**Abstract**

Contrary to the popular belief that “infinity is not a number; it’s a concept,” numbers that are not finite do exist. Mathematicians call them transfinite numbers. Just like ordinary numbers, some transfinite numbers are larger than others. This can be thought of as there being different levels of infinity, where some infinities are “more infinite” than others. If we draw a family tree in which every generation has finitely many offspring, and every chain of descendents is finite, then it is clear that we cannot have infinitely many family members. In the realm of the transfinite, things are not as intuitive. If we draw a family tree in which every generation has offspring at a certain level of infinity, and every chain of descendents is at that same level of infinity, it is possible (though not necessary) that the total number of family members is at a higher level of infinity.

**Motivation**

**Theorem.** There exists a tree of infinite height, whose branches are all of finite length.

**Proof.** Consider the set of all decreasing sequences of natural numbers, ordered by inclusion.

This example challenges our intuition about trees. König’s Lemma tells us that this unexpected behavior is due to the existence of levels that are too wide.

**Theorem (König’s Lemma).** If \( T \) is a tree of height \( \omega \), all of whose levels are finite, then \( T \) must have a branch of length \( \omega \).

In fact, this is true for an arbitrary cardinal.

**Theorem (The Generalized König’s Lemma).** Let \( \kappa \) be a cardinal. If \( T \) is a tree of height \( \kappa \), all of whose levels are finite, then \( T \) must have a branch of length \( \kappa \).

**An Aronszajn Tree**

**Theorem.** There is an Aronszajn tree.

**Proof.** Consider the set of all decreasing sequences of natural numbers, ordered by inclusion.

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**Proof outline.** We will define the levels \( L_\alpha \) by transfinite recursion. Let \( P(\alpha) \) be the logical conjunction of the following statements:

(i) \( L_\alpha \subseteq {}^\omega Q \).

(ii) \( |L_\alpha| \leq \aleph_0 \).

(iii) For every \( m \in L_\alpha \),

(a) \( m \) is increasing,

(b) \( \sup \text{ran} m \in Q \), and

(c) \( m \uparrow \beta \in L_\beta \) for all \( \beta < \alpha \).

(iv) For each \( n \in \bigcup_{\beta \leq \alpha} L_\beta \) and each \( q \in Q \) that satisfies \( q > \sup \text{ran} n \), there exists \( m(n,q,\alpha) \in L_\alpha \) such that \( m(n,q,\alpha) \supset n \) and \( \sup \text{ran} m(n,q,\alpha) = q \).

Define \( L_0 = \{ \varnothing \} \) so that \( P(0) \) holds trivially.

Given \( L_\alpha \), and that \( P(\alpha) \) is true, define \( L_{\alpha+1} = \{ n \cup \{ (n,q) : n \in L_\alpha \wedge q \in Q \wedge q > \sup \text{ran} n \} \} \), with the convention that \( \sup \varnothing = \aleph_0 \). We can verify that \( P(n+1) \) is true as well.

Now, let \( \lambda < \omega_1 \) be a limit ordinal and suppose that \( P(\beta) \) is satisfied for all \( \beta < \lambda \). We define \( L_\lambda = \{ m(n,q,\lambda) : n \in \bigcup_{\beta < \lambda} L_\beta \wedge q \in Q \wedge q > \sup \text{ran} n \} \), where \( (n,q,\lambda) \) is constructed in the following manner.

First, choose increasing sequences \( \sigma : \omega \rightarrow Q \) and \( \tau : \omega \rightarrow \lambda \) such that \( \sigma(0) = \sup \text{ran} n, \sup \sigma = q, \tau(0) = \text{dom} n, \) and \( \sup \tau = \lambda \).

Next, define \( \mu : \omega \rightarrow \bigcup_{\beta < \lambda} L_\beta \) recursively by \( \mu(0) = n \) and \( \mu(k+1) = m(\mu(k),\sigma(k+1),\tau(k+1)) \) for every \( k \in \omega \).

Finally, define \( m(n,q,\lambda) = \bigcup_{k \in \omega} \mu(k) \). It can be shown that \( P(\lambda) \) holds, thereby completing the recursion.

**Orders**

**Definition.** A set \( X \) is linearly ordered if and only if for all \( a, b \in X \), we have \( a < b, a = b, \) or \( b < a \).

**Definition.** A set \( W \) is well-ordered if and only if every nonempty subset of \( W \) has a least element.

**Definition.** Suppose \( W \) is a well-ordered set, \( \alpha \) is an ordinal, and \( f : W \rightarrow \alpha \) is a bijection such that for all \( a, b \in W \), we have \( a < b \) if and only if \( f(a) < f(b) \). Then \( W \) is said to be of order type \( \alpha \).

**Theorem.** Every well-ordered set has a unique order type.

**Trees**

**Definition.** A tree is an ordered set \( (T,\prec) \) with the property that for each \( x \in T \), the set \( \{ y \in T : y < x \} \) is well-ordered.

- The height of \( x \in T \) is \( h(x) = \text{order type of } \{ y \in T : y < x \} \).
- The height of \( T \) is \( h(T) = \sup \{ h(x) + 1 : x \in T \} \).
- The \( \alpha \)-th level in \( T \) is \( L_\alpha = \{ x \in T : h(x) = \alpha \} \).
- The width of \( L_\lambda \) is its cardinality \( |L_\lambda| \).
- A branch is a linearly ordered subset \( B \subseteq T \), such that if \( B \subseteq C \subseteq T \), then \( C \) is not linearly ordered. The length of a branch is its order type.

**Definition.** An Aronszajn tree is a tree of height \( \omega_1 \), such that none of its levels has uncountable width, and none of its branches has uncountable length.

**References**

