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Three point boundary value problems for ordinary differential equations, uniqueness implies existence

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. We consider a family of three point $n-2,1,1$ conjugate boundary value problems for nth order nonlinear ordinary differential equations and obtain conditions in terms of uniqueness of solutions imply existence of solutions. A standard hypothesis that has proved effective in uniqueness implies existence type results is to assume uniqueness of solutions of a large family of $n-$point boundary value problems. Here, we replace that standard hypothesis with one in which we assume uniqueness of solutions of large families of two and three point boundary value problems. We then close the paper with verifiable conditions on the nonlinear term that in fact imply global uniqueness of solutions of the large family of three point boundary value problems.

Keywords: uniqueness implies existence, nonlinear interpolation, ordinary differential equations, three point boundary value problems.

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1 Introduction

In a seminal paper, [23], Lasota and Opial proved that for second order ordinary differential equations, global existence and uniqueness of solutions of initial value problems and uniqueness of solutions of two point conjugate (Dirichlet) boundary value problems implies existence of solutions of two point conjugate boundary value problems. A vast study of problems referred to as uniqueness implies existence for higher order ($n-$th order) nonlinear problems was initiated. Following this work many related results were obtained; see for example, [3,8,9,15,19,21,22,24]. Henderson and many different co-authors have obtained analogous results for nonlocal boundary value problems, [2,14,16], for example, as well as boundary value problems for finite difference equations [11–13] for example, and boundary value problems

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for dynamic equations on time scales [17, 18], for example. Recently, these types of results were gathered in the monograph [4].

The results for n-th order problems, referred to above, all assumed a baseline unique solvability criterion for n-point Dirichlet type boundary conditions (n-point conjugate type boundary conditions.) Recently, the authors [5] revisited these uniqueness implies existence arguments with the baseline of a unique solvability criterion for two-point $n-1,1$ conjugate type boundary conditions. In this paper, we continue to develop the ideas initiated in [5] and begin with a baseline of unique solvability for two-point $n-1,1$ conjugate type boundary conditions and unique solvability criterion for two-point $n-2,1,1$ conjugate type boundary conditions.

Let $n \geq 2$ denote an integer and let $a < T_1 < T_2 < T_3 < b$. Let $a_i \in \mathbb{R}$, $i = 1, \dots, n$. Throughout this work, we shall consider the ordinary differential equation

$$y^{(n)}(t) = f(t, y(t), \dots, y^{(n-1)}(t)), \quad t \in [T_1, T_3], \tag{1.1}$$

where $f : (a, b) \times \mathbb{R}^n \to \mathbb{R}$, or the ordinary differential equation

$$y^{(n)}(t) = f(t, y(t)), \quad t \in [T_1, T_3], \tag{1.2}$$

where $f : (a, b) \times \mathbb{R} \to \mathbb{R}$. We shall consider three point boundary value problems for either (1.1) or (1.2) with the boundary conditions, for $j \in \{1, 2\}$,

$$y^{(i-1)}(T_1) = a_i, \quad i = 1, \dots, n-2, \quad y(T_2) = a_{n-1}, \quad y^{(j-1)}(T_3) = a_n, \tag{1.3}$$

and we shall consider two point boundary value problems for either (1.1) or (1.2) with the boundary conditions, for $j \in \{1, 2\}$,

$$y^{(i-1)}(T_1) = a_i, \quad i = 1, \dots, n-1, \quad y^{(j-1)}(T_2) = a_n. \tag{1.4}$$

For expository reasons only we state the $n-$point conjugate boundary conditions,

$$y(T_i) = a_i, \quad i \in \{1, \dots, n\}, \tag{1.5}$$

where $a < T_1 < \dots < T_n < b$.

The intent of this work is to show that under the assumptions of uniqueness of solutions of the boundary value problems (1.1), (1.3) and of the boundary value problems (1.1), (1.4), then there exists a solution of the boundary value problem (1.1) with boundary conditions (1.3) in the case $j = 1$.

With respect to (1.1), common assumptions for the types of results that we consider are:

(A) $f(t, y_1, \dots, y_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous;

(B) Solutions of initial value problems for (1.1) are unique and extend to (a, b);

With respect to (1.2), the assumptions (A) and (B) are replaced, respectively, by

(A') $f(t, y) : (a, b) \times \mathbb{R} \to \mathbb{R}$ is continuous;

(B') Solutions of initial value problems for (1.2) are unique and extend to (a, b).

There are two main purposes of this work. The first purpose is to obtain uniqueness of solutions for the boundary value problems (1.1), (1.3) and (1.1), (1.4) implies existence of solutions for the family of two-point boundary value problems (1.1), (1.3) in the case $j = 1$, and the primary tool will be a modification of the original sequential compactness argument provided by Lasota and Opial [23]. The second purpose is to obtain verifiable hypotheses that imply the uniqueness of solutions for the boundary value problems (1.2), (1.3) and (1.2), (1.4); hence, as a corollary, these verifiable hypotheses imply existence of solutions for the family of two-point boundary value problems (1.1), (1.3) in the case $j = 1$. And as it turns out, the existence will be global in $T_2 < T_3 < b$.

In Section 2, we remind the reader of a generalized mean value theorem for higher order derivatives that is commonly used in interpolation theory. It is this generalized mean value theorem that allows the Lasota and Opial argument [23] to be modified. Then in Section 3, we shall consider the general ordinary differential equation (1.1) with the boundary conditions (1.3) or (1.4). It is in Section 3 where we carry out the first main purpose of this work; in particular we produce hypotheses such that uniqueness of solutions for the boundary value problems (1.1), (1.3) and (1.1), (1.4) implies existence of solutions for the family of two-point boundary value problems (1.1), (1.3) in the case $j = 1$.

To implement the results in the literature cited above or likewise for the main result in Section 3, bounds on $T_3 - T_1$ are often required so that the contraction mapping principle can be employed to obtain the appropriate uniqueness criteria. This has led to the concept of best interval lengths for Lipschitz equations [6, 10, 20]. So in Section 4, to carry out the second purpose of this work to produce verifiable hypotheses, we consider the ordinary differential equation (1.2) with boundary conditions (1.3) or (1.4) and we assume f satisfies a Lipschitz condition in y. We construct Green's functions and estimates so that the contraction mapping principle can apply. Then in Section 5, we impose monotonicity hypotheses on f (in addition to the Lipschitz assumption) to produce the verifiable hypotheses to fulfill the second purpose of the article. In doing so, we obtain a type of global uniqueness implies existence result as will be discussed further in Section 5.

We state three further common assumptions, two of which are used throughout the paper.

(C) Solutions of the $n-$point boundary value problems (1.1), (1.5) are unique if they exist.

(D) Solutions of the two-point boundary value problems (1.1), (1.4) are unique if they exist.

(E) Solutions of the three point boundary value problems (1.1), (1.3) are unique if they exist.

We do not assume Condition (C) in this work; we state it to clearly see the contrast between this work and those cited in the first paragraph.

2 A review of divided differences

Lasota and Opial [23] literally employed the mean value theorem to construct a sequential compactness argument for the the second order conjugate boundary value problem. To modify that construction, we introduce a divided difference construction that is employed to derive an error bound for interpolating polynomials. An extension of the mean value theorem is the result. For the sake of self containment, we provide the following details. We refer the reader to the text by Conte and de Boor [1]. Let $t_0, \ldots, t_i$ denote $i + 1$ distinct real numbers and let

$z : \mathbb{R} \to \mathbb{R}$. Define $z[t_l] = z(t_l)$, $l = 0, \ldots, i$ and if $t_l, \ldots, t_{k+1}$ denote $k - l + 2$ distinct points, define

$$z[t_l, \ldots, t_{k+1}] = \frac{z[t_{l+1}, \ldots, t_{k+1}] - z[t_l, \ldots, t_k]}{t_{k+1} - t_l}.$$

The following theorem is obtained by repeated applications of Rolle's theorem to the difference of z and the polynomial that interpolates z at the $i + 1$ distinct points $t_0, \ldots, t_i$; a proof can be found in [1, Theorem 2.2].

Theorem 2.1. *Assume $z(t)$ is a real-valued function, defined on $[a, b]$ and i times differentiable in (a, b). If $t_0, \ldots, t_i$ are $i + 1$ distinct points in $[a, b]$, then there exists*

$$c \in (\min\{t_0, \ldots, t_i\}, \max\{t_0, \ldots, t_i\})$$

such that

$$z[t_0, \ldots, t_i] = \frac{z^{(i)}(c)}{i!}.$$

In Section 3, we shall set $h > 0$ and choose $t_0 = T, t_1 = T + h, \ldots, t_i = T + ih$ to be equally spaced. In this setting

$$z[T, T + h, \ldots, T + ih] = \frac{\sum_{l=0}^{i} (-1)^{i-l} \binom{i}{l} z(T + lh)}{i! h^i}.$$

For example, if $i = 1$, Theorem 2.1 is the mean value theorem and if $i = 2$, there exists $c \in (T, T + 2h)$ such that

$$\frac{z(T) - 2z(T + h) + z(T + 2h)}{2! h^2} = \frac{z''(c)}{2!}.$$

So, in general there exists $c \in (T_1, T_1 + ih)$ such that

$$\frac{\sum_{l=0}^{i} (-1)^{i-l} \binom{i}{l} z(T + ih)}{h^i} = z^{(i)}(c). \tag{2.1}$$

3 Uniqueness of solutions implies existence of solutions

In this section we consider the families of boundary value problems (1.1), (1.3) and (1.1), (1.4). We shall provide two preliminary results, Lemma 3.1 and Theorem 3.3, one addressing the continuous dependence of solutions of (1.1) on initial conditions and another addressing the continuous dependence of solutions of (1.1) on two point boundary conditions.

We state the first lemma without proof. See [7, page 14].

Lemma 3.1. *Assume that with respect to (1.1), Conditions (A) and (B) are satisfied. Then, given a solution y of (1.1), given $t_0 \in (a, b)$, given any compact interval $[c, d] \subset (a, b)$, and given $\epsilon > 0$, there exists $\delta > 0$ such that if z is a solution of (1.1) satisfying $|y^{(i-1)}(t_0) - z^{(i-1)}(t_0)| < \delta$, $i = 1, \ldots, n$, then $|y^{(i-1)}(t) - z^{(i-1)}(t)| < \epsilon$, $i = 1, \ldots, n$, for all $t \in [c, d]$.*

For the sake of self-containment, we also state the Brouwer invariance of domain theorem.

Theorem 3.2. *If $\mathcal{U} \subset \mathbb{R}^k$ is open, $\phi : \mathcal{U} \to \mathbb{R}^k$ is one-to-one and continuous on $\mathcal{U}$, then ϕ is a homeomorphism and $\phi(\mathcal{U})$ is open in $\mathbb{R}^k$.*

In [5], the authors employed the Brouwer invariance of domain theorem to prove continuous dependence of solutions on the boundary conditions (1.4); in particular, they proved the following theorem.

Theorem 3.3. *Assume that with respect to* (1.1) *Conditions (A), (B), and (D) are satisfied. Let* $j \in \{1, 2\}$.

(i) Given any $a < T_1 < T_2 < b$*, and any solution* y *of* (1.1)*, there exists* $\epsilon > 0$ *such that if* $|T_{11} - T_1| < \epsilon$*,* $|y^{(i-1)}(T_1) - y_{i1}| < \epsilon$*,* $i = 1, \ldots, n-1$*, and* $|T_{21} - T_2| < \epsilon$ $|y^{(j-1)}(T_2) - y_{n1}| < \epsilon$*, then there exists a solution* z *of* (1.1) *such that* $z^{(i-1)}(T_{11}) = y_{l1}$*,* $i = 1, \ldots, n-1$*,* $z^{(j-1)}(T_{21}) = y_{n1}$.

(ii) If $T_{1k} \to T_1$*,* $T_{2k} \to T_2$*,* $y_{ik} \to y_i$*,* $i = 1, \ldots, n$ *and* z_k *is a sequence of solutions of* (1.1) *satisfying* $z_k^{(i-1)}(T_{1k}) = y_{ik}$*,* $i = 1, \ldots, n-1$*,* $z_k^{(j-1)}(T_{2k}) = y_{nk}$*, then for each* $i \in \{1, \ldots, n\}$*,* $z_k^{(i-1)}$ *converges uniformly to* $y^{(i-1)}$ *on compact subintervals of* (a, b).

Here, we shall employ the Brouwer invariance of domain theorem to prove continuous dependence of solutions on the boundary conditions (1.3).

Theorem 3.4. *Assume that with respect to* (1.1) *Conditions (A), (B), and (E) are satisfied. Let* $j \in \{1, 2\}$.

(i) Given any $a < T_1 < T_2 < T_3 < b$*, and any solution* y *of* (1.1)*, there exists* $\epsilon > 0$ *such that if* $|T_{11} - T_1| < \epsilon$*,* $|y^{(i-1)}(T_1) - y_{i1}| < \epsilon$*,* $i = 1, \ldots, n-2$*,* $|T_{21} - T_2| < \epsilon$*, and* $|T_{31} - T_3| < \epsilon$*,* $|y(T_2) - y_{(n-1)1}| < \epsilon$*,* $|y(T_3) - y_{n1}| < \epsilon$*, then there exists a solution* z *of* (1.1) *such that* $z^{(i-1)}(T_{11}) = y_{l1}$*,* $i = 1, \ldots, n-2$*,* $z(T_{21}) = y_{(n-1)1}$*, and* $z^{(j-1)}(T_{31}) = y_{n1}$.

(ii) If $T_{1k} \to T_1$*,* $T_{2k} \to T_2$*,* $T_{3k} \to T_3$*,* $y_{ik} \to y_i$*,* $i = 1, \ldots, n$ *and* z_k *is a sequence of solutions of* (1.1) *satisfying* $z_k^{(i-1)}(T_{1k}) = y_{ik}$*,* $i = 1, \ldots, n-2$*,* $z_k(T_{2k}) = y_{(n-1)k}$*,* $z_k^{(j-1)}(T_{3k}) = y_{nk}$*, then for each* $i \in \{1, \ldots, n\}$*,* $z_k^{(i-1)}$ *converges uniformly to* $y^{(i-1)}$ *on compact subintervals of* (a, b).

Proof. Let $j \in \{1, 2\}$. Define $\mathcal{U} \subset \mathbb{R}^{n+3}$ to be the open set

$$\mathcal{U} = \{(T_1, T_2, T_3, c_1, \ldots, c_n) : a < T_1 < T_2 < T_3 < b, c_i \in \mathbb{R}, i = 1, \ldots, n\}.$$

Let $t_0 \in (a, b)$. Define $\phi : \mathcal{U} \to \mathbb{R}^{n+3}$ by

$$\phi(T_1, T_2, T_3, c_1, \ldots, c_n) = (T_1, T_2, T_3, y(T_1), \ldots, y^{(n-3)}(T_1), y(T_2), y^{(j-1)}(T_3)),$$

where y is the unique solution of (1.1) satisfying the initial conditions $y^{(i-1)}(t_0) = c_i$, $i = 1, \ldots, n$. Then by Lemma 3.1, ϕ is continuous on $\mathcal{U}$.

To see that ϕ is a $1-1$ map on $\mathcal{U}$ let

$$(t_1, t_2, t_3, c_1, \ldots, c_n), (s_1, s_2, s_3, d_1, \ldots, d_n) \in \mathcal{U}$$

and assume

$$\phi(t_1, t_2, t_3, c_1, \ldots, c_n) = \phi(s_1, s_2, s_3, d_1, \ldots, d_n).$$

By the definition of ϕ, $t_i = s_i$, $i = 1, 2, 3$. It follows by Condition (E) that $c_i = d_i$, $i = 1, \ldots, n$, since if y, z are solutions of (1.1) and $y^{(i-1)}(T_1) = z^{(i-1)}(T_1)$, $i = 1, \ldots, n-2$, $y(T_2) = z(T_2)$, $y^{(j-1)}(T_3) = z^{(j-1)}(T_3)$, then $y \equiv z$ on (a, b); in particular, $c_i = y^{(i-1)}(t_0) = z^{(i-1)}(t_0) = d_i$, $i = 1, \ldots, n$. Apply Brouwer's invariance of domain theorem to obtain that $\phi(\mathcal{U})$ is open in $\mathbb{R}^{n+3}$ which proves (i), and to obtain that ϕ^{-1} is continuous on $\mathcal{U}$ which proves (ii). □

Finally we state the uniqueness implies existence theorem proved by the authors in [5].

Theorem 3.5. *Assume that with respect to* (1.1)*, Conditions (A), (B), and (D) are satisfied. Then for each* $a < T_1 < T_2 < b$, $a_i \in \mathbb{R}$, $i = 1, \dots, n$, *the two point boundary value problem* (1.1)*,* (1.4) *has a solution.*

We are now in a position to adapt the method of Lasota and Opial [23] and show that the uniqueness of solutions of the boundary value problems (1.1), (1.3) and (1.1), (1.4) implies the existence of solutions of the boundary value problem (1.1), (1.3) for $j = 1$.

Theorem 3.6. *Assume that with respect to* (1.1)*, Conditions (A), (B), (D) and (E) are satisfied. Then for each* $a < T_1 < T_2 < T_3 < b$, $a_i \in \mathbb{R}$, $i = 1, \dots, n$, *then for* $j = 1$, *the three point boundary value problem* (1.1)*,* (1.3) *has a solution.*

Proof. Let $m \in \mathbb{R}$ and denote by $y(t; m)$ the solution of the two-point boundary value problem (1.1), with boundary conditions

$$y^{(i-1)}(T_1; m) = a_i, \quad i = 1, \dots, n-2, \quad y^{(n-2)}(T_1; m) = m, \quad y(T_2) = a_{n-1}.$$

Let

$$\Omega = \{p \in \mathbb{R} : \text{ there exists } m \in \mathbb{R} \text{ with } y(T_3; m) = p\}.$$

So the theorem is proved by showing $\Omega = \mathbb{R}$. By Theorem 3.5, $\Omega \neq \emptyset$, so the theorem is proved by showing Ω is opened and closed. That Ω is open follows from Theorem 3.4.

To show Ω is closed, let p_0 denote a limit point of Ω and without loss of generality let p_k denote a strictly increasing sequence of reals in Ω converging to p_0. Assume $y(T_3; m_k) = p_k$ for each $k \in \mathbb{N}_1$. It follows by the uniqueness of solutions, Condition (E), that

$$y^{(j-1)}(t; m_{k_1}) \neq y^{(j-1)}(t; m_{k_2}), \quad t \in (T_2, b), \tag{3.1}$$

for each $j \in \{1, 2\}$, if $k_1 < k_2$ and in particular,

$$y(t; m_1) < y(t; m_k) \quad t \in (T_2, b), \tag{3.2}$$

for each k.

Either $y'(T_3; m_k) \leq 0$ infinitely often or $y'(T_3; m_k) \geq 0$ infinitely often. Relabel if necessary and assume $y'(T_3; m_k) \leq 0$ or $y'(T_3; m_k) \geq 0$ for each k. Finally note that (3.1) implies that we may assume $y'(T_3; m_k) < 0$ or $y'(T_3; m_k) > 0$ for each k.

We first assume the case $y'(T_3; m_k) < 0$ for each k. Find $T_3 < T_4 < b$ such that $y'(t; m_1) \leq 0$, for $t \in [T_3, T_4]$. Then $y(t; m_1)$ is decreasing on $[T_3, T_4]$. By (3.2), if $t \in [T_3, T_4]$ and $k \geq 1$, then

$$L = y(T_4; m_1) \leq y(t; m_1) \leq y(t; m_k). \tag{3.3}$$

Fix k and find $T_3 < T_{4k} \leq T_4$ such that $y'(t; m_k) < 0$ on $[T_3, T_{4k}]$. Then $y(t; m_k)$ is decreasing on $[T_3, T_{4k}]$; in particular

$$L \leq y(T_{4k}; m_1) < y(T_{4k}; m_k) \leq y(t; m_k) \leq y(T_3; m_k) \leq p_0 \tag{3.4}$$

for $t \in [T_3, T_{4k}]$.

The observation employed by Lasota and Opial [23] is

$$0 > \frac{y(T_{4k}; m_k) - y(T_3; m_k)}{T_{4k} - T_3} \geq \frac{L - p_0}{T_{4k} - T_3} \geq \frac{L - p_0}{T_4 - T_3} = K_1. \tag{3.5}$$

Apply the mean value theorem (or (2.1) in the case $i = 1$ to the left hand side of (3.5), to see that

$$S_{k1} = \{t \in [T_3, T_{4k}] : K_1 - 1 \leq y'(t; m_k) < 0\} \neq \emptyset;$$

by the continuity of $y'(t; m_k)$, there exists a closed interval of positive length,

$$I_1 = [T_{3k1}, T_{4k1}] \subset S_{k1} \subset [T_3, T_{4k}].$$

To outline an induction argument in i, the order of the derivative $y^{(i-1)}$, set $h = \frac{T_{4k1} - T_{3k1}}{2}$ and consider

$$\frac{y(T_{3k1}; m_k) - 2y(T_{3k1} + h; m_k) + y(T_{3k1} + 2h; m_k)}{h^2}.$$

Then, continuing to observe that $y(t, m_k)$ is decreasing on I_1,

$$\frac{y(T_{31}; m_k) - 2y(T_{31} + h) + y(T_{31} + 2h)}{h^2} \geq \frac{2(L - p_0)}{h^2} = \frac{2^3(L - p_0)}{(T_{4k1} - T_{3k1})^2} \geq \frac{2^3(L - p_0)}{(T_4 - T_3)^2} = K_2$$

and

$$\frac{y(T_{31}; m_k) - 2y(T_{31} + h) + y(T_3 + 2h)}{h^2} \leq \frac{2(p_0 - L)}{h^2} \leq -K_2.$$

In particular,

$$\Big|\frac{y(T_{31}; m_k) - 2y(T_{31} + h) + z(T_{31} + 2h)}{h^2}\Big| \leq K_2.$$

Apply (2.1) in the case $i = 2$ and the set

$$S_{k2} = \{t \in [T_{3k1}, T_{4k1}] : |y''(t; m_k)| \leq -K_2 + 1\} \neq \emptyset$$

and contains a closed interval of positive length

$$I_2 = [T_{3k2}, T_{4k2}] \subset S_{k2} \subset [T_{3k1}, T_{43k1}] \subset [T_3, T_4].$$

The induction hypothesis is then, for $i \in \{2, \ldots n-2\}$ assume there exist $T_{3ki} < T_{4ki}$ such that $I_i = [T_{3ki}, T_{4ki}] \subset [T_{3k(i-1)}, T_{4k(i-1)}] \subset [T_3, T_4]$ and

$$|y^{(i)}(t; m_k)| \leq -K_i + 1, \quad t \in I_i$$

where

$$K_i = \frac{i^i 2^{i-1}(L - p_0)}{(T_4 - T_3)^i}.$$

Set $h = \frac{T_{4ki} - T_{3ki}}{i+1}$. Then,

$$\Big|\frac{\sum_{l=0}^{i+1}(-1)^{i+1-l}\binom{i+1}{l}y(T_{3ki} + lh)}{h^{i+1}}\Big| \geq \frac{(i+1)^{i+1}2^i(L - p_0)}{(T_{4ki} - T_{3ki})^{i+1}} \geq \frac{(i+1)^{i+1}2^i(L - p_0)}{(T_4 - T_3)^{i+1}} = -K_{i+1}.$$

Apply (2.1) in the case $i + 1$ and the set,

$$S_{k(i+1)} = \{t \in [T_{3ki}, T_{4ki}] : |y^{(i+1)}(t; m_k)| \leq -K_{i+1} + 1\} \neq \emptyset$$

and contains a closed interval of positive length

$$I_{i+1} = [T_{3(i+1)}, T_{4(i+1)}] \subset [T_{3i}, T_{4i}] \subset [T_3, T_4].$$

Recall, k is fixed. For this fixed k, choose $t_k \in I_{n-1}$. Then

$$(t_k, y(t_k; m_k), y'(t_k; m_k), \ldots, y^{(n-1)}(t_k; m_k)) \in [T_3, T_4] \times [L, p_0] \times \Pi_{i=1}^{n-1}[-K_i - 1, K_i + 1].$$

The set on the righthand side is a compact subset of $\mathbb{R}^{n+1}$ and independent of k. Perform this process for each k and generate a sequence

$$\{(t_k, y(t_k; m_k), y'(t_k; m_k), \ldots, y^{(n-1)}(t_k; m_k))\}_{k=1}^{\infty} \subset [T_3, T_4] \times [L, p_0] \times \Pi_{i=1}^{n-1}[-K_i - 1, K_i + 1].$$

In particular, there exists a convergent subsequence (relabeling if necessary)

$$\{(t_k, y(t_k; m_k), y'(t_k; m_k), \ldots, y^{(n-1)}(t_k; m_k))\} \to (t_0, c_1, \ldots, c_n)$$

where $t_0 \in [T_3, T_4]$. Since $t_0 \in (a, b)$ and by the continuous dependence of solutions of initial value problems, Lemma 3.1, $y(t; m_k)$ converges in $C^{n-1}[T_1, T_3]$ to a solution, say $z(t)$, of the initial value problem (1.1), with initial conditions, $y^{(i-1)}(t_0) = c_i$, $i = 1, \ldots, n$. Thus, $p_0 = z(T_3)$ which implies $p_0 \in \Omega$ and Ω is closed. This completes the proof if $y'(T_3; m_k) < 0$ for each k.

If $y'(T_3; m_k) > 0$ for each k, find $T_2 < T_4 < T_3$ such that $y'(t; m_1) \geq 0$, for $t \in [T_4, T_3]$. Then

$$L = y(T_4; m_1) < y(T_4; m_k) \leq y(t; m_k) \leq p_0, \quad T_4 \leq t \leq T_3,$$

and the above argument can be modified to apply on $[T_4, T_3]$. This completes the proof. □

4 Local uniqueness of solutions

In this section, we state conditions on $f(t, y)$ such that solutions of a boundary value problem (1.2), (1.3) are unique, if they exist, for $T_3 - T_1$ sufficiently small. The ideas here are not new and the result we state is standard, but the estimates that are employed are possibly new and the construction is provided for the sake of self containment. Assume that $f : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous and that there exists a positive constant, P such that

$$|f(t, y) - f(t, z)| \leq P|y - z| \tag{4.1}$$

for all $(t, y), (t, z) \in (a, b) \times \mathbb{R}$.

We require specific estimates for the Green's function for the boundary value problem (1.2), (1.3) for each $j = 1, 2$.

For $j = 1$, the Green's function, $G(1; t, s)$ for the boundary value problem (1.2), (1.3) has the following representation. If $T_1 \leq s \leq T_2$,

$$G(1; t, s) = \begin{cases} \frac{(t-T_1)^{n-2}}{(T_3-T_2)(n-1)!}\left[\frac{(T_2-s)^{n-1}(t-T_3)}{(T_2-T_1)^{n-2}} + \frac{(T_3-s)^{n-1}(T_2-t)}{(T_3-T_1)^{n-2}}\right], & T_1 \leq t \leq s \leq T_2, \\ \frac{(t-T_1)^{n-2}}{(T_3-T_2))(n-1)!}\left[\frac{T_2-s)^{n-1}(t-T_3)}{(T_2-T_1)^{n-2}} + \frac{(T_3-s)^{n-1}(T_2-t)}{(T_3-T_1)^{n-2}}\right] + \frac{(t-s)^{n-1}}{(n-1)!}, & T_1 \leq s \leq t \leq T_3, \end{cases}$$

and if $T_2 \leq s \leq T_3$,

$$G(1; t, s) = \begin{cases} \frac{(t-T_1)^{n-2}(T_3-s)^{n-1}(T_2-t)}{(T_3-T_2)(T_3-T_1)^{n-2}(n-1)!}, & T_1 \leq t \leq s \leq T_2, \\ \frac{(t-T_1)^{n-2}(T_3-s)^{n-1}(T_2-t)}{(T_3-T_2)(T_3-T_1)^{n-2}(n-1)!} + \frac{(t-s)^{n-1}}{(n-1)!}, & T_1 \leq s \leq t \leq T_3. \end{cases}$$

The Green's function is constructed in the following way. If (1.1) or (1.2) is a nonhomogenous linear equation, then the general solution is

$$y(t) = \sum_{i=1}^{n} c_i (t - T_1)^{i-1} + \int_{T_1}^{t} \frac{(t-s)^{n-1}}{(n-1)!} f(s) ds.$$

The homogeneous boundary conditions at T_1 imply $c_i = 0$, $i = 1, \ldots, n-2$. The homogeneous boundary conditions at T_2 and T_3 imply

$$\begin{cases} 0 = c_{n-1} + c_n (T_2 - T_1) + \int_{T_1}^{T_2} \frac{(T_2-s)^{n-1}}{(T_2-T_1)^{n-2}(n-1)!} f(s) ds, \\ 0 = c_{n-1} + c_n (T_3 - T_1) + \int_{T_1}^{T_3} \frac{(T_3-s)^{n-1}}{(T_3-T_1)^{n-2}(n-1)!} f(s) ds. \end{cases}$$

We now seek a bound on $|G(1; t, s)|$ on $[T_1, T_3] \times [T_1, T_3]$. The term $(T_3 - T_2)$ in the common denominator is apparently problematic. We provide algebraic details to show the term is not problematic. First note that if $T_1 \leq s$, then usual calculus methods imply that the function

$$h(\alpha) = \frac{(\alpha - s)^{n-1}}{(\alpha - T_1)^{n-2}}$$

is increasing in α for $s \leq \alpha$. In particular,

$$\frac{(T_2 - s)^{n-1}}{(T_2 - T_1)^{n-2}} < \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}}.$$

If $T_1 \leq t \leq T_2$,

$$\begin{aligned} \frac{(T_2 - s)^{n-1}}{(T_2 - T_1)^{n-2}} (t - T_3) &> \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}} (t - T_3) \\ &= \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}} (t - T_2) + \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}} (T_2 - T_3). \end{aligned}$$

So,

$$\frac{(T_2 - s)^{n-1}}{(T_2 - T_1)^{n-2}} (t - T_3) + \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}} (T_2 - t) > \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}} (T_2 - T_3).$$

Similarly, if $T_2 \leq t \leq T_3$,

$$\frac{(T_2 - s)^{n-1}}{(T_2 - T_1)^{n-2}} (t - T_3) + \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}} (T_2 - t) < \frac{(T_2 - s)^{n-1}}{(T_2 - T_1)^{n-2}} (T_2 - T_3).$$

Keeping in mind that the function $h(\alpha)$ is increasing we have, for $T_1 \leq s \leq T_2$, $T_1 \leq t \leq T_3$,

$$\left| \frac{(T_2 - s)^{n-1}}{(T_2 - T_1)^{n-2}} (t - T_3) + \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}} (T_2 - t) \right| \leq \frac{(T_3 - s)^{n-1}}{(T_3 - T_1)^{n-2}} (T_3 - T_2). \tag{4.2}$$

Now with the help of (4.2) it is now clear to see that

$$|G(1; t, s)| \leq \frac{2(T_3 - T_1)^{n-1}}{(n-1)!}, \quad (t, s) \in [T_1, T_2] \times [T_1, T_2]. \tag{4.3}$$

For $j = 2$, to construct the Green's function, $G(2;t,s)$, we solve a similar system of two equations to compute c_{n-1} and c_n for the boundary value problem (1.2), (1.4) and obtain the following representation. Let $D = (T_3 - T_1) + (n-2)(T_3 - T_2)$. Define

$$g(t,s) = \frac{(T_2-s)^{n-1}}{(n-1)!(T_2-T_1)^{n-2}}(-(n-1)(T_3-T_1) + (n-2)(t-T_1)) + \frac{(T_3-s)^{n-2}}{(n-2)!(T_3-T_1)^{n-3}}(T_2-t).$$

If $T_1 \le s \le T_2$,

$$G(2;t,s) = \begin{cases} \frac{(t-T_1)^{n-2}g(t,s)}{D}, & T_1 \le t \le s \le T_2, \\ \frac{(t-T_1)^{n-2}g(t,s)}{D} + \frac{(t-s)^{n-1}}{(n-1)!}, & T_1 \le s \le t \le T_3, \end{cases}$$

and if $T_2 \le s \le T_3$,

$$G(2;t,s) = \begin{cases} \frac{(t-T_1)^{n-2}(T_3-s)^{n-2}}{D(n-2)!(T_3-T_1)^{n-3}}(T_2-t), & T_1 \le t \le s \le T_2, \\ \frac{(t-T_1)^{n-2}(T_3-s)^{n-2}}{D(n-2)!(T_3-T_1)^{n-3}}(T_2-t) + \frac{(t-s)^{n-1}}{(n-1)!}, & T_1 \le s \le t \le T_3. \end{cases}$$

Now the term $T_3 - T_2$ in D is not problematic since $D > T_3 - T_1$.

To bound $|G(2;t,s)|$, we keep in mind that $h(\alpha)$ is increasing and write

$$|-(n-1)(T_3-T_1) + (n-2)(t-T_1)| = |(n-2)(t-T_3) - (T_3-T_1)| \le (n-1)(T_3-T_1).$$

Then,

$$\left| \frac{(T_2-s)^{n-1}}{(n-1)!(T_2-T_1)^{n-2}}(-(n-1)(T_3-T_1) + (n-2)(t-T_1)) \right| \le \frac{(T_3-T_1)^{n-1}}{(n-2)!}$$

and

$$\left| \frac{(T_3-s)^{n-2}}{(n-2)!(T_3-T_1)^{n-3}}(T_2-t) \right| \le \frac{(T_3-T_1)^{n-1}}{(n-2)!}.$$

Thus,

$$|G(2;t,s)| \le \frac{(2n-1)(T_3-T_1)^{n-1}}{(n-1)!}, \quad (t,s) \in [T_1,T_2] \times [T_1,T_2]. \tag{4.4}$$

For each $a < T_1 < T_2 < T_3 < b$, consider the usual Banach space $C[T_1,T_3]$ with norm

$$\|y\| = \max_{T_1 \le t \le T_3} |y(t)|.$$

For each $j \in \{1,2\}$, define the fixed point operator $T(j;\cdot) : C[T_1,T_3] \to C[T_1,T_3]$ by

$$T(j;y)(t) = p_{cj}(t) + \int_{T_1}^{T_3} G(j;t,s) f(s,y(s))ds,$$

where p_{cj} denotes the $n-1$ order polynomial satisfying the boundary conditions (1.3). Then (4.1), (4.3) and (4.4) are readily employed to see that if $y,z \in C[T_1,T_3]$, then for $T_1 \le t \le T_3$,

$$\begin{aligned} |T(j;y)(t) - T(j;z)(t)| &\le \int_{T_1}^{T_3} |G(j;t,s)||f(s,y(s) - f(s,z(s))|ds \\ &\le \max\left\{ \frac{2(T_3-T_1)^n}{(n-1)!}, \frac{(2n-1)(T_3-T_1)^n}{(n-1)!} \right\} P\|y-z\|. \end{aligned} \tag{4.5}$$

Choose

$$\delta = \left(\frac{(n-1)!}{(2n-1)P} \right)^{\frac{1}{n}} = \min \left\{ \left(\frac{(n-1)!}{2P} \right)^{\frac{1}{n}}, \left(\frac{(n-1)!}{(2n-1)P} \right)^{\frac{1}{n}} \right\}$$

and assume $|T_3 - T_1| < \delta$. Then the each fixed point map $T(j;\cdot)$ for $j \in \{1,2\}$ is a contraction map on $C[T_1, T_3]$.

Theorem 4.1. *Assume that $f : (a,b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous and that there exists positive constant P such that f satisfies (4.1) for all $(t,y),(t,z) \in (a,b) \times \mathbb{R}^n$. Assume $|T_3 - T_1| < \delta$ where*

$$\delta = \left(\frac{(n-1)!}{(2n-1)P} \right)^{\frac{1}{n}}.$$

Then for each $j \in \{1,2\}$ there exists a unique solution of the boundary value problem (1.2), (1.3).

The following information about the boundary value problem (1.2), (1.4) will be required in the next section so we state it here. For each $j \in \{1,2\}$, it was shown in [5] that the corresponding Green's function $\mathcal{G}(j;t,s)$ for the boundary value problem (1.2), (1.4) has the following representation and satisfies the following estimate:

$$\mathcal{G}(j;t,s) = \begin{cases} -\frac{(t-T_1)^{n-1}(T_2-s)^{n-j}}{(n-1)!(T_2-T_1)^{n-j}}, & T_1 \le s \le t \le T_2, \\ -\frac{(t-T_1)^{n-1}(T_2-s)^{n-j}}{(n-1)!(T_2-T_1)^{n-j}} + \frac{(t-s)^{n-1}}{(n-1)!}, & T_1 \le s \le t \le T_2. \end{cases} \tag{4.6}$$

Note that for each $i \in \{1,\ldots,n\}$,

$$|\mathcal{G}(j;t,s)| \le \frac{2|(T_2-T_1)|^{n-1}}{(n-1)!}, \quad (t,s) \in [T_1,T_2] \times [T_1,T_2]. \tag{4.7}$$

5 A type of global uniqueness of solutions implies existence of solutions for $n = 3$

In this section we consider the boundary value problem (1.2), (1.3) or the boundary value problem (1.2), (1.4), for $j \in \{1,2\}$ in the specific case that $n = 3$. We assume f continues to satisfy a Lipschitz condition in y; we shall also impose a new monotonicity condition on f. We shall assume that f is monotone decreasing in y for $t \in (T_1, T_2)$ and that f is monotone increasing in y for $t \in (T_2, T_3)$. Since the monotonicity of f depends on T_2, beginning with Theorem 5.2 we shall assume that T_2 is fixed and f is a function of $(T_2;t,y)$. For sake of exposition, we shall also assume that T_1 is fixed.

For $j \in \{1,2\}$, we first briefly address the local uniqueness of solutions for the boundary value problem, (1.2), (1.4). Continuing in the framework of the contraction mapping principle, employ the Banach space $\mathcal{B} = C[T_1,T_2]$ with the usual supremum norm. Then the fixed point operator

$$\mathcal{T}(j;y)(t) = p_c(t) + \int_{T_1}^{T_2} \mathcal{G}(j;t,s) f(s,y(s)) ds,$$

maps $\mathcal{B}$ into $\mathcal{B}$ if f is continuous and fixed points are 3 times continuously differentiable. By the estimates obtained in the preceding section, if each operator $T(j;y)$ is a contraction map, then each operator $\mathcal{T}(j;y)$ is a contraction map.

Theorem 5.1. *Assume that $f : (a,b) \times \mathbb{R} \to \mathbb{R}$ is continuous and assume there exists a positive constant, P, such that*

$$|f(t,y) - f(t,z)| \le P|y - z|$$

for all $(t,y),(t,z) \in (a,b) \times \mathbb{R}$. Assume $a < T_1 < T_2 < T_3 < b$ and and $T_3 - T_1 < \delta$ where

$$\delta = \left(\frac{(3-1)!}{(6-1)P}\right)^{\frac{1}{3}}.$$

Then for each $j \in \{1,2\}$ there exists a unique solution of the boundary value problem (1.2), (1.3) and there exists a unique solution of the boundary value problem (1.2), (1.4).

In the next result, we assume, in addition, that f is increasing in y and we prove a type of global uniqueness of solutions of the boundary value problem (1.2), (1.3). By global, we mean that although there is a constraint on $T_2 - T_1$, there is no local constraint on $T_3 - T_2$.

Theorem 5.2. *Assume $a < T_1 < T_2 < b$ and assume T_1 and T_2 are fixed. Assume that $f : (a,b) \times \mathbb{R} \to \mathbb{R}$ is continuous and assume there exists a positive constant, P, such that*

$$|f(t,y) - f(t,z)| \le P|y - z|$$

for all $(t,y),(t,z) \in (a,b) \times \mathbb{R}$. Assume $a < T_1 < T_2 < T_3 < b$. Set

$$\delta = \left(\frac{(3-1)!}{(6-1)P}\right)^{\frac{1}{3}}$$

and assume $T_2 - T_1 < \delta$. Assume

$$\begin{aligned} f(t,y) &\ge f(t,z), \quad t \in (T_1,T_2], \quad y < z, \\ f(t,y) &\le f(t,z), \quad t \in [T_2,b), \quad y < z. \end{aligned} \tag{5.1}$$

Then solutions of the boundary value problem (1.2), (1.3) are unique if they exist.

Proof. Assume for the sake of contradiction that y_1 and y_2 are distinct solutions of the boundary value problem (1.2), (1.3). We first argue that there exists $T_4 \in (T_1,T_2) \cup (T_2,T_3)$ such that $(y_1 - y_2)(T_4) = 0$. So, for the sake of contradiction, assume $y_1 - y_2$ is of constant sign on $(T_1,T_2) \cup (T_2,T_3)$ and without loss of generality assume $(y_1 - y_2)(t) > 0$ for $T_2 < t < T_3$. Set $u(t) = (y_1 - y_2)(t)$ and so $u(t) > 0$ on (T_2,T_3).

To obtain the contradiction, we shall consider multiple cases.

First assume $u(t) < 0$ on (T_1,T_2). Then by (5.1), $u'''(t) > 0$ on $(T_1,T_2) \cup (T_2,T_3)$. Thus, u'' is monotone increasing on (T_1,T_3). Apply Rolle's theorem to u which satisfies $u(T_1) = 0, u(T_2) = 0, u(T_3) = 0$ to obtain T_{11}, T_{12} and T_{21} satisfying

$$T_1 < T_{11} < T_2 < T_{12} < T_3, \quad T_{11} < T_{21} < T_{12}$$

such that

$$u'(T_{1i}) = 0, \quad i = 1,2, \quad u''(T_{21}) = 0.$$

Since u'' is monotone, there are no other roots of u'' or u' in (T_1,T_3). Since u'' is monotone increasing, $u''(t) > 0$ for $T_{21} < t$, this in turn implies u' is increasing for $T_{21} < t$. As $T_{21} < T_{12}$, this implies $u'(T_3) > 0$ which contradicts the hypothesis $u(t) > 0$ on (T_2,T_3).

Second, assume $u(t) > 0$ on (T_1, T_2). So now, $u'''(t) < 0$ on (T_1, T_2), $u'''(t) > 0$ on (T_2, T_3). In particular, u'' is decreasing on (T_1, T_2) and increasing on (T_2, T_3). We know u'' has at least one root in $[T_1, T_3]$ by Rolle's theorem and u'' has at most two roots in $[T_1, T_3]$ by the monotonicity property we have just observed on u''. Three more cases to consider are introduced.

Assume u'' has precisely one root, $T_{21} \in [T_1, T_3]$. By Rolle's theorem, $T_{21} \in [T_1, T_3]$ and u' has precisely two roots, T_{11}, T_{12} in $[T_1, T_3]$ satisfying

$$T_1 < T_{11} < T_{21} < T_{12} < T_3.$$

Since u'' is decreasing on (T_1, T_2) and increasing on (T_2, T_3) it must be the case that $T_{21} \le T_2$. (If $T_{21} = T_2$, then T_{21} is a repeated root. The argument works here too, so we are not counting multiplicity in the assumption u'' has precisely one root.) In particular, $u''(t) > 0$, on $[T_1, T_{21})$. Thus u' is increasing on $[T_1, T_{21})$ and $u'(T_{11}) = 0$, where $T_{11} < T_{21}$. From here, we conclude $u'(T_1) < 0$. This yields a contradiction because it is assumed that $u(t) > 0$ on (T_1, T_2).

We now come to the possibility that u'' has two distinct roots, $T_{21} < T_{22}$ in $[T_1, T_3]$. By Rolle's theorem, either $T_{11} < T_{21}$ or $T_{22} < T_{21}$. These are the final two cases to consider.

Assume $T_{11} < T_{21}$. Now T_{11} has been generated by Rolle's theorem and u'' has no roots in $(T_1, T_{21}]$. So we can conclude that $u'(T_1) \neq 0$. So u'' is decreasing on (T_1, T_2) again implies $u''(t) > 0$ on $[T_1, T_{21})$. This in turn implies u' is increasing on (T_1, T_{21}) and so $u'(t) < 0$ on $[T_1, T_{11})$. We conclude that $u'(T_1) < 0$ contradicts $u(t) > 0$ on (T_1, T_2).

For the final case, assume $T_{22} < T_{12}$. Due due the monotone nature of u'' it is clear that $u''(t) > 0$ on $(T_1, T_{21}) \cup (T_{22}, T_3)$ and $u''(t) < 0$ on (T_{21}, T_{22}). (It could be the case that $T_1 = T_{21}$. In this case, $u''(t) < 0$ on (T_1, T_{22}) and $u''(t) > 0$ on (T_{22}, T_3).) Regardless, $u''(t) > 0$ on (T_{22}, T_3), which implies u' is increasing on (T_{22}, T_3). Finally, $T_{22} < T_{12}$ implies $u'(T_3) > 0$. This produces the final contradiction since it is assume throughout that $u(t) > 0$ on (T_1, T_2).

Thus there exists $T_4 \in (T_1, T_2) \cup (T_2, T_3)$ such that $y_1(T_4) = y_2(T_4)$. It is clear by Theorem 4.1 in the case $n = 3$ and the hypothesis $|T_2 - T_1| < \delta$ that $T_4 \notin (T_1, T_2)$. So, $T_4 \in (T_2, T_3)$.

Let

$$S = \{t \in (T_2, T_3) : (y_1 - y_2)(t) = 0\}.$$

We have just shown $S \neq \emptyset$. Let $\tau = \inf S$. If $\tau > T_2$, argue that $(y_1 - y_2)(\tau) = 0$. This follows by continuity if τ is a limit point of S and by definition if τ is an isolated point of S. Thus if $\tau > T_1$, y_1 and y_2 are distinct solutions of a boundary value problem (1.2), (1.3) for $T_3 = \tau$. Apply the argument that employed four cases to conclude there exists $T_4 \in (T_2, \tau)$ such that $(y_1 - y_2)(T_3) = 0$; in particular, the assumption that $\tau = \inf S > T_1$ is false.

So, $\inf S = T_2$. Find $T \in S$ such that $0 < T - T_1 < \delta$. Then Theorem 5.1 implies $y_1 \equiv y_2$ on $[T_1, T]$. Now Condition (B) implies $y_1 \equiv y_2$ on (a, b). □

We close the article with a corollary, which represents the main result addressing the second purpose of this work.

Corollary 5.3. *Assume $a < T_1 < T_2 < b$ and assume T_1 and T_2 are fixed. Assume that $f : (a, b) \times \mathbb{R} \to \mathbb{R}$ is continuous and assume there exists a positive constant, P, such that*

$$|f(t, y) - f(t, z)| \le P|y - z|$$

for all $(t, y), (t, z) \in (a, b) \times \mathbb{R}$. Assume $a < T_1 < T_2 < T_3 < b$. Set

$$\delta = \left(\frac{(3-1)!}{(6-1)P} \right)^{\frac{1}{3}}$$

and assume $T_2 - T_1 < \delta$. Assume f satisfies (5.1)*. Assume that with respect to* (1.2)*, Conditions (A') and (B') are satisfied. Then for $j = 1$, and for each $T_2 < T_3 < b$, the three point boundary value problem* (1.2)*,* (1.3) *has a solution.*

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