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An Aronszajn Tree

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Definition. Every well-ordered set has a unique order type.

Theorem. There exists a tree of infinite height, whose branches are all of finite length.

Proof outline. We will define the levels $L_n$ by transfinite recursion.

Let $P(\alpha)$ be the logical conjunct of the following statements:

(i) $L_\alpha \subseteq {}^\alpha Q$.

(ii) $|L_\alpha| \leq \aleph_0$.

(iii) For every $m \in L_\alpha$,

(a) $m$ is increasing,

(b) sup ran $m \in Q$, and

(c) $m \upharpoonright \beta \in L_\beta$ for all $\beta < \alpha$.

(iv) For each $n \in \bigcup_{\beta < \alpha} L_\beta$ and each $q \in Q$ that satisfies $q \sup ran n$, there exists $m(n, q, \alpha) \in L_\alpha$ such that $m(n, q, \alpha) \sup ran n$ and sup ran $m(n, q, \alpha) = q$.

Define $L_0 = \{\varnothing\}$ so that $P(0)$ holds trivially.

Given $L_\alpha$ and that $P(\alpha)$ is true, define $L_{\alpha+1} = \{n \cup \{(n, q) : n \in L_\alpha \land q \in Q \land \neg \text{sup ran } n\} \cup \{n \} : |n| \leq \aleph_0\}$, with the convention that sup ran $\varnothing = \infty$.

By induction, we can verify that $P(n+1)$ is true as well.

Now, let $\lambda < \omega_1$ be a limit ordinal and suppose that $P(\beta)$ is satisfied for each $\beta < \lambda$. We define $L_\lambda = \{m(n, q, \lambda) : n \in \bigcup_{\beta < \lambda} L_\beta \land q \in Q \land q \sup ran n\}$, where $m(n, q, \lambda)$ is constructed in the following manner.

First, choose increasing sequences $\sigma : \omega \to Q$ and $\tau : \omega \to \lambda$ such that $\sigma(0) = \sup ran n$, $\sup \sigma = q$, $\tau(0) = \sup \tau$, and $\sup \tau = \lambda$.

Next, define $\mu : \omega \to \bigcup_{k<\lambda} L_\beta$ recursively by $\mu(0) = n$ and $\mu(k+1) = m(\mu(k), \sigma(k+1), \tau(k+1))$ for every $k \in \omega$.

Finally, define $m(n, q, \lambda) = \bigcup_{k<\lambda} \mu(k)$.

It can be shown that $P(\lambda)$ holds, thereby completing the recursion.

Theorem (König’s Lemma). If $T$ is a tree of height $\omega$, all of whose levels are finite, then $T$ must have a branch of length $\omega$.

In fact, this is true for an arbitrary cardinal.

Theorem (The Generalized König’s Lemma). Let $\kappa$ be a cardinal. If $T$ is a tree of height $\kappa$, all of whose levels are finite, then $T$ must have a branch of length $\kappa$.

Definition (König’s Lemma). If $T$ is a tree of height $\omega$, all of whose levels are finite, then $T$ must have a branch of length $\omega$.

Orders

Definition. A set $X$ is linearly ordered if and only if for all $a, b, c \in X$, we have $a < b$, $a = b$, or $b < a$.

Definition. A set $W$ is well-ordered if and only if every nonempty subset of $W$ has a least element.

Definition. Suppose $W$ is a well-ordered set, $\alpha$ is an ordinal, and $f : W \to \alpha$ is a bijection such that for all $a, b \in W$, we have $a < b$ if and only if $f(a) < f(b)$. Then $W$ is said to be of order type $\alpha$.

Theorem. Every well-ordered set has a unique order type.

Trees

Definition. A tree is an ordered set $(T, \lt)$ with the property that for each $x \in T$, the set $\{y \in T : y < x\}$ is well-ordered.

• The height of $x \in T$ is $h(x) = \text{order type of } \{y \in T : y < x\}$.

• The height of $T$ is $h(T) = \sup(h(x) + 1 : x \in T)$.

• The $\alpha$-th level in $T$ is $L_\alpha = \{x \in T : h(x) = \alpha\}$.

The width of $L_\alpha$ is its cardinality $|L_\alpha|$.

• A branch is a linearly ordered subset $B \subseteq T$, such that if $B \subseteq C \subseteq T$, then $C$ is not linearly ordered. The length of a branch is its order type.

Definition. An Aronszajn tree is a tree of height $\omega_1$, such that none of its levels has uncountable width, and none of its branches has uncountable length.

References

