]$, there exists $c \in (0, t)$ such that

$$\begin{aligned} u(t) &= \frac{I_0^{2-\alpha}u(0)}{\Gamma(\alpha-1)}t^{\alpha-2} + \frac{D_0^{\alpha-1}u(c)}{\Gamma(\alpha)}t^{\alpha-1} \\ &= \frac{g(0)}{\Gamma(\alpha-1)}t^{\alpha-2} + \frac{g'(c)}{\Gamma(\alpha)}t^{\alpha-1} = \frac{g'(c)}{\Gamma(\alpha)}t^{\alpha-1} > 0. \end{aligned}$$

In particular, $y_1(0) = y_2(0)$ and $y_1(t) > y_2(t)$ on $(0, 1]$. Consequently,

$$g''(t) = D_0^\alpha u(t) = f(c, y_1(t)) - f(c, y_2(t)) > 0, \quad 0 < t \leq 1, \quad (6)$$

and $g''(0) = 0$. This implies g' is an increasing function on $[0, 1]$ and hence $g'(0) < g'(1)$; that is,

$$D_0^{\alpha-1}u(0) < D_0^{\alpha-1}u(1).$$

This contradicts the second boundary condition satisfied by u . Thus, $u(t)$ does not have a positive maximum at 1 and the proof is complete. \square

Definition 3.2. We say $w \in C[0, 1]$ is a lower solution of the fractional boundary value problem (3) - (4) if $w(0) = 0$, $D_0^{\alpha-1}w(0) = D_0^{\alpha-1}w(1)$ and

$$D_0^\alpha w(t) \geq f(t, w(t)), \quad 0 \leq t \leq 1.$$

We say $v \in C[0, 1]$ is an upper solution of the fractional boundary value problem (3) - (4) if $w(0) = 0$, $D_0^{\alpha-1}w(0) = D_0^{\alpha-1}w(1)$ and

$$D_0^\alpha v(t) \leq f(t, v(t)), \quad 0 \leq t \leq 1.$$

Theorem 3.3. Assume $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ and $\frac{\partial f}{\partial y} = f_y : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous and assume that $f_y > 0$ on $[0, 1] \times \mathbb{R}$. Also, assume w and v are lower and upper solutions of the fractional boundary value problem (3) - (4). Then,

$$w(t) \leq v(t), \quad 0 \leq t \leq 1.$$

Proof. The proof of this theorem is very similar to the proof of the uniqueness theorem, Theorem 3.1. Assume w is a lower solution and v is an upper solution of the fractional boundary value problem (3) - (4), respectively. Assume for the sake of contradiction that $w \leq v$ is false. Assume that $(w - v)(t)$ has a positive maximum at $t_0 \in [0, 1]$.

First, assume $t_0 \in (0, 1)$. Then, $(w - v)(t_0) > 0$. Using Theorem 2.4, we have

$$D_0^\alpha(w - v)(t_0) < 0.$$

However, w and v are respectively lower and upper solutions of the fractional boundary value problem (3) - (4), and

$$D_0^\alpha(w - v)(t_0) \geq f(t_0, w(t_0)) - f(t_0, v(t_0)) > 0,$$

since f is increasing in the second variable. In particular, the usual contradiction applies and $(w - v)(t)$ does not have a positive maximum at $t_0 \in (0, 1)$.

Next, we assume $t_0 = 0$. Since $(w - v)(0) = 0$, $(w - v)$ doesn't have a positive maximum at 0.

The proof for $t_0 = 1$ is similar to the proof of Theorem 3.1. It is simply a matter of replacing the second equality in each of (5) and (6) with the appropriate differential inequality. \square

We now address existence of solutions of the fractional boundary value problem (3) - (4). A shift argument [14] will be applied to obtain an equivalent boundary value problem that is not at resonance and then an appropriate Green's function is constructed, employing Mittag - Leffler functions. We use definitions and properties of Mittag-Leffler functions that are commonly used and refer the reader to [22] or [13].

Definition 3.4. Let $\alpha, \beta > 0$. A two-parameter function of the Mittag-Leffler type is defined by the series expansion given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

Let $\mathcal{L}\{y(t); s\} = Y(s)$ denote the Laplace transform of y and let $E_{\alpha,\beta}^{(k)}(z)$ denote $\frac{d^k}{dz^k} E_{\alpha,\beta}(z)$.

Lemma 3.5. *The following relations hold:*

1. $\mathcal{L}\{D_0^\alpha y(t); s\} = s^\alpha Y(s) - \sum_{k=0}^{n-1} s^k D_0^{\alpha-k-1} y(0)$, where $n-1 < \alpha \leq n$, $n \in \mathbb{N}$;
2. $\mathcal{L}\{t^{\alpha k + \beta - 1} E_{\alpha,\beta}^{(k)}(\pm at^\alpha); s\} = \frac{k! s^{\alpha - \beta}}{(s^\alpha \mp a)^{k+1}}$, $\Re(s) > |a|^{\frac{1}{\alpha}}$.

To obtain existence of solutions, apply a shift argument [14]. Assume $K \neq 0$ and consider the equivalent shifted equation

$$D_0^\alpha y(t) - K^2 y(t) = \hat{f}(t, y(t)) = f(t, y(t)) - K^2 y(t), \quad 0 \leq t \leq 1. \quad (7)$$

The fractional boundary value problem (7) - (4) is not at resonance since Theorem 3.1 implies that $y \equiv 0$ is the only solution of the homogeneous fractional problem

$$D_0^\alpha y(t) = K^2 y(t)$$

satisfying the boundary conditions, (4), for any $K \neq 0$.

Since the fractional boundary value problem (7) - (4) is not at resonance, we shall construct the corresponding Green's function of the shifted equation. To do so, apply the Laplace transform to

$$D_0^\alpha y(t) - K^2 y(t) = \hat{f}(t, y(t)) = h(t), \quad y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1) = 0,$$

to obtain

$$s^\alpha Y(s) - D_0^{\alpha-1} y(0) - s D_0^{\alpha-2} y(0) - K^2 Y(s) = H(s),$$

where $\mathcal{L}\{h(t); s\} = H(s)$. Thus,

$$Y(s) = \frac{D_0^{\alpha-1} y(0)}{s^\alpha - K^2} + \frac{H(s)}{s^\alpha - K^2}.$$

Apply the inverse Laplace transform to obtain

$$y(t) = (D_0^{\alpha-1} y(0)) t^{\alpha-1} E_{\alpha,\alpha}(K^2 t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(K^2(t-s)^\alpha) h(s) ds. \quad (8)$$

Set

$$G(t) = t^{\alpha-1} E_{\alpha,\alpha}(K^2 t^\alpha) = t^{\alpha-1} \sum_{n=0}^{\infty} \frac{K^{2n} t^{\alpha n}}{\Gamma(\alpha n + \alpha)}.$$

Apply $D_0^{\alpha-1}$ to (8) and obtain

$$D_0^{\alpha-1} y(t) = (D_0^{\alpha-1} y(0)) D_0^{\alpha-1} G(t) + D_0^{\alpha-1} \left(\int_0^t G(t-s) h(s) ds \right). \quad (9)$$

Note that

$$\begin{aligned} D_0^{\alpha-1} G(t) &= \sum_{n=0}^{\infty} \frac{K^{2n}}{\Gamma(\alpha n + \alpha)} D_0^{\alpha-1} t^{\alpha n + \alpha - 1} \\ &= \sum_{n=0}^{\infty} \frac{K^{2n}}{\Gamma(\alpha n + \alpha)} \frac{\Gamma(\alpha n + \alpha)}{\Gamma(\alpha n + 1)} t^{\alpha n} = \sum_{n=0}^{\infty} \frac{K^{2n}}{\Gamma(\alpha n + 1)} t^{\alpha n} = E_{\alpha,1}(K^2 t^\alpha), \end{aligned} \quad (10)$$

and

$$D_0^{\alpha-1} \left(\int_0^t G(t-s)h(s)ds \right) = \int_0^t (D_0^{\alpha-1}G(s))h(t-s)ds + h(t) \lim_{t \rightarrow 0^+} [I_0^{2-\alpha}G(t)]. \quad (11)$$

The property (11) is observed in [17] and [22]. For the sake of self-containment, we provide some details.

$$\begin{aligned} D_0^{\alpha-1} \left(\int_0^t G(t-s)h(s)ds \right) &= DI_0^{2-\alpha} \left(\int_0^t G(t-s)h(s)ds \right) \\ &= D \int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)} \int_0^s G(s-r)h(r)drds \\ &= D \int_0^t \left(\int_r^t \frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)} G(s-r)ds \right) h(r)dr \\ &= D \int_0^t \left(\int_0^{t-r} \frac{(t-r-s)^{1-\alpha}}{\Gamma(2-\alpha)} G(s)ds \right) h(r)dr = D \int_0^t I_0^{2-\alpha}G(t-r)h(r)dr \\ &= \int_0^t (D_0^{\alpha-1}G(s))h(t-s)ds + h(t) \lim_{t \rightarrow 0^+} [I_0^{2-\alpha}G(t)]. \end{aligned}$$

Note that

$$\begin{aligned} I_0^{2-\alpha}G(t) &= \sum_{n=0}^{\infty} \frac{K^{2n}}{\Gamma(\alpha n + \alpha)} I_0^{2-\alpha} t^{\alpha n + \alpha - 1} \\ &= \sum_{n=0}^{\infty} \frac{K^{2n}}{\Gamma(\alpha n + \alpha)} \frac{\Gamma(\alpha n + \alpha)}{\Gamma(\alpha n + 2)} t^{\alpha n + 1} \\ &= t \sum_{n=0}^{\infty} \frac{K^{2n}}{\Gamma(\alpha n + 2)} t^{\alpha n} = tE_{\alpha,2}(K^2 t^\alpha). \end{aligned} \quad (12)$$

Substitute (10) and (12) into (11) and we obtain

$$\begin{aligned} D_0^{\alpha-1} \left(\int_0^t G(t-s)h(s)ds \right) &= \int_0^t E_{\alpha,1}(K^2 s^\alpha)h(t-s)ds + h(t) \lim_{t \rightarrow 0^+} [tE_{\alpha,2}(K^2 t^\alpha)] \\ &= \int_0^t E_{\alpha,1}(K^2(t-s)^\alpha)h(s)ds. \end{aligned} \quad (13)$$

Now substitute (10) and (13) into (9) and we obtain

$$D_0^{\alpha-1}y(t) = (D_0^{\alpha-1}y(0))E_{\alpha,1}(K^2 t^\alpha) + \int_0^t E_{\alpha,1}(K^2(t-s)^\alpha)h(s)ds.$$

Since $D_0^{\alpha-1}y(0) = D_0^{\alpha-1}y(1)$,

$$D_0^{\alpha-1}y(0) = \frac{1}{1 - E_{\alpha,1}(K^2)} \int_0^1 E_{\alpha,1}(K^2(1-s)^\alpha)h(s)ds. \quad (14)$$

Now substitute (14) into (8) to obtain

$$y(t) = \frac{G(t)}{1 - E_{\alpha,1}(K^2)} \int_0^1 E_{\alpha,1}(K^2(1-s)^\alpha)h(s)ds + \int_0^t G(t-s)h(s)ds.$$

Define a Green's function

$$G(K; t, s) = \begin{cases} G_1(t, s), & 0 \leq t \leq s \leq 1, \\ G_2(t, s), & 0 \leq s \leq t \leq 1, \end{cases} \quad (15)$$

where

$$G_1(t, s) = \frac{G(t)}{1 - E_{\alpha,1}(K^2)} E_{\alpha,1}(K^2(1-s)^\alpha)$$

and

$$G_2(t, s) = G(t-s) + \frac{G(t)}{1 - E_{\alpha,1}(K^2)} E_{\alpha,1}(K^2(1-s)^\alpha).$$

and write (8) as

$$y(t) = \int_0^1 G(K; t, s) h(s) ds.$$

We derive two standard properties of the Green's function, $G(K; t, s)$. First, we show that for K sufficiently small

$$G(K; t, s) \leq 0, \quad (t, s) \in [0, 1] \times [0, 1].$$

Clearly,

$$\lim_{K \rightarrow 0} G(t-s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, \quad (t, s) \in [0, 1] \times [0, 1], \quad (16)$$

$$\lim_{K \rightarrow 0} G(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t \in [0, 1], \quad (17)$$

$$\lim_{K \rightarrow 0} E_{\alpha,1}(K^2(1-s)^\alpha) = 1, \quad s \in [0, 1], \quad (18)$$

and

$$\lim_{K \rightarrow 0} E_{\alpha,1}(K^2) = 1. \quad (19)$$

Since $G(t) > 0$, $E_{\alpha,1}(K^2(1-s)^\alpha) > 0$ for $(t, s) \in [0, 1] \times [0, 1]$ and $E_{\alpha,1}(K^2) > 1$, it follows that

$$G_1(t, s) \leq 0, \quad (t, s) \in [0, 1] \times [0, 1]. \quad (20)$$

Consider

$$\begin{aligned} & \lim_{K \rightarrow 0} \left[(E_{\alpha,1}(K^2) - 1) G_2(t, s) \right] \\ &= \lim_{K \rightarrow 0} \left[(E_{\alpha,1}(K^2) - 1) G(t-s) \right] - \lim_{K \rightarrow 0} \left[G(t) E_{\alpha,1}(K^2(1-s)^\alpha) \right] \\ &= -\frac{t^{\alpha-1}}{\Gamma(\alpha)} \leq 0, \quad t \in [0, 1]. \end{aligned}$$

Then, for K sufficiently small,

$$G_2(t, s) \leq 0, \quad (t, s) \in [0, 1] \times [0, 1], \quad (21)$$

and $G(K; t, s) \leq 0$ for K sufficiently small and $(t, s) \in [0, 1] \times [0, 1]$.

Second, we bound

$$\max_{t \in [0, 1]} \left[\int_0^1 |G(K; t, s)| ds \right].$$

Note that

$$G_1(t, s) \leq G_2(t, s) \leq 0, \quad 0 \leq s \leq t \leq 1$$

implies

$$G_1(t, s) \leq G(K; t, s) \leq 0, \quad (t, s) \in [0, 1] \times (0, 1].$$

Thus,

$$\max_{t \in [0,1]} \left[\int_0^1 |G(K; t, s)| ds \right] \leq \max_{t \in [0,1]} \left[\int_0^1 |G_1(t, s)| ds \right].$$

Let

$$A = \frac{G(t)}{E_{\alpha,1}(K^2) - 1}.$$

Then,

$$\begin{aligned} \int_0^1 |G_1(t, s)| ds &= A \int_0^1 E_{\alpha,1}(K^2(1-s)^\alpha) ds = A \int_0^1 \sum_{n=0}^{\infty} \frac{K^{2n}(1-s)^{\alpha n}}{\Gamma(\alpha n + 1)} ds \\ &= A \sum_{n=0}^{\infty} \frac{K^{2n}}{\Gamma(\alpha n + 1)} \int_0^1 (1-s)^{\alpha n} ds = A \sum_{n=0}^{\infty} \frac{K^{2n}}{\Gamma(\alpha n + 1)} \frac{1}{(\alpha n + 1)} \\ &= AE_{\alpha,2}(K^2). \end{aligned}$$

Thus,

$$\begin{aligned} \max_{t \in [0,1]} \left[\int_0^1 |G(K; t, s)| ds \right] &\leq \max_{t \in [0,1]} \left[\frac{G(t)E_{\alpha,2}(K^2)}{E_{\alpha,1}(K^2) - 1} \right] = \frac{E_{\alpha,2}(K^2)}{E_{\alpha,1}(K^2) - 1} \max_{t \in [0,1]} G(t) \\ &= \frac{E_{\alpha,\alpha}(K^2)E_{\alpha,2}(K^2)}{E_{\alpha,1}(K^2) - 1}. \end{aligned} \quad (22)$$

Theorem 3.6. Assume $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ and $\frac{\partial f}{\partial y} = f_y : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous and assume that $f_y > 0$ on $[0, 1] \times \mathbb{R}$. Assume w and v are lower and upper solutions of the fractional boundary value problem (3) - (4), respectively. Then, there exists a unique solution $y \in C[0, 1]$ of (3) - (4) satisfying

$$w(t) \leq y(t) \leq v(t), \quad 0 \leq t \leq 1.$$

Proof. Let $K \neq 0$ and define a truncation of $\hat{f}(t, y(t)) = f(t, y(t)) - K^2 y(t)$ by

$$F(t, y(t)) = \begin{cases} f(t, v(t)) - K^2 v(t) + \frac{y(t) - v(t)}{1 + y(t) - v(t)}, & \text{if } y(t) > v(t), \\ f(t, y(t)) - K^2 y(t), & \text{if } w(t) \leq y(t) \leq v(t), \\ f(t, w(t)) - K^2 w(t) + \frac{y(t) - w(t)}{1 + w(t) - y(t)}, & \text{if } y(t) < w(t). \end{cases}$$

Define an operator $T : C[0, 1] \rightarrow C[0, 1]$ by

$$Ty(t) = \int_0^1 G(K; t, s) F(s, y(s)) ds \quad (23)$$

where $G(K; t, s)$ is given by (15). Then, y is a solution of the fractional boundary value problem

$$D_0^\alpha y(t) - K^2 y(t) = F(t, y(t)), \quad 0 \leq t \leq 1, \quad y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1), \quad (24)$$

if, and only if, $y \in C[0, 1]$ and

$$y(t) = \int_0^1 G(K; t, s) F(s, y(s)) ds, \quad 0 \leq t \leq 1.$$

Note that $F(t, y(t)) \in C[0, 1]$ for any $y(t) \in C[0, 1]$. Moreover, $F : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is bounded. So, it is a straightforward application of the Schauder fixed point

theorem to show that the fractional boundary value problem (24) has a solution. To see this, let

$$M = \sup \{ |F(t, y(t))| : 0 \leq t \leq 1, y \in \mathbb{R} \}$$

and let

$$\max_{t \in [0,1]} \left[\int_0^1 |G(K; t, s)| ds \right] \leq \frac{E_{\alpha,\alpha}(K^2)E_{\alpha,2}(K^2)}{E_{\alpha,1}(K^2) - 1} = \Omega.$$

Then,

$$\begin{aligned} \|Ty\|_C &= \max_{t \in [0,1]} |Ty(t)| = \max_{t \in [0,1]} \left| \int_0^1 G(K; t, s)F(s, y(s))ds \right| \\ &\leq \max_{t \in [0,1]} \int_0^1 |G(K; t, s)| |F(s, y(s))| ds \leq M\Omega. \end{aligned}$$

Define

$$\mathcal{U} = \{y \in C[0, 1] : \|y\|_C \leq M\Omega\}.$$

Then, \mathcal{U} is a closed convex subset of $C[0, 1]$ and $T : \mathcal{U} \rightarrow \mathcal{U}$. It can be shown that T is a completely continuous map on $C[0, 1]$ and so, the Schauder fixed point theorem implies there exists a fixed point, $y \in \mathcal{U}$ of the operator T .

Let y denote a continuous fixed point of the operator T . Then, y satisfies (24). Since $F(t, y(t))$ is continuous, then $y \in C[0, 1]$ and $D_0^\alpha y \in C[0, 1]$ and Theorems 2.5 and 2.4 can be employed as in the proof of Theorem 3.3 to show that

$$w(t) \leq y(t) \leq v(t), \quad 0 \leq t \leq 1.$$

Then, $F(t, y(t)) = f(t, y(t))$ and it follows that a continuous fixed point y of T is a solution of the original fractional boundary value problem (3) - (4).

We show the details that $(y-v)(t)$ doesn't have a positive maximum at $t_0 \in [0, 1]$. Assume for the sake of contradiction that $(y-v)(t)$ have a positive maximum at $t_0 \in (0, 1)$. Then,

$$D_0^\alpha y(t_0) - K^2 y(t_0) = f(t_0, v(t_0)) - K^2 v(t_0) + \frac{y(t_0) - v(t_0)}{1 + y(t_0) - v(t_0)}.$$

Since v is an upper solution of (3) - (4), it follows that

$$\begin{aligned} D_0^\alpha (y-v)(t_0) &\geq f(t_0, v(t_0)) + K^2(y(t_0) - v(t_0)) + \frac{y(t_0) - v(t_0)}{1 + y(t_0) - v(t_0)} - f(t_0, v(t_0)) \\ &= K^2(y(t_0) - v(t_0)) + \frac{y(t_0) - v(t_0)}{1 + y(t_0) - v(t_0)} > 0. \end{aligned}$$

This contradicts Theorem 2.4 and so, $(y-v)(t)$ does not have a positive maximum at $t_0 \in (0, 1)$.

The arguments to show $(y-v)(t)$ does not have a positive maximum at $t_0 = 0$ and $t_0 = 1$ follow analogously to the proof in Theorem 3.1.

The uniqueness of the continuous solution, y , follows immediately since the hypotheses of Theorem 3.1 are contained in the hypotheses of Theorem 3.6. \square

4. The monotone method and quadratic convergence. In this section, we briefly present the monotone method and obtain a quadratic rate of convergence; Once the uniqueness and existence results from Section 3 have been obtained, the implementation of the quasilinearization algorithm is routine.

To obtain the monotone method, assume one further condition on f , that f_{yy} exists and $f_{yy} \geq 0$.

Theorem 4.1. *Assume $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ and $\frac{\partial f}{\partial y} = f_y : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous and assume that $f_y > 0$ on $[0, 1] \times \mathbb{R}$. Assume in addition that f_{yy} exists and $f_{yy} \geq 0$ on $[0, 1] \times \mathbb{R}$. Also, assume w_0 and v_0 are lower and upper solutions of the fractional boundary value problem (3) - (4), respectively. Then, there exists a unique solution $y \in C[0, 1]$ of (3) - (4) satisfying*

$$w_0(t) \leq y(t) \leq v_0(t), \quad 0 \leq t \leq 1.$$

Moreover, there exist sequences $\{w_n\}$, $\{v_n\}$ of lower and upper solutions of the fractional boundary value problem (3) - (4), respectively, each of which converges to the unique solution y of the fractional boundary value problem (3) - (4) and satisfy

$$w_n(t) \leq w_{n+1}(t) \leq y(t) \leq v_{n+1}(t) \leq v_n(t), \quad 0 \leq t \leq 1, \quad n = 0, 1, \dots$$

The rate of convergence of each sequence $\{w_n\}$, $\{v_n\}$ is quadratic.

Proof. Let w_0 , v_0 denote a lower and an upper solution of (3) - (4), respectively. So, under the assumption that $f_y > 0$ on $[0, 1] \times \mathbb{R}$, we have

$$w_0(t) \leq v_0(t), \quad 0 \leq t \leq 1.$$

Define the function $h(w_0, v_0; t)$ on $[0, 1]$ by

$$h(w_0, v_0; t, y(t)) = f(t, w_0(t)) + f_y(t, v_0(t))(y - w_0)(t)$$

and consider the boundary value problem for the linear non-homogeneous fractional differential equation

$$D_0^\alpha y(t) = h(w_0, v_0; t, y(t)), \quad 0 \leq t \leq 1, \quad y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1). \quad (25)$$

Note that

$$h(w_0, v_0; t, w_0(t)) = f(t, w_0(t)), \quad 0 \leq t \leq 1,$$

and so,

$$D_0^\alpha w_0(t) \geq f(t, w_0(t)) = h(w_0, v_0; t, w_0(t)), \quad 0 \leq t \leq 1. \quad (26)$$

Moreover, since

$$f(t, v_0(t)) = f(t, w_0(t)) + f_y(t, c(t))(v_0 - w_0)(t),$$

there exists $c(t)$ satisfying $w_0(t) \leq c(t) \leq v_0(t)$ such that

$$\begin{aligned} f(t, w_0(t)) + f_y(t, c(t))(v_0 - w_0)(t) &\leq f(t, w_0(t)) + f_y(t, v_0(t))(v_0 - w_0)(t) \\ &= h(w_0, v_0; t, v_0(t)) \quad 0 \leq t \leq 1, \end{aligned}$$

since f_y is increasing in y for each $t \in [0, 1]$. Thus,

$$h(w_0, v_0; t, v_0(t)) \geq f(t, v_0(t)) \geq D_0^\alpha v_0(t), \quad 0 \leq t \leq 1. \quad (27)$$

In particular, (26) and (27) imply w_0 , v_0 are lower and upper solutions of (25) respectively as well. Since, h satisfies the hypotheses of Theorem 3.6, there exists a continuous solution, $w_1(t)$, of (25) satisfying

$$w_0(t) \leq w_1(t) \leq v_0(t), \quad 0 \leq t \leq 1.$$

Next, we observe that w_1 is a lower solution (3) - (4). To see this, note that there exists $w_0(t) \leq c(t) \leq w_1(t) \leq v_0(t)$ such that

$$f(t, w_1(t)) - f(t, w_0(t)) = f_y(t, c(t))(w_1(t) - w_0(t)) \leq f_y(t, v_0(t))(w_1(t) - w_0(t))$$

and so,

$$D_0^\alpha w_1(t) = h(w_0, v_0; t, w_1(t)) \geq f(t, w_1(t)), \quad 0 \leq t \leq 1.$$

Since $w_1 \in C[0, 1]$, w_1 is a lower solution (3) - (4) since $w_1 \in C[0, 1]$.

Now define the function $k(v_0; t)$ on $[0, 1]$ by

$$k(v_0; t, y(t)) = f(t, v_0(t)) + f_y(t, v_0(t))(y - v_0)(t)$$

and consider the boundary value problem for the linear nonhomogeneous fractional differential equation

$$D_0^\alpha y(t) = k(v_0; t, y(t)), \quad 0 \leq t \leq 1, \quad y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1). \quad (28)$$

Note that

$$k(v_0; t, v_0(t)) = f(t, v_0(t)), \quad 0 \leq t \leq 1,$$

and

$$D_0^\alpha v_0(t) \leq f(t, v_0(t)) = k(v_0; t, v_0(t)), \quad 0 \leq t \leq 1.$$

Thus, v_0 is an upper solution of (28). Note that there exists $c(t)$ satisfying $w_0(t) \leq c(t) \leq v_0(t)$ such that, for $0 \leq t \leq 1$,

$$\begin{aligned} D_0^\alpha w_0(t) &\geq f(t, w_0(t)) = f(t, v_0(t)) + f_y(t, c(t))(w_0(t) - v_0(t)) \\ &\geq f(t, v_0(t)) + f_y(t, v_0(t))(w_0(t) - v_0(t)) = k(v_0; t, w_0(t)), \end{aligned}$$

and so, w_0 is a lower solution of (28). Since k satisfies the hypotheses of Theorem 3.6 there exists a continuous solution, $v_1(t)$, of (28) satisfying

$$w_0(t) \leq v_1(t) \leq v_0(t), \quad 0 \leq t \leq 1.$$

An application of the mean value theorem again will give,

$$k(v_0; t, v_1(t)) \leq f(t, v_1(t)), \quad 0 \leq t \leq 1.$$

Thus,

$$D_0^\alpha v_1(t) = k(v_0; t, v_1(t)) \leq f(t, v_1(t)), \quad 0 \leq t \leq 1.$$

Again, since $v_1 \in C[0, 1]$, v_1 is an upper solution of (3) - (4).

Finally, apply Theorem 3.3 to obtain

$$w_1(t) \leq v_1(t), \quad 0 \leq t \leq 1;$$

in particular,

$$w_0(t) \leq w_1(t) \leq v_1(t) \leq v_0(t), \quad 0 \leq t \leq 1.$$

Apply Theorem 3.6 with lower and upper solutions, w_1 and v_1 , respectively, and keeping in mind that the solution y obtained in Theorem 3.6 is unique, we obtain

$$w_0(t) \leq w_1(t) \leq y(t) \leq v_1(t) \leq v_0(t), \quad 0 \leq t \leq 1,$$

where y is the unique solution of the fractional boundary value problem, (3) - (4).

For the inductive step, assume the sequences $\{w_k\}_{k=1}^n$ and $\{v_k\}_{k=1}^n$ have been constructed such that for each $k = 1, \dots, n$,

$$h(w_k, v_k; t, y(t)) = f(t, w_k(t)) + f_y(t, v_k(t))(y - w_k)(t),$$

$$k(v_k; t, y(t)) = f(t, v_k(t)) + f_y(t, v_k(t))(y - v_k)(t),$$

where w_k is the solution of the fractional boundary value problem

$$D_0^\alpha y(t) = h(w_{k-1}, v_{k-1}; t, y(t)), \quad 0 \leq t \leq 1, \quad y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1),$$

v_k is the solution of the fractional boundary value problem

$$D_0^\alpha y(t) = k(v_{k-1}; t, y(t)), \quad 0 \leq t \leq 1, \quad y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1),$$

and

$$w_{k-1}(t) \leq w_k(t) \leq y(t) \leq v_k(t) \leq v_{k-1}(t), \quad 0 \leq t \leq 1, \quad k = 0, \dots, n.$$

Moreover, $w_k, v_k, k = 1, \dots, n$ denote lower and upper solutions, respectively of (3) - (4), and y is the unique solution of the fractional boundary value problem (3) - (4).

To complete the induction argument, consider the boundary value problem for the linear nonhomogeneous fractional differential equation

$$D_0^\alpha y(t) = h(w_n, v_n; t, y(t)), \quad 0 \leq t \leq 1, \quad y(0) = 0, \quad D_0^{\alpha-1} y(0) = D_0^{\alpha-1} y(1). \quad (29)$$

Note that

$$h(w_n, v_n; t, w_n(t)) = f(t, w_n(t)), \quad 0 \leq t \leq 1,$$

and

$$h(w_n, v_n; t, v_n(t)) \geq f(t, v_n(t)), \quad 0 \leq t \leq 1.$$

So, w_n, v_n denote a lower and an upper solution of (29) respectively as well.

Let y be the unique solution of the fractional boundary value problem (3) - (4). The arguments above to show the existence of $w_1(t)$ and $v_1(t)$ and the inequalities

$$w_0(t) \leq w_1(t) \leq y(t) \leq v_1(t) \leq v_0(t), \quad 0 \leq t \leq 1,$$

are readily adapted to show the existence of $w_{n+1}(t)$ and $v_{n+1}(t)$ and the inequalities

$$w_n(t) \leq w_{n+1}(t) \leq y(t) \leq v_{n+1}(t) \leq v_n(t), \quad 0 \leq t \leq 1.$$

To complete the proof of monotone convergence, $\{w_n\}$ and $\{v_n\}$ are monotone sequences of continuous functions bounded above and below, respectively, on a compact domain. So by Dini's theorem, each converges uniformly to continuous functions w and v respectively on $[0, 1]$. Since

$$k(v_n; t, v_{n+1}(t)) = f(t, v_n(t)) + f_y(t, v_0(t))(v_{n+1} - v_n)(t) \rightarrow f(t, v) \text{ as } n \rightarrow \infty,$$

where the convergence is uniform on $[0, 1]$, and

$$D_0^\alpha v_{n+1}(t) - K^2 v_{n+1}(t) = k(v_n; t, v_{n+1}(t)) - K^2 v_{n+1}(t), \quad 0 \leq t \leq 1,$$

it follows that

$$v_{n+1}(t) = T v_{n+1}(t) = \int_0^1 G(K; t, s)(k(v_n; s, v_{n+1}(s)) - K^2 v_{n+1}(s)) ds, \quad 0 \leq t \leq 1,$$

where T is defined by (23), it follows that $v = Tv$ and v is the unique solution, y , of (3) - (4). Similarly, w is the unique solution, y , of (3) - (4).

We now obtain quadratic convergence and to do so, for each n , define the error e_n by

$$e_n(t) = v_n(t) - w_n(t), \quad 0 \leq t \leq 1.$$

So, $0 \leq e_n(t)$ for $0 \leq t \leq 1$. Denote by $\|e_n\|_C$ the error bound

$$\|e_n\|_C = \max_{t \in [0, 1]} |e_n(t)|.$$

Assume without loss of generality that $K > 0$ is sufficiently small such that

$$f_y(t, y) \geq K^2, \quad \text{on } [0, t] \times \left[\min_{t \in [0, 1]} w_0(t), \max_{t \in [0, 1]} v_0(t) \right],$$

and

$$G_K(t, s) \leq 0, \quad (t, s) \in [0, 1] \times [0, 1],$$

where $G(K; t, s)$ is defined by (15).

Recall

$$\begin{aligned} D_0^\alpha w_{n+1}(t) &= h(w_n, v_n; t, w_{n+1}(t)) = f(t, w_n(t)) + f_y(t, v_n(t))(w_{n+1}(t) - w_n(t)), \\ D_0^\alpha v_{n+1}(t) &= k(v_n; t, v_{n+1}(t)) = f(t, v_n(t)) + f_y(t, v_n(t))(v_{n+1}(t) - v_n(t)). \end{aligned}$$

Then

$$\begin{aligned} D_0^\alpha e_n(t) &= D_0^\alpha v_{n+1}(t) - D_0^\alpha w_{n+1}(t) \\ &= [f(t, v_n(t)) - f(t, w_n(t))] + f_y(t, v_n(t))[e_{n+1}(t) - e_n(t)]. \end{aligned}$$

By the mean value theorem, there exists $c(t)$ satisfying $w_n(t) < c_n(t) < v_n(t)$ such that

$$f(t, v_n(t)) - f(t, w_n(t)) = f_y(t, c_n(t))e_n(t).$$

Thus,

$$\begin{aligned} D_0^\alpha e_{n+1}(t) &= f_y(t, c_n(t))e_n(t) + f_y(t, v_n(t))e_{n+1}(t) - f_y(t, v_n(t))e_n(t) \\ &= f_y(t, v_n(t))e_{n+1}(t) + [f_y(t, c_n(t)) - f_y(t, v_n(t))]e_n(t). \end{aligned}$$

Employ the mean value theorem again for $f_y(t, c_n(t)) - f_y(t, v_n(t))$ and there exists $\hat{c}_n(t)$ satisfying

$$c_n(t) < \hat{c}_n(t) < v_n(t)$$

such that

$$f_y(t, c_n(t)) - f_y(t, v_n(t)) = f_{yy}(t, \hat{c}_n(t))(c_n(t) - v_n(t)).$$

Then

$$D_0^\alpha e_{n+1}(t) = f_y(t, v_n(t))e_{n+1}(t) + f_{yy}(t, \hat{c}_n(t))(c_n(t) - v_n(t))e_n(t).$$

Apply the shift argument, assume $K \neq 0$, and

$$D_0^\alpha e_{n+1}(t) - K^2 e_{n+1}(t) = (f_y(t, v_n(t)) - K^2)e_{n+1}(t) + f_{yy}(t, \hat{c}_n(t))(c_n(t) - v_n(t))e_n(t).$$

Note that e_{n+1} satisfies the boundary conditions (4) and employ the Green's function (15). Since

$$G(K; t, s)(f_y(s, v_n(s)) - K^2)e_{n+1}(s) \leq 0, \quad 0 \leq s \leq 1,$$

it follows that

$$\begin{aligned} 0 &\leq e_{n+1}(t) \\ &= \int_0^1 G(K; t, s)[(f_y(s, v_n(s)) - K^2)e_{n+1}(s) \\ &\quad + f_{yy}(s, \hat{c}_n(s))(c_n(s) - v_n(s))e_n(s)]ds \\ &\leq \int_0^1 |G(K; t, s)|f_{yy}(s, \hat{c}_n(s))(v_n(s) - c_n(s))e_n(s)ds \\ &\leq \int_0^1 |G(K; t, s)|f_{yy}(s, \hat{c}_n(s))e_n^2(s)ds. \end{aligned} \tag{30}$$

Let

$$M = \max\{|f_{yy}(t, y(t))|, \quad w_0(t) \leq y(t) \leq v_0(t), \quad 0 \leq t \leq 1\}$$

and recall (22)

$$\max_{t \in [0, 1]} \left[\int_0^1 |G(K; t, s)|ds \right] = \frac{E_{\alpha, \alpha}(K^2)E_{\alpha, 2}(K^2)}{E_{\alpha, 1}(K^2) - 1} = \Omega.$$

Then, from (30)

$$\begin{aligned} |e_{n+1}(t)| &\leq \int_0^1 |G(K; t, s)| |f_{yy}(s, \hat{c}_n(s))| e_n^2(s) ds \\ &\leq M\Omega \|e_n^2\|_C, \end{aligned}$$

implies

$$\|e_{n+1}\|_C \leq M\Omega \|e_n^2\|_C$$

and hence the rate of convergence is quadratic. \square

5. Concluding discussion. We have studied a boundary value problem at resonance for a Riemann-Liouville fractional differential equation. Under mild conditions, uniqueness of solutions is initially established by a method of upper and lower solutions. With a shift method, an equivalent boundary value problem, not at resonance, is constructed. A method of upper and lower solutions and the Schauder fixed point theorem are employed to obtain existence of solutions. With the development to obtain the uniqueness and existence of solutions, the quasilinearization method can be applied and a numerical algorithm generating sequences of lower and upper solutions converging monotonically and quadratically to a unique solution is constructed. The uniqueness and existence of solution theorem and the application of the numerical algorithm are both dependent on the existence of upper and lower solutions.

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