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Investigation of negative index in dispersive, chiral materials via contra-propagating velocities under second-order dispersion (GVD)

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ABSTRACT

Negative refractive index arises typically in metamaterials via multiple routes. One such avenue is the condition where the Poynting vector of the electromagnetic wave is in opposition to the group velocity in the material. An earlier work along this route in a chiral material led to the well-known result of requiring very large (non-realizable) chirality. Thereafter, a combination of chirality together with first-order dispersion was examined using plane wave electromagnetic analysis. To arrive at the conclusions in that approach, the three wave velocities (energy, group and phase) were derived under first-order dispersion in permittivity, permeability and chirality. Negative index in this approach was established under the condition of contra-propagating group and phase velocities. Regions of negative index were found analytically by assuming standard dispersive models (such as Condon). In this paper, we will re-visit the negative index problem under higher-order dispersion. In addition, we will re-examine the plane wave propagation model under parametric dispersion where each material parameter (ϵ , μ , κ) is dispersively expanded up to the second order in frequency. Such a physical effect may be traced to group velocity dispersion (GVD) in the material. Field solutions are then obtained under the GVD effect, and extended to the evaluation of the energy, phase and group velocities.

Keywords: second order dispersion, chirality, negative index, contra-propagation, material dispersion, phase velocity, group velocity, energy velocity, higher order dispersion, parametric dispersion

1. INTRODUCTION AND BACKGROUND

Recently, the negative refractive index was examined under a combination of first-order dispersion and chirality. Conditions were established for the counter-propagation of group and phase velocities. Negative index was realizable for moderate chirality coefficients with appropriate dispersive parameters using standard models. Banerjee and Chatterjee [1] have shown that in the presence of chirality, the electric and magnetic fields are circularly polarized. They demonstrated that chirality is responsible for the circularly polarized eigenstates. Lakhtakia [2] discusses at length that a chiral material may be characterized by either a left-handedness or a right-handedness in its microstructure. The study of phase, group, and energy velocities inside a dispersive material is important because negative refractive index conditions require contra-propagation of such quantities as the Poynting vector and the electromagnetic (EM) velocities in the medium. In section 2 we rewrite the electromagnetic field solutions that were previously derived by Banerjee and Chatterjee [1]. We then examine the effect on the field components when dispersion via material parameters is considered up to second order, i.e., up to order $O(\Omega^2)$ in sideband frequency. We show that once again the fields exhibit circular polarization even under second-order dispersion. It is shown that the derived results are consistent with the cases of zero and first-order dispersion. Additionally, we find that in order to reduce the dispersive material problem to the *non-chiral* limit, one

cannot simply set the chirality parameter to zero and expect the usual results to follow. Instead, one needs to begin with the standard constitutive relations and apply those to the Helmholtz to obtain the correct result. In section 3 we formulate an expression for the energy velocity (\tilde{v}_{e3}) based on the Poynting vector and the stored energy density inside the medium. We find that the ratio of these two quantities can no longer be made independent of the (unknown) field component (\tilde{E}_{px}). As a result, when it is necessary to use \tilde{v}_{e3} , a valid form needs to be assigned to \tilde{E}_{px} in order to fully derive the dispersive characteristics of. The phase and group velocities (\tilde{v}_{p3} and \tilde{v}_{g3}) are obtained after considerable algebra in section 4; it is evident that the resulting dependence on Ω is much more extensive than in the first-order case [3]. We readily verify that the derived results converge back to the non-dispersive and first-order cases in the limit. Section 5 concludes this paper with a summary of the second-order fields and velocities and their similarities and differences w.r.t. the standard dispersionless and first-order results. Some mathematically long derivations have been appended to the end of this paper (see Appendices A and B). Extension of this work includes applying the time-dependent phasor analysis (with triple second-order dispersion) to specific cases of known dispersive models [4-6], and thereby establish regimes of negative index and the overall dependence on frequency and practical realizability of such materials. At this time, work is in progress in this direction.

2. EM SOLUTIONS IN THE PRESENCE OF DISPERSION

In this section we will re-derive briefly the possible propagation constant s and the corresponding field transformations using slowly-time-varying phasors and Fourier spectra carried out previously by Banerjee and Chatterjee [1]. To do this, we utilize the main equations and steps necessary to derive conditions under second-order dispersion. An important factor in the derivation is the use of *chiral* constitutive relations (as opposed to standard non-chiral materials) as follows:

$$\tilde{D} = \tilde{\epsilon}\tilde{E} - j\tilde{\kappa}\sqrt{\mu_0\epsilon_0}\tilde{H} \quad (1a)$$

$$\tilde{B} = j\tilde{\kappa}\sqrt{\mu_0\epsilon_0}\tilde{E} + \tilde{\mu}\tilde{H} \quad (1b)$$

where \tilde{D} , \tilde{B} , \tilde{E} and \tilde{H} are the usual field vectors written in terms of frequency-dependent material parameters; likewise, $\tilde{\kappa}(\omega)$ is the frequency-dependent chirality parameter, which is dimensionless; and $\tilde{\epsilon}(\omega)$ is called frequency dependent electric permittivity and also $\tilde{\mu}(\omega)$ is the magnetic permeability.

Start from the Maxwell's curl equations via Fourier transforms and using the constitutive relations, we obtain for the phasor fields ($\omega > 0$) the vector relation:

$$\bar{k} \times \tilde{E}_P(\Omega) = \omega[J\tilde{\kappa}_P(\Omega)\sqrt{\mu_0\epsilon_0}\tilde{E}_P(\Omega) + \tilde{\mu}_P(\Omega)\tilde{H}_P(\Omega)] \quad (2)$$

Similarly, from the other Maxwell curl equation, we obtain:

$$\bar{k} \times \tilde{H}_P(\Omega) = \omega[\tilde{\epsilon}_P(\Omega)\tilde{E}_P(\Omega) - J\tilde{\kappa}_P(\Omega)\sqrt{\mu_0\epsilon_0}\tilde{H}_P(\Omega)] \quad (3)$$

To obtain a valid set of plane-wave field solutions from the above, we assume a propagation vector \bar{k}_z strictly in the Z-direction for simplicity. This approach simplifies the resulting characteristic matrix to a 4×4 instead of a 6×6 . Using the above, it is simple to show that the longitudinal components of the fields vanish, resulting in purely transverse propagation in a bulk chiral material with triple material dispersion. The following set of well-known homogenous equations for the field components is then obtained from Eqs. (2) and (3):

$$j\tilde{\alpha}_p \tilde{E}_{px} + \tilde{k}_z \tilde{E}_{py} + \omega \tilde{\mu}_p \tilde{H}_{px} = 0, \quad (4a)$$

$$\tilde{k}_z \tilde{E}_{px} - j\tilde{\alpha}_p \tilde{E}_{py} - \omega \tilde{\mu}_p \tilde{H}_{py} = 0, \quad (4b)$$

$$\omega \tilde{\epsilon}_p \tilde{E}_{px} - j\tilde{\alpha}_p \tilde{E}_{px} - \tilde{k}_z \tilde{H}_{py} = 0, \quad (4c)$$

$$\omega \tilde{\epsilon}_p \tilde{E}_{py} - \tilde{k}_z \tilde{H}_{px} - j\tilde{\alpha}_p \tilde{H}_{py} = 0, \quad (4d)$$

where the parameter $\tilde{\alpha}_p = \omega \sqrt{\mu_0 \epsilon_0}$, which has the dimension $\text{rad } m^{-1}$, is the *chiral wavenumber*. In Eqs. (4a) - (4d), all quantities with tildes are function of the sideband frequency spread Ω about the center frequency ω_0 . Thus, $\omega (= \omega_0 + \Omega)$ is the total instantaneous frequency centered at ω_0 with Ω as the sideband. Finding non-trivial solutions for the homogenous Eqs. (4a) - (6d) requires that the determinant of the coefficient matrix must vanish. This leads to the well-known expressions for the wavenumbers in the chiral medium:

$$\tilde{k}_{z1} = +\omega \tilde{\kappa}_p \sqrt{\mu_0 \epsilon_0} + \omega \sqrt{\tilde{\mu}_p \tilde{\epsilon}_p}, \quad (5a)$$

$$\tilde{k}_{z2} = +\omega \tilde{\kappa}_p \sqrt{\mu_0 \epsilon_0} - \omega \sqrt{\tilde{\mu}_p \tilde{\epsilon}_p}, \quad (5b)$$

$$\tilde{k}_{z3} = -\omega \tilde{\kappa}_p \sqrt{\mu_0 \epsilon_0} + \omega \sqrt{\tilde{\mu}_p \tilde{\epsilon}_p}, \quad (5c)$$

$$\tilde{k}_{z4} = -\omega \tilde{\kappa}_p \sqrt{\mu_0 \epsilon_0} - \omega \sqrt{\tilde{\mu}_p \tilde{\epsilon}_p}, \quad (5d)$$

This indicates a set of four possible values of the wavenumber that satisfy the nontrivial field solutions. We may note that the \tilde{k}_z values depend on the chirality parameter $\tilde{\kappa}_p$.

We next assume that the field component \tilde{E}_{px} is known from the homogeneous set of equations (4a)-(4d). Assuming the known \tilde{E}_{px} value, the field solutions are obtained after some algebra as follows [1]:

$$\tilde{E}_{px} \text{ arbitrary}, \quad (6a)$$

$$\tilde{E}_{py} = \frac{-\omega^2 \tilde{\mu}_p \tilde{\epsilon}_p + \tilde{\alpha}_p^2 + \tilde{k}_z^2}{2j\tilde{\alpha}_p \tilde{k}_z} \tilde{E}_{px}, \quad (6b)$$

$$\tilde{H}_{px} = \frac{-\tilde{k}_z^2 + \omega^2 \tilde{\mu}_p \tilde{\epsilon}_p + \tilde{\alpha}_p^2}{2j\tilde{\alpha}_p \omega \tilde{\mu}_p} \tilde{E}_{px}, \quad (6c)$$

$$\tilde{H}_{py} = \frac{-\tilde{\alpha}_p^2 + \tilde{k}_z^2 + \omega^2 \tilde{\mu}_p \tilde{\epsilon}_p}{2\omega \tilde{\mu}_p \tilde{k}_z} \tilde{E}_{px}. \quad (6d)$$

By using the above relations for the field components, where excursion of the material parameters around a carrier frequency is assumed, we may anticipate that the field components will assume the following forms on the basis of a nonlinear expansion (up to Ω^2) in the sideband frequency:

$$\tilde{E}_{px}(\Omega), \text{ arbitrary}$$

$$\tilde{E}_{py}(\Omega) = j(A_1 + B_1\Omega + C_1\Omega^2)\tilde{E}_{px}(\Omega),$$

$$\tilde{H}_{px}(\Omega) = j(A_2 + B_2\Omega + C_3\Omega^2)\tilde{E}_{px}(\Omega),$$

$$\tilde{H}_{py}(\Omega) = (A_3 + B_3\Omega + C_3\Omega^2)\tilde{E}_{px}(\Omega). \quad (7)$$

We next use inverse Fourier transforms and write the corresponding time-dependent phasor fields *in terms of the component* $E_{px}(t)$ as follows:

$$E_{px}(t) \text{ arbitrary,}$$

$$E_{py}(t) = (jA_1\tilde{E}_{px}(t) + B_1\partial\tilde{E}_{px}(t)/\partial t - jC_1\partial^2\tilde{E}_{px}(t)/\partial t^2),$$

$$H_{px}(t) = (jA_2\tilde{E}_{px}(t) + B_2\partial\tilde{E}_{px}(t)/\partial t - jC_3\partial^2\tilde{E}_{px}(t)/\partial t^2),$$

$$H_{py}(t) = (A_3\tilde{E}_{px}(t) - jB_3\partial\tilde{E}_{px}(t)/\partial t - C_3\partial^2\tilde{E}_{px}(t)/\partial t^2), \quad (8)$$

where t is the inverse transform variable corresponding to Ω , and hence denotes “slow time .” The coefficients A_{1-3} , B_{1-3} and C_{1-3} are evaluated next.

$$\tilde{k}_{z3} = -\omega\tilde{\kappa}_p\sqrt{\mu_0\epsilon_0} + \omega\sqrt{\tilde{\mu}_p\tilde{\epsilon}_p} = -\tilde{\alpha}_p + \omega\sqrt{\tilde{\mu}_p\tilde{\epsilon}_p} \quad (9)$$

Proper dispersion results for the wavenumber \tilde{k}_{z3} are obtained via power series expansions of each of the material parameters ($\tilde{\epsilon}_p, \tilde{\mu}_p, \tilde{\kappa}_p$) up to second-order in Ω . Likewise the unknown coefficients A_{1-3} , B_{1-3} and C_{1-3} are derived to second-order. Thus we expand the material parameters in second-order Taylor expansions around $\Omega = 0$ as

$$\begin{bmatrix} \tilde{\varepsilon}_p(\Omega) \\ \tilde{\mu}_p(\Omega) \\ \tilde{\alpha}_p(\Omega) \end{bmatrix} = \begin{bmatrix} \tilde{\varepsilon}_p(\Omega) \\ \tilde{\mu}_p(\Omega) \\ \omega\sqrt{\mu_0\varepsilon_0}\tilde{\kappa}_p(\Omega) \end{bmatrix} = \left. \begin{bmatrix} \tilde{\varepsilon}_{p0} + \Omega\tilde{\varepsilon}'_{p0} + \frac{\Omega^2}{2}\tilde{\varepsilon}''_{p0} \\ \tilde{\mu}_{p0} + \Omega\tilde{\mu}'_{p0} + \frac{\Omega^2}{2}\tilde{\mu}''_{p0} \\ \omega\sqrt{\mu_0\varepsilon_0}(\tilde{\kappa}_{p0} + \Omega\tilde{\kappa}'_{p0} + \Omega^2\frac{\tilde{\kappa}''_{p0}}{2}) \end{bmatrix} \right\}. \quad (10)$$

From the second of the relations in Eq. (6b),

$$\tilde{E}_{py} = \frac{-\omega^2\tilde{\mu}_p\tilde{\varepsilon}_p + \tilde{\alpha}_p^2 + \tilde{k}_z^2}{2j\tilde{\alpha}_p\tilde{k}_z}\tilde{E}_{px}. \quad (11)$$

Incorporating the parameter expansions and carrying out extensive algebra, we may finally show:

$$A_{13} = 1, B_{13} = 0, C_{13} = 0. \quad (12)$$

If we substitute the values of A_{13} , B_{13} and C_{13} in the second relation of equation (8a), the electric field is found to be circularly polarized. This result essentially shows that whenever a medium possesses chirality, the corresponding EM fields are always circularly polarized, independent of the nature of dispersion in the material.

Proceeding similarly, we may obtain the results for the other field components as follows:

$$\tilde{H}_{px} = -j\frac{\sqrt{\tilde{\varepsilon}_p}}{\sqrt{\tilde{\mu}_p}}\tilde{E}_{px}, \quad (13)$$

so that, using Eq. (10), the corresponding coefficients are found to be:

$$\begin{aligned} A_{23} &= -\sqrt{\frac{\tilde{\varepsilon}_{p0}}{\tilde{\mu}_{p0}}}, & B_{23} &= -\sqrt{\frac{\tilde{\varepsilon}_{p0}}{\tilde{\mu}_{p0}}}\left(\frac{\tilde{\varepsilon}'_{p0}}{2\tilde{\varepsilon}_{p0}} - \frac{\tilde{\mu}'_{p0}}{2\tilde{\mu}_{p0}}\right), \text{ and} \\ C_{23} &= -\sqrt{\frac{\tilde{\varepsilon}_{p0}}{\tilde{\mu}_{p0}}}\left(\frac{1}{4}\frac{\tilde{\varepsilon}''_{p0}}{\tilde{\varepsilon}_{p0}} - \frac{1}{8}\frac{\tilde{\varepsilon}'_{p0}{}^2}{\tilde{\varepsilon}_{p0}^2} - \frac{\tilde{\mu}'_{p0}}{2\tilde{\mu}_{p0}}\frac{\tilde{\varepsilon}'_{p0}}{2\tilde{\varepsilon}_{p0}} - \frac{1}{4}\frac{\tilde{\mu}''_{p0}}{\tilde{\mu}_{p0}} + \frac{3}{8}\frac{\tilde{\mu}'_{p0}{}^2}{\tilde{\mu}_{p0}^2}\right). \end{aligned} \quad (14)$$

Finally, from the fourth relation in Eq. (6d), it follows after straightforward algebra, that

$$\begin{aligned} A_{33} &= -A_{23} = \sqrt{\frac{\tilde{\varepsilon}_{p0}}{\tilde{\mu}_{p0}}}, & B_{33} &= -B_{23} = \sqrt{\frac{\tilde{\varepsilon}_{p0}}{\tilde{\mu}_{p0}}}\left(\frac{\tilde{\varepsilon}'_{p0}}{2\tilde{\varepsilon}_{p0}} - \frac{\tilde{\mu}'_{p0}}{2\tilde{\mu}_{p0}}\right), \text{ and} \\ C_{33} &= -C_{23} = \sqrt{\frac{\tilde{\varepsilon}_{p0}}{\tilde{\mu}_{p0}}}\left(\frac{1}{4}\frac{\tilde{\varepsilon}''_{p0}}{\tilde{\varepsilon}_{p0}} - \frac{1}{8}\frac{\tilde{\varepsilon}'_{p0}{}^2}{\tilde{\varepsilon}_{p0}^2} - \frac{\tilde{\mu}'_{p0}}{2\tilde{\mu}_{p0}}\frac{\tilde{\varepsilon}'_{p0}}{2\tilde{\varepsilon}_{p0}} - \frac{1}{4}\frac{\tilde{\mu}''_{p0}}{\tilde{\mu}_{p0}} + \frac{3}{8}\frac{\tilde{\mu}'_{p0}{}^2}{\tilde{\mu}_{p0}^2}\right). \end{aligned} \quad (15)$$

We may readily show that in general $\tilde{H}_{py} = j\tilde{H}_{px}$, indicating once again circular polarization, as indicated earlier. Further, we may show that the A , B and C coefficients are mutually related as:

$$\begin{aligned} A_{11} &= -A_{13}, & B_{11} &= B_{13}, & C_{11} &= C_{13}, \\ A_{21} &= -A_{23}, & B_{21} &= -B_{23}, & C_{21} &= -C_{23}, & \text{and} \\ A_{31} &= A_{33}, & B_{31} &= B_{33}, & C_{31} &= C_{33}. \end{aligned} \quad (16)$$

It is still true that the opposite circular polarization can be observed for as Banerjee and Chatterjee explained in [1]. We note from the above that the EM fields do not exhibit any explicit dependence on the chirality; the chirality is, however, responsible for the circularly polarized eigenstates, and manifests itself explicitly in the wavenumber eigenvalues and also the velocities of propagation.

3. ENERGY VELOCITY AND POYNTING VECTOR

Using the expressions for the electric and magnetic field components expressed as time-varying phasors, we may write down an expression for the slowly time-varying Poynting vector, averaged over fast time or the carrier period. The slowly time-varying Poynting vector is given as:

$$\bar{S}_{av}(t) = \frac{1}{2} [E_p(t) \times H_p^*(t)] \hat{a}_z = \frac{1}{2} [E_{px}(t)H_{py}^*(t) - E_{py}(t)H_{px}^*(t)] \hat{a}_z. \quad (17)$$

Using the field solutions in the above relation, the corresponding slowly-varying Poynting vector becomes:

$$\bar{S}_{av3}(t) = \frac{1}{2} [E_{px}(t)H_{py}^*(t) - E_{py}(t)H_{px}^*(t)] = \frac{1}{2} \left[\begin{array}{l} \{E_{px}(t)(A_{33}E_{px}^*(t) + jB_{33}\frac{\partial E_{px}^*(t)}{\partial t} - C_{33}\frac{\partial^2 E_{px}^*(t)}{\partial t^2})\} - \\ \{jA_{13}E_{px}(t) + B_{13}\frac{\partial E_{px}(t)}{\partial t} - jC_{13}\frac{\partial^2 E_{px}(t)}{\partial t^2}\} \\ (-jA_{23}E_{px}^*(t) + B_{23}\frac{\partial E_{px}^*(t)}{\partial t} + jC_{23}\frac{\partial^2 E_{px}^*(t)}{\partial t^2}) \end{array} \right] \hat{a}_z. \quad (18)$$

Using the relations

$$A_{23} = -A_{33}, \quad B_{23} = -B_{33}, \quad C_{23} = -C_{33},$$

The Poynting vector reduces to:

$$\bar{S}_{av}(t) = \frac{1}{2} [E_{px}(t)H_{py}^* - E_{py}(t)H_{px}^*] = \{A_{33}E_{px}^2(t) + jB_{33}\frac{\partial E_{px}^2(t)}{2\partial t} - C_{33}E_{px}(t)\frac{\partial^2 E_{px}(t)}{\partial t^2}\} \hat{a}_z.$$

We can rewrite the above equation to be:

$$\bar{S}_{av3}(t) = \left\{ -A_{23} E_{px}^2(t) - j \frac{B_{23}}{2} \frac{\partial E_{px}^2(t)}{\partial t} + C_{23} E_{px}(t) \frac{\partial^2 E_{px}(t)}{\partial t^2} \right\} \hat{a}_z. \quad (19)$$

We note here that the Poynting vector above has been expressed entirely in terms of the quantity $E_{px}^2(t)$. The advantage in doing so (as was demonstrated in ref. [1]) is that by writing the total stored energy *also* in terms of the transform of $E_{px}^2(t)$, the resulting energy velocity written as a ratio of Poynting vector and stored energy then becomes independent of the field amplitude.

Assuming that $E_{px}(t)$ is real, and taking the Fourier transform of $S_{av3}(t)$, we obtain the following result in the frequency domain:

$$\tilde{S}_{av3}(\Omega) = \left\{ -A_{23} E_{px}^2(\Omega) + \Omega \frac{B_{23} E_{px}^2(\Omega)}{2} + C_{23} \frac{1}{2} [-\Omega^2 E_{px}^2(\Omega) - 2(\Omega E_{px}(\Omega) * \Omega E_{px}(\Omega))] \right\}. \quad (20)$$

To calculate the energy velocity we next need to compute the total stored energy and thereafter divide the average Poynting vector by the stored energy to find \tilde{v}_{e3} as defined below:

$$\tilde{v}_{e3}(\Omega) = \tilde{S}_{av3}(\Omega) / \tilde{w}_{e3}(\Omega). \quad (21)$$

The total stored energy is given in terms of the electric and magnetic energies as:

$$\tilde{w}_{e3}(\Omega) = \tilde{w}_{e3}(\Omega) + \tilde{w}_{m3}(\Omega). \quad (22)$$

To calculate the total stored energy we have to compute the stored electric and magnetic energies using the relations:

$$w_e(t) = \frac{1}{4} D_p^*(t) \cdot E_p(t), \quad w_m(t) = \frac{1}{4} B_p(t) \cdot H_p^*(t) \quad (23)$$

Since

$$D_p^*(t) = \left\{ [\tilde{\epsilon}_{p0} + j \frac{\partial}{\partial t} \tilde{\epsilon}'_{p0} - 2\tilde{\epsilon}''_{p0} \frac{\partial^2}{\partial t^2}] E_p(t) + \sqrt{\mu_0 \epsilon_0} (-j\tilde{k}_{p0} - \tilde{k}'_{p0} \frac{\partial}{\partial t} + j \frac{\tilde{k}''_{p0}}{2} \frac{\partial^2}{\partial t^2}) H_p(t) \right\}, \quad (24)$$

The stored electric energy is then:

$$w_{e3}(t) = \frac{1}{2} \left(\left\{ [E_{px}^2 \tilde{\epsilon}_{p0} - j E_{px}^2 \frac{\partial}{\partial t} \tilde{\epsilon}'_{p0} - 2\tilde{\epsilon}''_{p0} E_{px}^2 \frac{\partial^2}{\partial t^2}] + \sqrt{\mu_0 \epsilon_0} (\tilde{k}_{p0} - j\tilde{k}'_{p0} \frac{\partial}{\partial t} + \frac{\tilde{k}''_{p0}}{2} \frac{\partial^2}{\partial t^2}) (A_{23} E_{px}^2(t) + j B_{23} E_{px}(t) \frac{\partial E_{px}(t)}{\partial t} - C_{23} E_{px}(t) \frac{\partial^2 E_{px}(t)}{\partial t^2}) \right\} \right). \quad (25)$$

Taking the Fourier transform and retaining only up to Ω^2 , we get:

$$w_{e3}(\Omega) = \frac{1}{2} \left(\begin{array}{l} \{[\tilde{\epsilon}_{p0} + \Omega\tilde{\epsilon}'_{p0} + \Omega^2 2\tilde{\epsilon}''_{p0}] \mathfrak{I}_t[E^2_{px}(t)] + \\ \mathfrak{I}_t \{ \sqrt{\mu_0 \epsilon_0} (\tilde{k}_{p0} - j\tilde{k}'_{p0} \frac{\partial}{\partial t} + \frac{\tilde{k}''_{p0}}{2} \frac{\partial^2}{\partial t^2}) \\ (A_{23}E^2_{px}(t) + jB_{23}E_{px}(t)\partial E^2_{px}/\partial t - C_{23}E_{px}(t)\partial^2 E^2_{px}/\partial t^2) \} \} \end{array} \right) \quad (26)$$

Proceeding similarly we obtain for the magnetic stored energy:

$$w_{m3}(t) = \frac{1}{2} \left(\begin{array}{l} \{[\sqrt{\mu_0 \epsilon_0} (j\tilde{k}_{p0} + \frac{\partial}{\partial t} \tilde{k}'_{p0} + j\frac{\partial^2}{\partial t^2} \frac{\tilde{k}''_{p0}}{2}) (E_{px} H^*_{px}) + \\ (\tilde{\mu}_{p0} - j\frac{\partial}{\partial t} \tilde{\mu}'_{p0} - \frac{\partial^2}{\partial t^2} \frac{1}{2} \tilde{\mu}''_{p0}) (H_{px} H^*_{px}) \} \end{array} \right) \quad (27)$$

Hence, we obtain the total stored energy by Eqs.(21), (22), and therefrom the energy velocity as:

$$\tilde{v}_{e3}(\Omega) = \left[\frac{\{-A_{23}E^2_{px}(\Omega) + \Omega \frac{B_{23}E^2_{px}(\Omega)}{2} + C_{23} \frac{1}{2} [-\Omega^2 E^2_{px}(\Omega) - 2(\Omega E_{px}(\Omega) * \Omega E_{px}(\Omega))]\}}{\tilde{w}_{l3}(\Omega)} \right] \quad (28)$$

It is clear from the above that the energy velocity contains the *unknown* field component \tilde{E}_{px} , and hence cannot be expressed solely as a function of the sideband frequency and the dispersive parameters. As a result, calculating the frequency-dependence of the energy velocity will require the use of assumed forms of the field \tilde{E}_{px} itself. This will be explored in future work.

4. PHASE AND GROUP VELOCITIES

Using this relation $\tilde{v}_p(\Omega) = 1/(K_z / \omega)$, we are able to calculate the phase velocity by using \tilde{k}_{z3} and retaining up to the second order in Ω :

$$\tilde{v}_{p3}(\Omega) = \frac{1}{\left(-\tilde{k}_p \sqrt{\mu_0 \epsilon_0} + \sqrt{\tilde{\mu}_p \tilde{\epsilon}_p} \right)} \quad (29)$$

Using the appropriate expansions, we finally get:

$$\tilde{v}_{p3}(\Omega) = \frac{1}{\left(\begin{aligned} &\{-\tilde{\kappa}_{p0}\sqrt{\mu_0\varepsilon_0} + \sqrt{\tilde{\mu}_{p0}\tilde{\varepsilon}_{p0}}\} + \\ &\Omega\left[\frac{1}{2}\frac{(\tilde{\varepsilon}_{p0}\tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0}\tilde{\mu}_{p0})\sqrt{\tilde{\mu}_{p0}\tilde{\varepsilon}_{p0}}}{\tilde{\mu}_{p0}\tilde{\varepsilon}_{p0}} - \sqrt{\mu_0\varepsilon_0}\tilde{\kappa}'_{p0}\right] + \\ &\Omega^2\left[-\frac{1}{2}\sqrt{\mu_0\varepsilon_0}\tilde{\kappa}''_{p0} + \frac{1}{4}\frac{(2\tilde{\varepsilon}'_{p0}\tilde{\mu}'_{p0} + \tilde{\varepsilon}_{p0}\tilde{\mu}''_{p0} + \tilde{\varepsilon}''_{p0}\tilde{\mu}_{p0})\sqrt{\tilde{\mu}_{p0}\tilde{\varepsilon}_{p0}}}{\tilde{\mu}_{p0}\tilde{\varepsilon}_{p0}} \right. \\ &\left. - \frac{1}{8}\frac{(\tilde{\varepsilon}_{p0}\tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0}\tilde{\mu}_{p0})^2\sqrt{\tilde{\mu}_{p0}\tilde{\varepsilon}_{p0}}}{\tilde{\mu}_{p0}^2\tilde{\varepsilon}_{p0}^2}\right] \end{aligned} \right)} \hat{a}_z \quad (30)$$

We note from the above that (i) if we eliminate dispersion, thereby discarding all Ω - and Ω^2 -dependent terms, we recover the expected non-dispersive limit of the phase velocity under chirality; likewise (ii) if we eliminate the 2nd-order terms in frequency, the resulting phase velocity matches the previously derived result [1].

We next calculate group velocity using the definition,

$$\tilde{v}_g(\Omega) = \frac{1}{(\partial k_{z3} / \partial \Omega)} = \frac{1}{(\partial k_{z3} / \partial \omega)} \quad (31)$$

Since the sideband frequency Ω is expressible as $\Omega = \omega - \omega_0$, we write the wavenumber directly as a function of Ω (this is to enable finding the derivative of Ω , wherein dependence explicitly on the center frequency is evident) as follows:

$$\tilde{k}_{z3} = \left(\begin{aligned} &\{-(\omega_0 + \Omega)\tilde{\kappa}_{p0} - (\omega_0\Omega\tilde{\kappa}'_{p0} + \Omega^2\tilde{\kappa}''_{p0}) - (\omega_0\frac{\Omega^2}{2}\tilde{\kappa}''_{p0} + \frac{\Omega^3}{2}\tilde{\kappa}'''_{p0})\}\sqrt{\mu_0\varepsilon_0} + \sqrt{\tilde{\mu}_{p0}\tilde{\varepsilon}_{p0}}\left[(\omega_0 + \Omega) + \frac{1}{2}\frac{(\omega_0\Omega + \Omega^2)(\tilde{\varepsilon}_{p0}\tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0}\tilde{\mu}_{p0})}{\tilde{\mu}_{p0}\tilde{\varepsilon}_{p0}}\right] \\ &+ \frac{1}{4}\frac{(\omega_0\Omega^2 + \Omega^3)(2\tilde{\varepsilon}'_{p0}\tilde{\mu}'_{p0} + \tilde{\varepsilon}_{p0}\tilde{\mu}''_{p0} + \tilde{\varepsilon}''_{p0}\tilde{\mu}_{p0})}{\tilde{\mu}_{p0}\tilde{\varepsilon}_{p0}} - \frac{1}{8}\frac{(\omega_0\Omega^2 + \Omega^3)(\tilde{\varepsilon}_{p0}\tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0}\tilde{\mu}_{p0})^2}{\tilde{\mu}_{p0}^2\tilde{\varepsilon}_{p0}^2} + \frac{1}{16}\left[\frac{\tilde{\varepsilon}_{p0}\tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0}\tilde{\mu}_{p0}}{\tilde{\mu}_{p0}\tilde{\varepsilon}_{p0}}\right]^3\Omega^3 \end{aligned} \right) \quad (32)$$

Using the above expression for the wavenumber and its derivative, the group velocity is obtained after considerable algebra as:

$$\tilde{v}_{g3}(\Omega) = \left[\begin{aligned} & \{ -\tilde{\kappa}_{p0} \sqrt{\mu_0 \varepsilon_0} - \omega_0 \tilde{\kappa}'_{p0} \sqrt{\mu_0 \varepsilon_0} + \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} \frac{\omega_0}{2} \frac{(\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})}{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} + \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} \} + \\ & \Omega \{ -2\tilde{\kappa}'_{p0} \sqrt{\mu_0 \varepsilon_0} - \omega_0 \tilde{\kappa}''_{p0} \sqrt{\mu_0 \varepsilon_0} + \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} \frac{(\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})}{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} \\ & + \frac{\sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}}}{2} \frac{(\omega_0)(2\tilde{\varepsilon}'_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}_{p0} \tilde{\mu}''_{p0} + \tilde{\varepsilon}''_{p0} \tilde{\mu}_{p0})}{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} - \frac{1}{4} \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} \frac{(\omega_0)(\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})^2}{\tilde{\mu}_{p0}^2 \tilde{\varepsilon}_{p0}^2} \} + \\ & \Omega^2 \{ -\frac{3}{2} \sqrt{\mu_0 \varepsilon_0} \tilde{\kappa}''_{p0} + \frac{3}{4} \frac{(2\tilde{\varepsilon}'_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}_{p0} \tilde{\mu}''_{p0} + \tilde{\varepsilon}''_{p0} \tilde{\mu}_{p0}) \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}}}{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} - \\ & \frac{3}{8} \frac{(\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})^2 \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}}}{\tilde{\mu}_{p0}^2 \tilde{\varepsilon}_{p0}^2} + \frac{3}{16} [\frac{\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0}}{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}}]^3 \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} \} \end{aligned} \right] \hat{a}_z \quad (33)$$

We find from the above that under zeroth-order expansion, the group velocity appears to retain a few dispersive coefficients. This may seem to be confusing at first; however, remembering that for the non-dispersive limit, *all dispersive coefficients must vanish*, we obtain a limiting group velocity that matches the non-dispersive phase velocity. Similarly, eliminating the second-order frequency terms, the above group velocity is seen to reduce to the first-order result as predicted in ref. [1].

In specific follow-up to the phase and group velocities, we will apply known dispersive models to combinations of the permittivity, permeability and chirality, and thereby ascertain if regions of opposition in these velocities exist within the material. Finding such regions would then establish appropriate environments for negative refractive index. Additionally, we will need to determine if such negative index conditions are physically realizable.

5. CONCLUSIONS

We have extended the results from ref. [1] to include dispersive effects on the phase, group and energy velocities in a general, dispersive material up to the second order. A slowly varying phasor approach is combined with Fourier transforms over the sideband frequency, along with dispersive parameters expanded to second-order in frequency. It is shown that the resulting plane wave solutions are circularly polarized fields, independent of dispersion. Fields are expressed as power series expansions in terms of a common field component (the X-component of the electric field in this work). The field and the wavenumber solutions are then used to derive dispersive phase and group velocities. It is found that calculating the energy velocity would require assumed valid forms for the common field component. Future work will involve examining the dispersive results applied to known dispersive models, including Condon and Lorenz, and thereby determining conditions for the emergence of negative refractive index for such materials.

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APPENDIX A

- To prove that $\tilde{H}_{py} = j\tilde{H}_{px}$ under second- order dispersion

$$\tilde{H}_{py} = \frac{j\tilde{\alpha}_p(-\tilde{\alpha}_p^2 + \tilde{k}_z^2 + \omega^2 \tilde{\mu}_p \tilde{\varepsilon}_p)}{\tilde{k}_z(-\tilde{k}_z^2 + \omega^2 \tilde{\mu}_p \tilde{\varepsilon}_p + \tilde{\alpha}_p^2)} \tilde{H}_{px} \quad (I)$$

$$\sqrt{\tilde{\mu}_p \tilde{\varepsilon}_p} = \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} \left[1 + \frac{1}{2} \frac{\Omega(\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})}{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} + \frac{1}{4} \Omega^2 \frac{(2\tilde{\varepsilon}'_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}_{p0} \tilde{\mu}''_{p0} + \tilde{\varepsilon}''_{p0} \tilde{\mu}_{p0})}{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} - \frac{1}{8} \frac{(\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})^2}{\tilde{\mu}_{p0}^2 \tilde{\varepsilon}_{p0}^2} \Omega^2 \right],$$

$$\tilde{\alpha}_p^2 = \omega^2 \tilde{k}_p^2 \mu_0 \varepsilon_0$$

$$\tilde{\varepsilon}_p = \tilde{\varepsilon}_{p0} + \Omega \tilde{\varepsilon}'_{p0} + \frac{\Omega^2}{2} \tilde{\varepsilon}''_{p0} \quad \text{----- (1)}$$

$$\tilde{\mu}_p = \tilde{\mu}_{p0} + \Omega \tilde{\mu}'_{p0} + \frac{\Omega^2}{2} \tilde{\mu}''_{p0} \quad \text{----- (2)}$$

$$\tilde{k}_p = \tilde{k}_{p0} + \Omega \tilde{k}'_{p0} + \frac{\Omega^2}{2} \tilde{k}''_{p0} \quad \text{----- (3)}$$

$$\tilde{\mu}_p \tilde{\varepsilon}_p = \tilde{\mu}_{p0} \tilde{\varepsilon}_{p0} \left[1 + \frac{\Omega(\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})}{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} + \frac{1}{4} \Omega^2 \frac{(2\tilde{\varepsilon}'_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}_{p0} \tilde{\mu}''_{p0} + \tilde{\varepsilon}''_{p0} \tilde{\mu}_{p0})}{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} - \frac{1}{8} \frac{(\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})^2}{\tilde{\mu}_{p0}^2 \tilde{\varepsilon}_{p0}^2} \Omega^2 + \frac{1}{4} \frac{\Omega^2 (\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})^2}{\tilde{\mu}_{p0}^2 \tilde{\varepsilon}_{p0}^2} + \frac{1}{4} \Omega^2 \frac{(2\tilde{\varepsilon}'_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}_{p0} \tilde{\mu}''_{p0} + \tilde{\varepsilon}''_{p0} \tilde{\mu}_{p0})}{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} - \frac{1}{8} \frac{(\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})^2}{\tilde{\mu}_{p0}^2 \tilde{\varepsilon}_{p0}^2} \Omega^2 \right]$$

Let $N = j\tilde{\alpha}_p (-2\omega\tilde{\alpha}_p \sqrt{\tilde{\mu}_p \tilde{\varepsilon}_p} + 2\omega^2 \tilde{\mu}_p \tilde{\varepsilon}_p)$ and $D = \tilde{k}_z (-\tilde{k}_z^2 + \omega^2 \tilde{\mu}_p \tilde{\varepsilon}_p + \tilde{\alpha}_p^2)$.

Where N is the numerator and D is the denominator for equation (I)

$$ND^{-1} = j \left\{ \left[(-2\omega^3 \mu_0 \varepsilon_0 \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}}) \tilde{k}_{p0}^2 + 2\omega^3 \tilde{\mu}_{p0} \tilde{\varepsilon}_{p0} \sqrt{\mu_0 \varepsilon_0} \tilde{k}_{p0} \right] \frac{1}{[-2\omega^3 \mu_0 \varepsilon_0 \tilde{k}_{p0}^2 \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} + 2\omega^3 \tilde{\mu}_{p0} \tilde{\varepsilon}_{p0} \tilde{k}_{p0} \sqrt{\mu_0 \varepsilon_0}]} - \Omega \left[\left[(-2\omega^3 \mu_0 \varepsilon_0 \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}}) \tilde{k}_{p0}^2 + 2\omega^3 \tilde{\mu}_{p0} \tilde{\varepsilon}_{p0} \sqrt{\mu_0 \varepsilon_0} \tilde{k}_{p0} \right] \left[(-2\omega^3 \mu_0 \varepsilon_0 \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}}) \tilde{k}_{p0}^2 + 2\omega^3 \tilde{\mu}_{p0} \tilde{\varepsilon}_{p0} \sqrt{\mu_0 \varepsilon_0} \tilde{k}_{p0} \right]^{-1} \right] \right\}$$

$$\frac{-2\omega^3 \mu_0 \varepsilon_0 \tilde{k}_{p0}^2 \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} (\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})}{2 \tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} - 2\omega^3 \mu_0 \varepsilon_0 (2\tilde{k}_{p0} \tilde{k}'_{p0}) \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} + \frac{2\omega^3 \tilde{\mu}_{p0} \tilde{\varepsilon}_{p0} \tilde{k}_{p0} \sqrt{\mu_0 \varepsilon_0} (\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})}{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} + \tilde{k}_{p0}^2 \omega^3 \sqrt{\mu_0 \varepsilon_0} \tilde{\mu}_{p0} \tilde{\varepsilon}_{p0} \left[\right]$$

$$- \Omega^2 \left\{ \begin{aligned} & \left[\frac{(-2\omega^3 \mu_0 \varepsilon_0 \tilde{k}_{p0}^2 \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}}) (2\tilde{\varepsilon}'_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}_{p0} \tilde{\mu}''_{p0} + \tilde{\varepsilon}''_{p0} \tilde{\mu}_{p0})}{4 \tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} - \frac{(-2\omega^3 \mu_0 \varepsilon_0 \tilde{k}_{p0}^2 \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}}) (\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})^2}{8 \tilde{\mu}_{p0}^2 \tilde{\varepsilon}_{p0}^2} \right. \\ & \left. + \frac{(-2\omega^3 \mu_0 \varepsilon_0 (2\tilde{k}_{p0} \tilde{k}'_{p0})) \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} (\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})}{2 \tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} - 2\omega^3 \mu_0 \varepsilon_0 (\tilde{k}_{p0}^2 + \tilde{k}_{p0} \tilde{k}''_{p0}) \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} \right] \end{aligned} \right\}$$

APPENDIX B

- To prove that $\tilde{E}_{py} = j\tilde{E}_{px}$ under second- order dispersion

$$\tilde{E}_{py} = \frac{-\omega^2 \tilde{\mu}_p \tilde{\varepsilon}_p + \tilde{\alpha}_p^2 + \tilde{k}_z^2}{2j\tilde{\alpha}_p \tilde{k}_z} \tilde{E}_{px}$$

By using the same steps above when we calculated $\tilde{H}_{py} = j\tilde{H}_{px}$

Collect Ω terms:

$$\begin{aligned} & \{2\mu_0 \varepsilon_0 \tilde{k}_{p0} \tilde{k}'_{p0} - \frac{(\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})}{2\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} \tilde{k}_{p0} \sqrt{\mu_0 \varepsilon_0} \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} - \sqrt{\mu_0 \varepsilon_0} \tilde{k}'_{p0} \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}}\} \\ + \Omega j [& \frac{(\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})}{2\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} \tilde{k}_{p0} \sqrt{\mu_0 \varepsilon_0} \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} - \sqrt{\mu_0 \varepsilon_0} \tilde{k}'_{p0} \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} \\ & \frac{(\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})}{2\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} \tilde{k}_{p0} \sqrt{\mu_0 \varepsilon_0} \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} - \sqrt{\mu_0 \varepsilon_0} \tilde{k}'_{p0} \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} \\ & \frac{1}{2} \frac{(\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0}) \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} \sqrt{\mu_0 \varepsilon_0}}{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} + \tilde{k}'_{p0} \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} \sqrt{\mu_0 \varepsilon_0} - 2\mu_0 \varepsilon_0 \tilde{k}_{p0} \tilde{k}'_{p0} \} \\ & \frac{(\tilde{\varepsilon}_{p0} \tilde{\mu}'_{p0} + \tilde{\varepsilon}'_{p0} \tilde{\mu}_{p0})}{2\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} \tilde{k}_{p0} \sqrt{\mu_0 \varepsilon_0} \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}} - \sqrt{\mu_0 \varepsilon_0} \tilde{k}'_{p0} \sqrt{\tilde{\mu}_{p0} \tilde{\varepsilon}_{p0}}] = 0 \end{aligned}$$

And also Ω^2 terms equals to zero and we just have the constant term which equals to j.