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The Cantor Set, Trees, and Compact Metric Spaces

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Abstract

The Cantor Set is a famous topological set developed from an infinite process of starting with the interval [0,1] and, at each iteration, removing the middle third of the intervals remaining. Our goal is to determine some of the properties of this unintuitive set and to show that it is homeomorphic to any general compact metric space with similar properties. To do so, we show that the Cantor Set is topologically equivalent to a tree, a more familiar structure, and use this fact to establish a homeomorphism to the general compact metric space.

Dedication and Acknowledgements

I dedicate this paper to my mother Kathleen and my older brother Brian, their continued support and encouragement is always greatly appreciated. I would also like to thank my thesis advisors. Dr. Mashburn, I thank you for providing the material and expertise necessary for me to take on this thesis, and thank you Dr. Brown for taking the time and helping me see this through to the end.

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1. Topological Background

We will begin this paper by providing some background in the field of topology. This background includes definitions, lemmas, and theorems that a student in an introductory course in topology would likely see.

We begin with the notion of orders on sets. These relations allow us to organize the elements of a set as necessary.

Definition 1.1. A relation $<$ on a set X is an $order$ on X if and only if it satisfies the following properties.

(1) For every $a, b \in X$, if $a < b$ then $b \not< a$

(2) For every $a, b, c \in X$, if $a < b$ and $b < c$ then $a < c$

Lemma 1.2. *The subset relation on a collection of sets $\mathcal{A}$ is an order on $\mathcal{A}$.*

Proof. Let $X, Y, Z \in \mathcal{A}$. First, let $X < Y$ if and only if $X \subset Y$. Suppose $X < Y$, so $X \subset Y$. Since $X \neq Y$, there exists $a \in Y$ such that $a \notin X$, thus $Y \not\subset X$, so $Y \not< X$ Suppose $X < Y$ and $Y < Z$, so $X \subset Y \subset Z$, therefore $X < Z$.

Now let $X < Y$ if and only if $Y \subset X$. Suppose $X < Y$, so $Y \subset X$. Since $Y \neq X$, there exists $b \in X$ such that $b \notin Y$, thus $X \not\subset Y$ and $Y \not< X$. Finally suppose $X < Y$ and $Y < Z$. Then, $Z \subset Y \subset X$, so $X < Z$. Therefore, in either direction, the subset relation is an order on $\mathcal{A}$. □

Definition 1.3. An order $<$ on a set X is a $linear\ order$ if and only if for every $a, b \in X$, with $a \neq b$, either $a < b$ or $b < a$. If $<$ is a linear order on a set X then we say that $\langle X, < \rangle$ is a $linearly\ ordered\ set$.

Definition 1.4. Let X be an ordered set. An element a of X is a $minimal$ element of X if and only if there is no $b \in X$ such that $b < a$. An element a of X is the $minimum$ element of X if and only if $a < b$ for all $b \in X - \{a\}$.

Definition 1.5. A linearly ordered set X is $well-ordered$ if and only if every nonempty subset of X has a least, or minimum, element.

We now define what is meant by topology.

Definition 1.6. A *topology* on a set X is a collection τ of subsets of X satisfying the following properties

(1) $\emptyset, X \in \tau$

(2) If $\mathcal{U} \subseteq \tau$ then $\bigcup \mathcal{U} \in \tau$

(3) If $\mathcal{U}$ is a nonempty finite subset of τ then $\bigcap \mathcal{U} \in \tau$

A simple induction argument will show that property (3) can be replaced by the following statement. If $U, V \in \tau$ then $U \cap V \in \tau$. A subset of X is said to be *open* in τ if and only if it is an element of τ. A set with a topology will be known as a $topological\ space$.

Definition 1.7. A collection $\mathcal{A}$ of subsets of a set X is a $subbasis\ for\ a\ topology\ on\ X$ if and only if $\bigcup \mathcal{A} = X$.

Definition 1.8. Let X be a linearly ordered set. The $order\ topology$ on X is the topology generated by the subbasis $\mathcal{A} = \{(-\infty, a) : a \in X\} \cup \{(a, \infty) : a \in X\}$.

Lemma 1.9. *Let X be a linearly ordered set. The set defined above, $\mathcal{A} = \{(-\infty, a) : a \in X\} \cup \{(a, \infty) : a \in X\}$ is a subbasis for a topology on X.*

Proof. Let $b \in X$. Then, $(-\infty, b) \cup (b, \infty) \in \mathcal{A}$. Let $a \in X$ with $a \neq b$, and define $B = (-\infty, b) \cup (b, \infty)$ and $A = (-\infty, a) \cup (a, \infty)$. So, $B = X - \{b\}$ and $A = X - \{a\}$, so $A \cup B = (X - \{a\}) \cup (X - \{b\}) = X$. So, the union of any two sets in $\mathcal{A}$ is equal to X, therefore, $\bigcup \mathcal{A} = X$, so $\mathcal{A}$ is a subbasis for a topology on X, namely, the order topology. □

We say that a linear ordered set X with the order topology is a $linearly\ ordered\ topological\ space$.

Definition 1.10. A $neighborhood$ of a point p in a space X is an open subset of X that contains p.

Lemma 1.11. *Let τ be a topology on a set X. Let $U \subseteq X$. If for every $x \in U$ there is $V \in \tau$ such that $x \in V \subseteq U$ then $U \in \tau$.*

Proof. For every element $x \in U$, there is some $V(x) \in \tau$ such that $x \in V(x) \subseteq U$. So $U \subseteq \bigcup_{x \in U} V(x)$. But $V(x) \subseteq U$ for all $x \in U$, so $\bigcup_{x \in U} V(x) \subseteq U$ and thus $\bigcup_{x \in U} V(x) = U$. Since $V(x) \in \tau$ for all $x \in U$, then $\bigcup_{x \in U} V(x) \in \tau$, and so $U \in \tau$. □

Definition 1.12. A collection $\mathcal{B}$ of subsets of a set X is a $basis$ for a topology on X if and only if $\mathcal{B}$ satisfies the following properties.

(1) For every $p \in X$ there is $B \in \mathcal{B}$ such that $p \in B$ (In other words, $\bigcup \mathcal{B} = X$)

(2) For every $A, B \in \mathcal{B}$ and $x \in A \cap B$ there is $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$

Lemma 1.13. *If $\mathcal{B}$ is a collection of subsets of a set X such that $\bigcup \mathcal{B} = X$ and $A \cap B \in \mathcal{B}$ for all $A, B \in \mathcal{B}$ then $\mathcal{B}$ is a basis for a topology on X.*

Proof. $\bigcup \mathcal{B} = X$, so for every $p \in X$ there is $B \in \mathcal{B}$ such that $p \in B$ and thus the first property has been satisfied. Let $C = A \cap B$, then $C \in \mathcal{B}$. If $x \in A \cap B$ then $x \in C \subseteq A \cap B$. Therefore, $\mathcal{B}$ is a basis for a topology on X. □

Lemma 1.14. *If $\mathcal{B}$ is a basis for a topology on X then $\tau = \{\bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{B}\}$ is a topology on X. This is called the topology generated by $\mathcal{B}$.*

Proof. To satisfy the first property, observe that $\emptyset \subseteq \mathcal{B}$, then $\bigcup \emptyset = \emptyset \in \tau$. Also, $\mathcal{B} \subseteq \mathcal{B}$, so $\bigcup \mathcal{B} = X \in \tau$. Next, let $\mathcal{U} \subseteq \tau$, so $\mathcal{U} = \bigcup \mathcal{A}$ with $\mathcal{A} \subseteq \mathcal{B}$. Then, $\bigcup \mathcal{U} = \bigcup(\bigcup \mathcal{A}) = \bigcup \mathcal{A} \in \tau$. Finally, let $\mathcal{U}$ be a nonempty finite subset of τ, so $\bigcup \mathcal{U} \in \tau$. Then $\bigcap \mathcal{U} = \bigcap(\bigcup \mathcal{A}) = \bigcup \mathcal{A} \in \tau$. Therefore τ is a topology on X. □

Now we will define an important topology that is used to describe open sets of a product of any number of spaces.

Definition 1.15. Let X_n be a nonempty space with topology $\tau(X_n)$ for every n in a nonempty indexing set I. The $product\ topology\ on$ $\Pi_{n \in I} X_n$ is the topology generated by the subbasis $\mathcal{A}$ which consists of all sets of the form $\pi_n^{-1}[U_n]$ where π_n is the projection of $\Pi_{m \in I} X_m$ onto X_n and $U_n \in \tau(X_n)$.

To better visualize what the open sets of this topology look like, let $I = \omega$, where ω is the first inductive ordinal. Then $\pi_n^{-1}[U_n] = X_0 \times X_1 \times \cdots \times X_{n-1} \times U_n \times X_{n+1} \times \cdots$. In general, $\pi_n^{-1}[U] = \Pi_{m \in \omega} U_m$ where $U_m = X_m$ when $m \neq n$ and $U_n = U$. All of the factors of this set will be the whole space except the n^{th} factor, which will be the open set U. The elements of this set will be every function $f \in \Pi_{m \in \omega} X_m$ such that $f(n) \in U$.

The basis that we get from this sub-basis is the set of all intersections of nonempty finite subsets of the sub-basis. Let J be a nonempty finite subset of I and for every $n \in J$ let U_n be open in X_n. A typical basis element then has the form $\bigcap_{n \in J} \pi_n^{-1}[U_n]$. To get a better picture of this set, let us again use $I = \omega$ and let $J = \{2, 5, 7, 8\}$. Then $\bigcap_{n \in J} \pi_n^{-1}[U_n] = X_0 \times X_1 \times U_2 \times X_3 \times X_4 \times U_5 \times X_6 \times U_7 \times U_8 \times X_9 \times \cdots$.

An element of this set is a function f on ω such that $f(n)$ can be any element of X_n when $n \notin J$, and $f(n) \in U_n$ when $n \in J$. We can think of the U_n's as gates through which the function f must pass in order to be in this set. A subset V of $\Pi_{n \in I} X_n$ is open in the product topology if and only if for every function $f \in V$ there is a finite subset J of I and for every $n \in J$ there is an open subset U_n of X_n such that $f \in \Pi_{n \in J} \pi_n^{-1}[U_n] \subseteq V$.

Lemma 1.16. *If X is a space with topology $\tau(X)$ and $Y \subseteq X$ then the collection $\tau(Y) = \{U \cap Y : U \in \tau(X)\}$ is a topology on Y.*

Proof. First, $\emptyset \in \tau(X)$ and $\emptyset \cap Y = \emptyset$, so $\emptyset \in \tau(Y)$. Also, $X \in \tau(X)$ and $X \cap Y = Y$, so $Y \in \tau(Y)$. Now, let $\mathcal{V} \subseteq \tau(Y)$. So $\mathcal{V}$ is of the form $\mathcal{U} \cap Y$ where $\mathcal{U}$ is a collection of open sets U in $\tau(X)$. Then $\bigcup \mathcal{V} = \bigcup(\mathcal{U} \cap Y) = (\bigcup \mathcal{U}) \cap Y$. Since $\tau(X)$ is a topology, $\bigcup \mathcal{U} \in \tau(X)$. Therefore, $\bigcup \mathcal{V} \in \tau(Y)$. Lastly, let $\mathcal{V}$ be a nonempty finite subset of $\tau(Y)$, so $\bigcap \mathcal{V} = \bigcap(\mathcal{U} \cap Y) = (\bigcap \mathcal{U} \cap Y)$ where $\mathcal{U}$ is a nonempty finite subset of $\tau(X)$. Since $\tau(X)$ is a topology, $\bigcap \mathcal{U} \in \tau(X)$ so $\bigcap \mathcal{V} \in \tau(Y)$. Thus, $\tau(Y)$ is a topology on Y. $\square$

This topology $\tau(Y)$ is called the *subspace topology* on Y, or the topology that Y inherits from X. We say that Y is a subspace of X when it has this topology.

Lemma 1.17. *If X is a space with basis $\mathcal{B}$ and $Y \subseteq X$ then $\mathcal{A} = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.*

Proof. First let $p \in Y$, then $p \in X$. So, there is $B \in \mathcal{B}$ such that $p \in B$. Thus, $p \in B \cap Y$. Therefore, there is some $A \in \mathcal{A}$ such that $p \in A$. Now let $A_1, A_2 \in \mathcal{A}$ and let $p \in A_1 \cap A_2$. A_1 and A_2 are of the form $B \cap Y$, where $B \in \mathcal{B}$. Let $A_1 = B_1 \cap Y$ and $A_2 = B_2 \cap Y$, with $B_1, B_2 \in \mathcal{B}$. Thus, $p \in A_1 \cap A_2$, so $p \in (B_1 \cap Y) \cap (B_2 \cap Y) = B_1 \cap B_2 \cap Y$. There is some $B_3 \in \mathcal{B}$ such that $p \in B_3 \subseteq B_1 \cap B_2$, so $p \in B_3 \cap Y$, with $B_3 \in \mathcal{B}$. Therefore, there is $A_3 \in \mathcal{A}$, $A_3 = B_3 \cap Y$, such that $p \in A_3$. Since $B_3 \subseteq B_1 \cap B_2$, $B_3 \cap Y = A_3 \subseteq B_1 \cap B_2 \cap Y = A_1 \cap A_2$. Thus, $\mathcal{A}$ is a basis for the subspace topology on Y. $\square$

If Y is a subspace of X then $U \subseteq Y$ is open in Y if and only if U belongs to the subspace topology on Y. U is open in X if and only if it belongs to the topology of X.

Lemma 1.18. *Let X be a space and Y a subspace of X. Let $U \subseteq Y$. If U is open in Y and Y is open in X, then U is open in X.*

Proof. There is some V open in X such that $U = V \cap Y$, and $V \cap Y$ is a finite union of open sets in X so U is open in X. $\square$

Definition 1.19. A subset C of a space X is *closed* if and only if $X - C$ is open.

Theorem 1.20. *Let X be a space.*
(1) $\emptyset$ and X are closed subsets of X
(2) If $\mathcal{C}$ is a nonempty collection of closed subsets of X then $\bigcap \mathcal{C}$ is closed
(3) If $\mathcal{C}$ is a finite collection of closed subsets of X then $\bigcup \mathcal{C}$ is closed.

Proof. First, $X - \emptyset = X$ is open and $X - X = \emptyset$ is open, therefore $\emptyset$ and X are also both closed. Now let $\mathcal{C}$ be a nonempty collection of closed subsets of X. Then we have that $X - \bigcap \mathcal{C} = X - \bigcap_{C \in \mathcal{C}} C = \bigcup_{C \in \mathcal{C}} (X - C)$. Since each $X - C$ is open, the union of these sets is also open, and thus $\bigcap \mathcal{C}$ is closed. Finally, let $\mathcal{C}$ be a finite collection of closed subsets of X. Then, $X - \bigcup \mathcal{C} = \bigcap_{C \in \mathcal{C}} (X - C)$. Since a finite intersection of open sets is open, $\bigcup \mathcal{C}$ is closed. □

Theorem 1.21. *Let Y be a subspace of a space X. A subset A of Y is closed in Y if and only if there is a closed subset B of X such that $A = Y \cap B$.*

Proof. Let A be a closed subset of Y. Then $Y - A$ is open, so there is an open subset U of X such that $Y - A = Y \cap U$. Let $B = X - U$.

$$B \cap Y = (X - U) \cap Y = Y - U = Y - (Y \cap U) = Y - (Y - A) = A$$

Now assume that there is a closed subset B of X such that $A = Y \cap B$. Let $U = X - B$. Then U is an open subset of X.

$$Y - (Y \cap U) = Y - [Y \cap (X - B)] = Y - (X - B) = Y \cap B = A$$

Therefore, A is closed in Y. □

Theorem 1.22. *If Y is a closed subset of a space X and A is a closed subset of Y then A is closed in X.*

Proof. $X - Y$ is open in X and $Y - A$ is open in Y, so $Y - A$ is open in X. Since $X - A = (Y - A) \cup (X - Y)$, a union of two open sets, so $X - A$ is open in X, thus A is closed in X. □

Definition 1.23. A space X is $Hausdorff$ if and only if for every $p, q \in X$ there is a neighborhood U of p and a neighborhood V of q such that $U \cap V = \emptyset$.

Theorem 1.24. *Every linearly ordered topological space is Hausdorff.*

Proof. Let X be a linearly ordered topological space. Let $p, q \in X$ such that $p < q$. Let $A = \{x \in X : p < x < q\}$. If A is empty then then $p \in (-\infty, q)$ and $q \in (p, \infty)$ with $(-\infty, q) \cap (p, \infty) = \emptyset$, so X is Hausdorff. If A is nonempty then $p \in (-\infty, x)$ and $q \in (x, \infty)$ for any $x \in A$, and $(-\infty, x) \cap (x, \infty) = \emptyset$, so X is Hausdorff. □

Theorem 1.25. *Every subspace of a Hausdorff space is also Hausdorff.*

Proof. Let X be a Hausdorff space and let $Y \subseteq X$. Let $p, q \in Y$, so $p, q \in X$. Thus, there is a neighborhood U of p and a neighborhood V of q in $\tau(X)$ such that $U \cap V = \emptyset$. Then $U \cap Y \in, V \cap Y \tau(Y)$, with $p \in U \cap Y$ and $q \in V \cap Y$ and

$$(U \cap Y) \cap (V \cap Y) = U \cap V \cap Y = \emptyset \cap Y = \emptyset.$$

Thus, Y is also Hausdorff. □

Lemma 1.26. *If X_n is Hausdorff for all $n \in \omega$, then $\Pi_{n \in \omega} X_n$ is Hausdorff.*

Proof. Let f and g be distinct in $\Pi_{n\in\omega}X_n$. Then, there is $m \in \omega$ such that $f(m) \neq g(m)$. X_m is Hausdorff so there are open sets U_m and V_m in X_m such that $f(m) \in U_m$ and $g(m) \in V_m$ with $U_m \cap V_m = \emptyset$. Define $U = \Pi_{n\in\omega}U_n$ and $V = \Pi_{n\in\omega}V_n$ with $U_n = V_n = X_n$ for all $n \neq m$. Then, U and V are open in the product topology on $\Pi_{n\in\omega}X_n$ with $f \in U$ and $g \in V$. It must be that $U \cap V = \emptyset$, because if $h \in U \cap V$ then $h(m) \in U_m \cap V_m$, which cannot be true. Thus, $\Pi_{n\in\omega}X_n$ is also Hausdorff. $\square$

The concept of continuity is found in many fields of mathematics. A more abstract notion of continuity is the continuity of functions between topological spaces.

Definition 1.27. A function f from a space X into a space Y is *continuous* if and only if $f^{-1}[V]$ is open in X for every open subset V of Y.

Theorem 1.28. *Let X and Y be spaces. Let $f : X \to Y$. The following statements are equivalent.*
(1) *f is continuous*
(2) *For every closed subset C of Y, $f^{-1}[C]$ is closed in X*
(3) *For every $p \in X$ and every neighborhood V of $f(p)$ there is a neighborhood U of p such that $f[U] \subseteq V$.*

Proof. Let C be closed in Y, then $Y - C$ is open in Y.

$$\begin{aligned} p \in f^{-1}[Y - C] &\iff f(p) \in Y - C \\ &\iff f(p) \notin C \\ &\iff p \notin f^{-1}[C] \\ &\iff p \in X - f^{-1}[C] \end{aligned}$$

Thus, $f^{-1}[Y - C] = X - f^{-1}[C]$, an open set in X. Therefore f is continuous, so we have shown $(1) \iff (2)$.

Now we show $(1) \iff (3)$. First let f be continuous, and let $p \in X$ and V a neighborhood of $f(p)$ in Y. Then, $f^{-1}[V]$ is open in X and $p \in f^{-1}[V]$. Let $U = f^{-1}[V]$.

$$\begin{aligned} f(p) \in f[U] &\iff f(p) \in f[f^{-1}[V]] \\ &\implies f(p) \in V \end{aligned}$$

Thus, $f[U] \subset V$.

Now, let V be an open set in Y. Let $p \in f^{-1}[V]$. There exists U, such that $f(p) \in f[U] \subseteq V$. So, $p \in U \subseteq f^{-1}[V]$ and by Lemma 1.11, $f^{-1}[V]$ is open and thus, f is continuous. $\square$

From a topological point of view, the way we establish equivalence between two spaces is through a function known as a homeomorphism.

Definition 1.29. Let X and Y be spaces. A function $f : X \to Y$ is a *homeomorphism* if and only if f is one-to-one, onto, continuous, and f^{-1} is continuous.

The spaces X and Y are said to be *homeomorphic* if and only if there is some homeomorphism $f : X \to Y$. Note that if f is a homeomorphism, f^{-1} is also a homeomorphism.

If X and Y are homeomorphic, then they look exactly the same topologically. A subset U of X will be open in X if and only if $f[U]$ is open in Y.

Lemma 1.30. *If X and Y are homeomorphic and X is Hausdorff, then Y is Hausdorff.*

Proof. For every $p, q \in X$ there is a neighborhood U of p and V of q such that $U \cap V = \emptyset$. Let f be the homeomorphism between X and Y. Since f is continuous, $f[U]$ is a neighborhood of $f(p)$ and $f[V]$ a neighborhood of $f(q)$. Then $f[U] \cap f[V] = f[U \cap V] = f[\emptyset] = \emptyset$. Thus, Y is also Hausdorff. $\square$

Lemma 1.31. *Let X, Y and Z be spaces. If $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms then $g \circ f$ is a homeomorphism.*

Proof. Since f and g are both onto, then $g \circ f$ is onto. Let $a, b \in X$ with $a \neq b$. Then $f(a) \neq f(b)$ and $g(a) \neq g(b)$, so $g \circ f$ is one-to-one. Then, $(g \circ f)^{-1}$ is a function. Let U be a open subset in Z, then $g^{-1}[U]$ is open in Y, and $f^{-1}[g^{-1}[U]]$ is open in X, so $(g \circ f)^{-1}[U]$ is open in X and $g \circ f$ is continuous. Similarly, if U is an open set in X then $f[U]$ is open in Y, and so $g[f[U]]$ is open in Z. So, $(g \circ f)[U]$ is open in Z and $(g \circ f)^{-1}$ is also continuous. Thus, $g \circ f$ is a homeomorphism. $\square$

Another important topological concept is that of a metric on a set.

Definition 1.32. A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties.
(1) For every $p, q \in X$, $d(p, q) \geq 0$ and $d(p, q) = 0$ if and only if $p = q$.
(2) For every $p, q \in X$, $d(p, q) = d(q, p)$.
(3) (Triangle Inequality) For every $p, q, r \in X$, $d(p, r) \leq d(p, q) + d(q, r)$.

A common way of understanding a metric d on a set X is to think of $d(p, q)$ as the distance between p and q. A set X along with a metric d is called a *metric space*.

Definition 1.33. If (X, d) is a metric space, $p \in X$, and $\epsilon > 0$ then $B_d(p, \epsilon) = \{q \in X : d(p, q) < \epsilon\}$ is called the *open ball* of radius ϵ centered at p.

When the metric d is understood then we will use $B(p, \epsilon)$ rather than $B_d(p, \epsilon)$.

Lemma 1.34. *If (X, d) is a metric space then $\{B(p, \epsilon) : p \in X, \epsilon > 0\}$ is a basis for a topology on X.*

Proof. It is easy to see that $X = \bigcup_{p \in X} B(p, 1)$. Now let $p, q \in X$ and $\epsilon_1, \epsilon_2 > 0$. Let $r \in B(p, \epsilon_1) \cap B(q, \epsilon_2)$. Set $\epsilon = \min\{\epsilon_1 - d(p, r), \epsilon_2 - d(q, r)\}$. We will show that $B(r, \epsilon) \subseteq B(p, \epsilon_1) \cap B(q, \epsilon_2)$. Let $s \in B(r, \epsilon)$.

$$d(s, p) \leq d(s, r) + d(r, p) < \epsilon_1 - d(r, p) + d(r, p) = \epsilon_1$$

$$d(s, q) \leq d(s, r) + d(r, q) < \epsilon_2 - d(r, q) + d(r, q) = \epsilon_2$$

Therefore $s \in B(p, \epsilon_1) \cap B(q, \epsilon_2)$ and $B(r, \epsilon) \subseteq B(p, \epsilon_1) \cap B(q, \epsilon_2)$. It follows that $\{B(p, \epsilon) : p \in X, \epsilon > 0\}$ is a basis for a topology on X. $\square$

Definition 1.35. The *metric topology* on a metric space (X, d) is the topology generated by the basis $\{B(p, \epsilon) : p \in X, \epsilon > 0\}$.

Definition 1.36. A space X is said to be $zero-dimensional$ if and only if it has a basis consisting of sets which are both closed and open.

Lemma 1.37. *If X is a zero-dimensional space, and $Y \subseteq X$, then Y is zero-dimensional.*

Proof. Let $\mathcal{B}[X]$ be a basis of open and closed sets. Then, by Lemma 1.16, $\{B \cap Y : B \in \mathcal{B}[X]\}$ is a basis for the subspace topology on Y. Then, by Definition 1.6 and Theorem 1.21, $Y \cap B$ is both open and closed for all $B \in \mathcal{B}[X]$, so $\{B \cap Y : B \in \mathcal{B}[X]\}$ is a basis of open and closed sets and Y is zero-dimensional. □

Lemma 1.38. *Let $f : X \to Y$ be a homeomorphism between two spaces X and Y. If X is zero-dimensional then Y is zero-dimensional.*

Proof. Let $\mathcal{B}[X]$ be a basis of X which consists of both open and closed sets. We want to show that $\mathcal{B}[Y] = \{f[B] : B \in \mathcal{B}[X]\}$ is a basis, consisting of sets which are both open and closed, for the topology on Y, $\tau(Y)$.

First, we show it is a basis for a topology. Let $p \in Y$. Then there exists $q \in X$ such that $f(q) = p$. There is $B \in \mathcal{B}[X]$ such that $q \in B$, so $p \in f[B]$. Now let $p \in Y$ and $A, B \in \mathcal{B}[X]$ such that $p \in f[A] \cap f[B]$. There exists $C \in \mathcal{B}[X]$ such that $f^{-1}(p) \in C \subseteq A \cap B$, so $p \in f[C] \subseteq f[A \cap B] = f[A] \cap f[B]$. Thus, $\mathcal{B}[Y] = \{f[B] : B \in \mathcal{B}[X]\}$ is a basis on Y. Now we need to show that $\mathcal{B}[Y]$ consists of sets which are closed as well. Let $B \in \mathcal{B}[X]$, then B is closed. Since f is continuous, $f[B]$ is also closed. Thus, $\mathcal{B}[Y]$ consists of sets which are closed as well. Therefore, we have found a basis of closed and open sets in Y.

Now we show $\{f[B] : B \in \mathcal{B}[X]\}$ is a basis for $\tau(Y)$. Let $U \in \tau(Y)$. Since f is a homeomorphism, $f^{-1}[U]$ is open in X. So, there exists $\mathcal{A} \subseteq \mathcal{B}[X]$ such that $\bigcup \mathcal{A} = f^{-1}[U]$. So, $f[\bigcup \mathcal{A}] = \bigcup_{B \in \mathcal{A}} f[B] = U$. Thus, we have shown that $\{f[B] : B \in \mathcal{B}[X]\}$ is a basis for $\tau(Y)$. □

Definition 1.39. A point $p \in X$ is an *isolated point* of X if $\{p\}$ is open.

Lemma 1.40. *If X is a space with no isolated points, and $Y \subseteq X$, then Y has no isolated points.*

Proof. By way of contradiction, suppose Y has an isolated point p. Then, $\{p\}$ is open in Y. By Lemma 1.18, it must be that $\{p\}$ is open in X, but this is a contradiction because X contains no isolated points. Therefore, Y cannot contain any isolated points either. □

We will now introduce an important property of topological spaces known as compactness. First we must define a few terms.

Definition 1.41. A *cover* of a space X is a collection $\mathcal{C}$ of subsets of X such that $X = \bigcup \mathcal{C}$. We say that $\mathcal{C}$ covers X. If $Y \subseteq X$ and $Y \subseteq \bigcup \mathcal{C}$ then we say that $\mathcal{C}$ covers Y. If the elements of $\mathcal{C}$ are open then $\mathcal{C}$ is an open cover of X.

Definition 1.42. If $\mathcal{C}$ is a cover of a space X then a *subcover* of $\mathcal{C}$ is a subset $\mathcal{D}$ of $\mathcal{C}$ that covers X.

Definition 1.43. A space X is *compact* if and only every open cover of X has a finite subcover.

Example 1.44. We state without proof that the interval $[0, 1]$ is compact.

Theorem 1.45. *Every closed subset of a compact space is compact.*

Proof. Let C be a closed subset of the compact space X. Let $\mathcal{U}$ be a collection of open subsets of X that covers C. Then $\mathcal{U} \cup \{X - C\}$ is an open cover of X and has a finite subcover $\mathcal{V}$. Then $\mathcal{V} - \{X - C\}$ is a finite subcover of $\mathcal{U}$. □

Theorem 1.46. *Every compact subset of a Hausdorff space is closed.*

Proof. Let C be a compact subset of the Hausdorff space X and let $p \in X - C$. We will show that $X - C$ is open. For every $q \in C$ there are disjoint open subsets $U(q)$ and $V(q)$ of X such that $q \in U(q)$ and $p \in V(q)$. The set $\{U(q) : q \in C\}$ is an open cover of C so there is a finite subset A of C such that such that $\{U(q) : q \in A\}$ covers C. Let $U = \bigcup_{q \in A} U(q)$ and $V = \bigcap_{q \in A} V(q)$. Then $C \subseteq U$ and $p \in V$. Now we show that $V \subseteq X - C$. If $r \in U$ then there is a $q \in A$ such that $r \in U(q)$. Since $U(q) \cap V(q) = \emptyset$, r cannot be an element of $V(q)$. Therefore, $r \notin V$. So $U \cap V = \emptyset$ which means that $V \cap C = \emptyset$. Thus $V \subseteq (X - C)$ and $X - C$ is open, and so C is closed. □

Theorem 1.47. *The continuous image of a compact space is compact.*

Proof. Let X be compact. Let $f : X \to Y$ be a continuous function. Suppose $\mathcal{V}$ is an open cover of $f[X]$. Then $\{f^{-1}[V] : V \in \mathcal{V}\}$ is a collection of open sets in X. We have $X \subseteq f^{-1}[f[X]] \subseteq f^{-1}[\bigcup \mathcal{V}] = \bigcup f^{-1}[V]$. Thus $\bigcup f^{-1}[V]$ for $V \in \mathcal{V}$ is an open cover of X. There is $n \in \omega$ such that $\{f^{-1}[V_i] : 1 \leq i \leq n\}$ is a finite cover of X. Thus, $f[X] \subseteq f[\bigcup_{i=1}^{n} f^{-1}[V_i]] = \bigcup_{i=1}^{n} V_i$. So $\{V_i : 1 \leq i \leq n\}$ is a finite open cover of $f[X]$ and Y is compact. □

Corollary 1.48. *Let X and Y be spaces and let $f : X \to Y$ be continuous. If C is closed subset of X, X is compact, and Y is Hausdorff then $f[C]$ is a closed subset of Y.*

Proof. C is compact from Theorem 1.45 and $f[C]$ is compact from Theorem 1.47. And then by Theorem 1.46 $f[C]$ is closed in Y. □

Corollary 1.49. *Let X be a compact space and Y a Hausdorff space. If $f : X \to Y$ is continuous, one-to-one, and onto then f is a homeomorphism.*

Proof. To show that f is a homeomorphism, we need to show that f^{-1} is continuous. The function $f^{-1} : Y \to X$ is continuous if for every closed subset C of X, $f[C]$ is closed in Y. Let C be a closed subset of X. Then $f[C]$ is closed in Y from Corollary 1.48. Thus, f^{-1} is continuous and f is a homeomorphism. □

Definition 1.50. A nonempty subset $\mathcal{A}$ of subsets of a set X has the $finite\ intersection\ property$ if $\bigcap \mathcal{B} \neq \emptyset$ for every finite nonempty subset $\mathcal{B}$ of $\mathcal{A}$.

Theorem 1.51. *A space X is compact if and only if for every nonempty collection $\mathcal{C}$ of closed subsets of X having the finite intersection property, $\bigcap \mathcal{C} \neq \emptyset$.*

Proof. We will begin by proving the contrapositive of the reverse implication. Assume that X is not compact. Then there is an open cover $\mathcal{U}$ of X which does not have a finite subcover. Let $\mathcal{C} = \{X - U : U \in \mathcal{U}\}$. This is a nonempty collection of closed subsets of X. If $\mathcal{D}$ is a nonempty finite subset of $\mathcal{C}$ then there is a nonempty finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{D} = \{X - V : V \in \mathcal{V}\}$. Now $\bigcap_{V \in \mathcal{V}}(X - V) = X - \bigcup_{V \in \mathcal{V}} V$ and $\bigcup_{V \in \mathcal{V}} V \neq X$, since $\mathcal{U}$ has no finite subcover, so $\bigcap_{V \in \mathcal{V}}(X - V) \neq \emptyset$. Thus, $\mathcal{C}$ has the finite intersection property. But $\bigcap \mathcal{C} = \bigcap_{U \in \mathcal{U}}(X - U) = X - \bigcup_{U \in \mathcal{U}} U = X - X = \emptyset$. This shows that if $\bigcap \mathcal{C} \neq \emptyset$ for every nonempty collection $\mathcal{C}$ of closed subsets of X then X must be compact.

Now, we prove the contrapositive of the forward implication. Let $\mathcal{C}$ be a nonempty collection of closed subsets of X which satisfies the finite intersection property but has $\bigcap \mathcal{C} = \emptyset$. Let $\mathcal{C} = \{X - U : U \in \mathcal{U}\}$. Then $\bigcap \mathcal{C} = \bigcap_{U \in \mathcal{U}}(X - U) = X - \bigcup_{U \in \mathcal{U}} U = \emptyset$, so $\mathcal{U}$ is an open cover of X. Now let $\mathcal{V}$ be a nonempty finite subset of C. Following the finite intersection property, $\bigcap \mathcal{V} \neq \emptyset$. So, for every finite subset $\mathcal{F}$ of $\mathcal{U}$ such that $\bigcap \mathcal{V} = \bigcap_{U \in \mathcal{F}} (X - U) = X - \bigcup \mathcal{F} \neq \emptyset$. Therefore, X has no finite subcover of $\mathcal{U}$, and X is not compact. Thus, we have shown that if X is compact, then $\bigcap \mathcal{C} \neq \emptyset$ for every nonempty collection $\mathcal{C}$ of closed subsets of X having the finite intersection property. □

Corollary 1.52. *Let $\{C_n : n \in \omega\}$ be a collection of nonempty closed subsets of a space X. If X is compact and $C_{n+1} \subseteq C_n$ for every $n \in \omega$ then $\bigcap_{n \in \omega} C_n \neq \emptyset$.*

Proof. Let $\mathcal{C} = \{C_n : n \in \omega\}$. Let $\mathcal{V}$ be a nonempty finite subset of $\mathcal{C}$. Since every element of $\mathcal{C}$ is contained within its predecessor and $\mathcal{V}$ is nonempty, then $\bigcap \mathcal{V} \neq \emptyset$, so $\mathcal{C}$ satisfies the finite intersection property and $\bigcap \mathcal{C} = \bigcap_{n \in \omega} C_n \neq \emptyset$ from Theorem 1.51. □

If $\mathcal{C} = \{C_n : n \in \omega\}$ where $C_{n+1} \subseteq C_n$ for every $n \in \omega$, then we say that $\mathcal{C}$ is a collection of $nested$ sets.

2. Cantor Set

We are now ready to introduce the famous Cantor Set. With this background in mind, we will prove some of the interesting properties of the Cantor Set show some topological structures which it is homeomorphic to.

Definition 2.1. We define the $Cantor\ Set$ by defining its construction. First, set $C_0 = [0, 1]$. For every $n \in \omega$ set $C_{n+1} = C_n - \bigcup_{k=1}^{3^n} (\frac{3k-2}{3^{n+1}}, \frac{3k-2}{3^{n+1}})$. Finally, the Cantor Set is $C = \bigcap_{n \in \omega} C_n$.

Figure 1 below is a graphical representation of the first few iterations of the construction process.

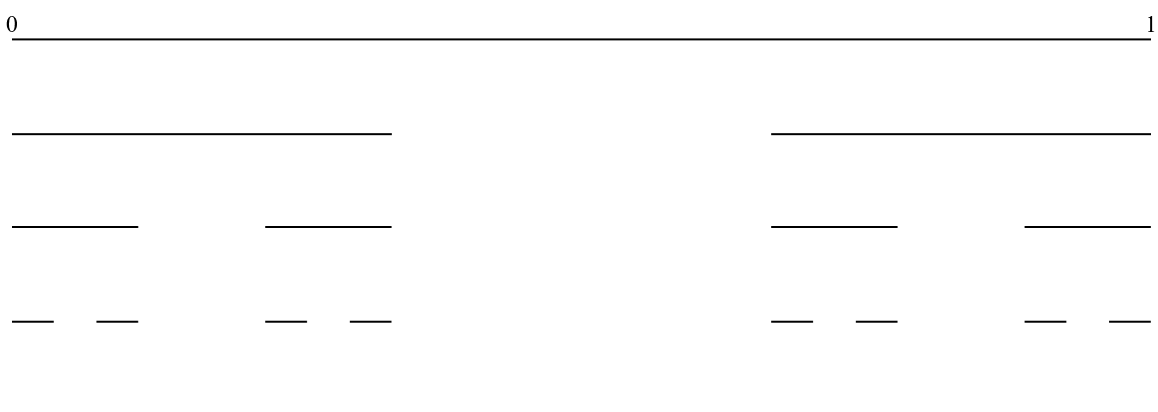

Figure 1. First iterations of construction outlined in Definition 2.1

The Cantor Set is defined through a process of removing the open middle thirds of remaining intervals. If $C_0 = [0, 1]$ then $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ because $(\frac{1}{3}, \frac{2}{3})$ was the first open interval removed. Then $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, where $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ where removed from the two intervals of C_1. The elements of the Cantor Set are the remaining points after repeating this process of removing open middle thirds an infinite number of times.

We know that C is not empty because the endpoints of an interval C_n remain in C_k for all $k > n$, as we are only removing the open middle third of that interval, so they must be in the final intersection. Also, each C_n is a finite union of closed intervals, and therefore, by Theorem 1.20 we know that each C_n is a closed subset of $[0, 1]$. We will now prove two properties of the Cantor Set, and then illustrate the set in a different manner.

Lemma 2.2. *The Cantor Set is compact.*

Proof. $C = \bigcap_{n\in\omega} C_n$, where each C_n is closed for all $n \in \omega$. Then, by Theorem 1.20 $C = \bigcap_{n\in\omega} C_n$ is closed. Since C is a subset of a compact space, $[0, 1]$, and it is closed, C is also a compact space by Theorem 1.45. $\square$

Lemma 2.3. *The Cantor Set has no isolated points.*

Proof. C_n is the union of a finite number of disjoint intervals, each with diameter $(\frac{1}{3})^n$. Let $x \in C$. Let $\epsilon > 0$. There is $k \in \omega$ such that $(\frac{1}{3})^k < \epsilon$. It must be that $x \in C_k$ and so x is in a closed interval of length $(\frac{1}{3})^k$. Let y be an endpoint of that interval. Then $y \in C$ and $|x - y| \leq (\frac{1}{3})^k < \epsilon$, so no $x \in C$ is isolated. $\square$

Now we would like to consider a different way to represent the Cantor Set. Recall that to define the Cantor Set, we defined the process of constructing it, so we can think of each point in the Cantor set as the end of a different path in the construction process. In each closed interval of C_n, we remove the open middle third, so we are left with two closed intervals, one lower in the initial $[0, 1]$ interval than the other. These are the left and right thirds of the interval divided. We can think of each point of the Cantor Set as an infinite sequence of choosing the left or right interval in the next iteration. We'll assign 0 to the act of choosing the left interval and 1 to the act of choosing the right interval so that we can represent each path as an infinite binary sequence.

For example, we know that $\frac{20}{27} \in C$ because it is an endpoint of an interval in C_3. Starting initially with $[0, 1]$, we first take the right interval, $[\frac{2}{3}, 1]$. We then take the left third of $[\frac{2}{3}, 1]$, which is $[\frac{2}{3}, \frac{7}{9}]$. Then we take the right interval of $[\frac{2}{3}, \frac{7}{9}]$, which is $[\frac{20}{27}, \frac{7}{9}]$. We have now reached the interval for which $\frac{20}{27}$ is an endpoint, the left endpoint specifically. This interval will be split into two in C_4 so we will take the left interval because it contains $\frac{20}{27}$. This interval in C_5 will again be split into two and again we take the left interval containing $\frac{20}{27}$. We can see that from now on we only take the left interval because it will always be the one which contains $\frac{20}{27}$. So, to get to $\frac{20}{27}$, we went right, left, right, and then left indefinitely. In other words, our sequence representing $\frac{20}{27}$ is $101000...$.

The set of infinite binary sequences is $\{0, 1\}^\omega$. We have $C = \bigcap_{n\in\omega} C_n$ where each C_n is a union of 2^n disjoint closed intervals of length $(\frac{1}{3})^n$. We will label the intervals which constitute each C_n from left to right with sequences of 0's and 1's of length n. For example $C_1 = C^0 \cup C^1$, $C_2 = C^{00} \cup C^{01} \cup C^{10} \cup C^{11}$, and $C_3 = C^{000} \cup C^{001} \cup C^{010} \cup C^{011} \cup C^{100} \cup C^{101} \cup C^{110} \cup C^{111}$, and so on. We will let $[0, 1] = C_0 = C^\emptyset$. Then we have $C_n = \bigcup_{a\in\{0,1\}^n} C^a$.

Definition 2.4. Define $\mathcal{A}_f$ to denote the collection of all finite binary sequences.

If $a_k, a_m \in \mathcal{A}_f$ of length k and m respectively, with $k > m$, then $C^{a_k} \subset C^{a_m}$ if a_k restricted to the first m numbers is equal to a_m, that is, $a_k|_m = a_m$. If $a_k|_m = a_m$ then we must have reached the interval C^{a_m} in our path to C^{a_k}, so it follows that $C^{a_k} \subset C^{a_m}$. So, if $a \in \{0, 1\}^\omega$, then $[0, 1] = C^\emptyset = C^{a|_0} \supset C^{a|_1} \supset C^{a|_2} \supset C^{a|_3} \supset$...

Because each interval in C_n, and for every $n \in \omega$, is a closed subset of a compact space, by Corollary 1.52 we have $\bigcap_{n\in\omega} C^{a|n} \neq \emptyset$. Since the length of each interval of C_n goes to zero as $n \to \infty$ it must be the case that $\bigcap_{n\in\omega} C^{a|n}$ is a single point.

Now we have an idea of how we will define our mapping, however, showing that our mapping is continuous is necessary in showing that it is a homeomorphism, and to show a function to be continuous, we need to get a better idea of what open sets in both C and in $\{0,1\}^\omega$ look like.

Definition 2.5. We will let the *Cantor topology* be the topology generated by the basis $\{C \cap C^{a_n} : a_n \in \mathcal{A}_f\}$.

Lemma 2.6. *The collection $\{C \cap C^{a_n} : a_n \in \mathcal{A}_f\}$ is a basis for a topology on C.*

Proof. Let $p \in C$. Then $p \in C_1$, so $p \in C^0$ or $p \in C^1$. If $p \in C^0$, let a_1 be the finite sequence of length one such that $a_1(0) = 0$. Then $p \in C \cap C^{a_1}$. If $p \in C^1$, let a_1 be the finite sequence of length one such that $a_1(0) = 1$. Then $p \in C \cap C^{a_1}$. In either case, there exists $a_0 \in \mathcal{A}_f$ such that $p \in C \cap C^{a_0}$. So there exists $C \cap C^{a_n} \in \{C \cap C^{a_n} : a_n \in \mathcal{A}_f\}$ which contains p for any $p \in C$.

Now let $a_n, a_m \in \mathcal{A}_f, a_n \neq a_m$, such that $p \in (C \cap C^{a_n}) \cap (C \cap C^{a_m}) = C \cap C^{a_n} \cap C^{a_m}$. It must be that $n < m$ or $n > m$. If $n = m$, then there is some $i \leq n = m$ such that $a_n(i) \neq a_m(i)$, so $C^{a_n|i} \cap C^{a_m|i} = \emptyset$, and p cannot be in both. Without loss of generality, suppose $n < m$. Then $C^{a_m} \subset C^{a_n}$. So $p \in (C \cap C^{a_m}) \subseteq (C \cap C^{a_n}) \cap (C \cap C^{a_m})$. Thus, $\{C \cap C^{a_n} : a_n \in \mathcal{A}_f\}$ is a basis for a topology on C. $\square$

Definition 2.7. We will let the *binary sequence topology* be the topology generated by the basis $\{a \in \{0,1\}^\omega : \forall i, a(i) = a_n(i), a_n \in \mathcal{A}_f\}$.

Lemma 2.8. *Then the collection of $\{a \in \{0,1\}^\omega : \forall i, a(i) = a_n(i), a_n \in \mathcal{A}_f\}$ is a basis for a topology on $\{0,1\}^\omega$.*

Proof. Let $p \in \{0,1\}^\omega$. We need to find an a_n in $\mathcal{A}_f$ such that for all i, $p(i) = a_n(i)$. Let a_0 be a sequence of length one and define $a_0(0) = p(0)$. Then $a_0 \in \mathcal{A}_f$ and $p \in \{a \in \{0,1\}^\omega : a(0) = a_0(0)\} \in \{a \in \{0,1\}^\omega : \forall i, a(i) = a_n(i), a_n \in \mathcal{A}_f\}$. So, there exists a set in $\{a \in \{0,1\}^\omega : \forall i, a(i) = a_n(i), a_n \in \mathcal{A}_f\}$ which contains p for any $p \in \{0,1\}^\omega$.

Now, we need to show that if $p \in \{0,1\}^\omega$ and $a_n, a_m \in \mathcal{A}_f, a_n \neq a_m$, such that $p(i) = a_n(i)$ for all $i \leq n$ and $p(i) = a_m(i)$ for all $i \leq m$, then there is $a_k \in \mathcal{A}_f$ such that $p \in \{a \in \{0,1\}^\omega : \forall i, a(i) = a_k(i)\} \subseteq \{a \in \{0,1\}^\omega : \forall i, a(i) = a_n(i)\} \cap \{a \in \{0,1\}^\omega : \forall i, a(i) = a_m(i)\}$. The sequence a_n is of length n and a_m of length m. Since $p(i) = a_n(i)$ for all $i \leq n$ and $p(i) = a_m(i)$ for all $i \leq m$, $n \neq m$. Otherwise, $a_n = a_m$. So, $n < m$ or $n > m$. Without loss of generality, suppose $n < m$. Then $p(i) = a_n(i) = a_m(i)$ for all $i \leq n$. So $\{a \in \{0,1\}^\omega : \forall i, a(i) = a_m(i)\} \subset \{a \in \{0,1\}^\omega : \forall i, a(i) = a_n(i)\}$. So $p \in \{a \in \{0,1\}^\omega : \forall i, a(i) = a_m(i)\} \subseteq \{a \in \{0,1\}^\omega : \forall i, a(i) = a_n(i)\} \cap \{a \in \{0,1\}^\omega : \forall i, a(i) = a_m(i)\}$. Therefore the collection $\{a \in \{0,1\}^\omega : \forall i, a(i) = a_n(i), a_n \in \mathcal{A}_f\}$ is a basis for a topology on $\{0,1\}^\omega$. $\square$

Lemma 2.9. *The basis for the binary sequence topology is also a basis for the product topology on $\{0,1\}^\omega$.*

Proof. Let U be open in the product topology on $\{0,1\}^\omega$. Every open set is a union of basis sets, so we can assume U is a basis set. Recall that these basis sets in the product topology on $\{0,1\}^\omega$ are all of the form $\bigcap_{i\in I} \pi_i^{-1}[U_i]$, where U_i is open in $\{0,1\}$ for all i

in the finite indexing set I. Let $U = \bigcap_{i\in I} \pi_i^{-1}[U_i]$. Since I is finite, let m be the greatest element of I. Let $U(a_m)$ be the set of all $a_m \in \mathcal{A}_f$ such that $a_m(i) \in U_i$ for all U_i. Then $U = \bigcap_{i\in I} \pi_i^{-1}[U_i] = \bigcup\{a \in \{0,1\}^\omega : \forall i, a(i) = a_m(i), a_m \in U(a_m)\}$. Therefore, $\{a \in \{0,1\}^\omega : \forall i, a(i) = a_n(i), a_n \in \mathcal{A}_f\}$ is also a basis for the product topology on $\{0,1\}^\omega$. □

We are now ready to define our function and prove it to be a homeomorphism.

Theorem 2.10. *The function $\alpha : \{0,1\}^\omega \to C$ defined as $\alpha(a) = \bigcap_{n\in\omega} C^{a|_n}$ is a homeomorphism between $\{0,1\}^\omega$ and the Cantor Set.*

We prove this by first showing that α is a bijection, that is, that it is one-to-one and onto.

Lemma 2.11. *α is one-to-one.*

Proof. Let $a, b \in \{0,1\}^\omega$ such that $a \neq b$. Then, there is $k \in \omega$ such that $a(k) \neq b(k)$. Then, in C_k, $\alpha(a)$ and $\alpha(b)$ are in different sets which have no intersection, so $\alpha(a) \neq \alpha(b)$. Thus, α is one-to-one. □

Lemma 2.12. *α is onto.*

Proof. Let $x \in C$. We will show that there is $a \in \{0,1\}^\omega$ such that $\alpha(a) = x$. We have $x \in C = \bigcap_{n\in\omega} C_n$. So, there is an interval C^{a_k} in C_k such that $x \in C^{a_k}$ for all $k \in \omega$. Define $a(n) = a_k(n)$ for all $n \in \omega$. Then $a \in \{0,1\}^\omega$ such that $\alpha(a) = x$, and α is onto. □

Thus, α^{-1} is also one-to-one and onto. To show that α is a homeomorphism, we show that α^{-1} is a homeomorphism, because if α^{-1} meets all the requirements of being a homeomorphism, then so will α.

First, we notice that set $\{0,1\}$ is Hausdorff because $\{0\}$ and $\{1\}$ are both open in the discrete topology on $\{0,1\}$. Then, by Lemma 1.26, we know that $\{0,1\}^\omega$ is also Hausdorff. Since C is compact, by Lemma 1.49, if α^{-1} is shown to be continuous then α^{-1} is a homeomorphism.

Lemma 2.13. *α^{-1} is continuous.*

Proof. Let $a_k \in \mathcal{A}_f$. Then $\{a \in \{0,1\}^\omega : \forall i, a(i) = a_k(i)\}$ is open in the binary sequence topology.

$$\begin{aligned}\alpha[\{a \in \{0,1\}^\omega : \forall i, a(i) = a_k(i)\}] &= \{\bigcap_{n\in\omega} C^{a|_n} : \bigcap_{n}^{k} C^{a|_n} = \bigcap_{n}^{k} C^{a_k|_n}\} \\ &= \{\bigcap_{n\in\omega} C^{a|_n} : C^{a|_k} = C^{a_k|_k}\}\end{aligned}$$

This is the set of paths in the construction of C which pass through C^{a_k}, in other words, the set of all points in C which are also in C^{a_k}. So $\alpha[\{a \in \{0,1\}^\omega : \forall i, a(i) = a_k(i)\}] = C \cap C^{a_k}$, an open set in the Cantor topology. Thus, α^{-1} is continuous. □

We have now proved Theorem 2.10, showing that $\alpha : \{0,1\}^\omega \to C$ is a homeomorphism, so $\{0,1\}^\omega$ and C are topologically equivalent spaces.

3. TREES

A commonly used and more intuitive structure in topology is that of a tree.

Definition 3.1. A $tree$, T, is a partially ordered set with the property that for every $x \in T$, $\{y \in T : y < x\}$ is totally ordered. Elements of T are vertices.

Note that in the definition of a tree above, the set $\{y \in T : y < x\}$ can be finite or infinite. When $\{y \in T : y < x\}$ is finite for all $x \in T$, then $\{y \in T : y < x\}$ is well-ordered, that is, is has a least element.

Definition 3.2. Let T be a tree such that if $x \in T$, then $\{y \in T : y \leq x\}$ is finite. We say T is a $rooted\ tree$ if there exists $x_0 \in T$ which is the least, or minimum, element of $\{y \in T : y \leq x\}$ for all $x \in T$.

The trees we work with in this paper are all rooted trees and will just be referred to as trees from now on for convenience. Figure 2 below depicts a special kind of tree that we will define later.

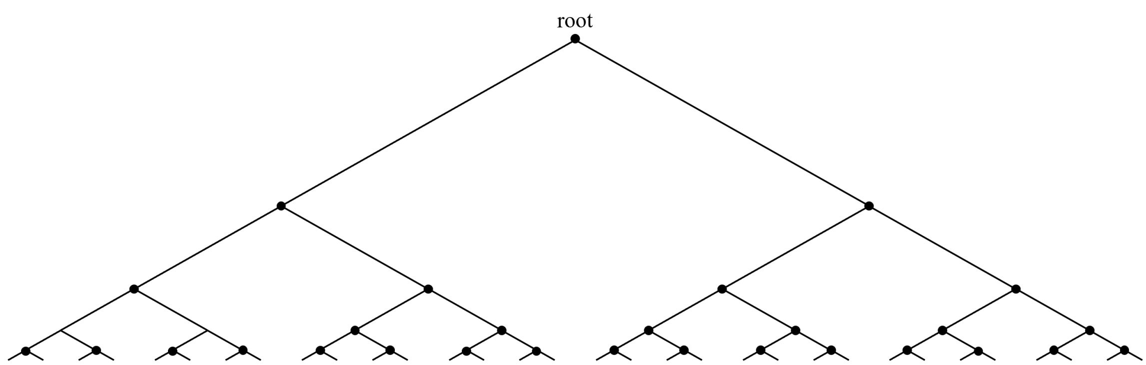

FIGURE 2. Example of a rooted tree

Definition 3.3. Let T be a tree. If $a, b \in T$ and $a < b$ or $b < a$, using the order $<$ on T, then a and b are said to be $comparable$. Two points are $incomparable$ if and only if they are not comparable.

Definition 3.4. A $chain$ is a set of comparable elements of a tree T. An $antichain$ is a set of incomparable elements of a tree T.

Definition 3.5. A $branch$ of a tree T is a maximal chain. That is, if B is the branch of T, there is no such $x \in T$ such that x is comparable to every element of B. A $maximal$ $antichain$ is an antichain, A, such that there is no $y \in T$ such that y is incomparable every element of A.

Definition 3.6. Let T be a tree. For every $n \in \omega$, the set of $a \in T$ such that $|\{b \in T : b < a\}| = n$ is the n^{th} $level$ of T. That is, every element in the n^{th} level of T has exactly n elements less than it. We denote the n^{th} level of T as $L_n(T)$. Note that $L_0(T)$ is the least element of T.

Lemma 3.7. *Let $L_n(T)$ be a level of a tree T. If $x, y \in L_n(T)$ then x and y are incomparable.*

Proof. By way of contradiction, suppose x and y are comparable. Then $x < y$ or $x > y$. Without loss of generality, suppose $x < y$. We know that $|\{p \in T : p < x\}| = n$, since $x < y$, then $|\{p \in T : p < y\}| > |\{p \in T : p < x\}| = n$, a contradiction because $y \in L_n(T)$. Therefore, x and y must be incomparable. $\square$

Definition 3.8. A $tall\ chain$, A, is a chain of a tree T such that for all $n \in \omega$, there is $y \in A$ such that $y > x$ for some $x \in L_n(T)$.

Definition 3.9. Let $x \in T$, then define $\downarrow x = \{y \in T : y < x\}$. Let $A \subseteq T$, then $\downarrow A = \{y \in T : \exists a \in A : y < a\}$. In other words, $\downarrow A$ is the set of all vertices in T which are less than some vertex in A.

Lemma 3.10. *If A is a tall chain in a tree T, then $\downarrow A$ is a branch.*

Proof. Let $y \in T - \downarrow A$. By way of contradiction, suppose y is comparable to every element of $\downarrow A$. There is $n \in \omega$ such that $y \in L_n(T)$. Since, A is a tall chain, there is $a \in A$ such that $a > x$ for some $x \in L_n(T)$. Then $x \in \downarrow A$ and y is comparable to x. This is a contradiction because $y, x \in L_n(T)$. Thus, $\downarrow A$ is a maximal chain and therefore is a branch in T. $\square$

Lemma 3.11. *Let T be a tree, A a tall chain in T, and B a branch in T. If $A \subset B$ then $\downarrow A = B$.*

Proof. We know that $\downarrow A$ is a branch in T. Let $x \in \downarrow A$. Then there is $a \in A$ such that $x < a$. Since $a \in A$ then $a \in B$, and so $x \in B$ because $\{y \in T : y < a\}$ is linearly ordered and B is a branch.

Now let $x \in B$. Then there is $n \in \omega$ such that $x \in L_n(T)$. Since A is a tall chain there is $a \in A$ such that $a > y$ for some $y \in L_n(T)$. Since $A \subset B$, $a \in B$. Since $x \in B$, it must be that $x = y$, otherwise B would contain two incomparable vertices.

Thus, we have shown that $\downarrow A \subseteq B$ and $B \subseteq \downarrow A$, so it must be that $\downarrow A = B$. $\square$

Definition 3.12. A point y of a tree T is a $successor$ of another point $x \in T$ if and only of $x < y$ and there is no $z \in T$ with $x < z < y$. We will let $s(x)$ denote the set of all successors of x, so $s : T \to \mathcal{P}(T)$. To map a set of elements to the union of their successors, we will use $S : \mathcal{P}(T) \to \mathcal{P}(T)$, where $S(X) = s[X] = \{s(x) : x \in X\}$ for some $X \subseteq T$.

Note that $L_{n+1}(T) = S(L_n(T))$.

Lemma 3.13. *Let T be a tree, with $x \in T$. There is $n \in \omega$ such that $x \in L_n(T)$.*

Proof. Since $x \in T$, the set $\{y \in T : y < x\}$ is well-ordered. If x is the least element of T then $x \in L_0(T)$. If x is not the least element of T then $|\{y \in T : y < x\}| = n$ for some $n \in \omega$ and by definition $x \in L_n(T)$. $\square$

Lemma 3.14. *If B is a branch and A is a finite maximal antichain of a tree T, then B intersects A at exactly one point.*

Proof. We know B cannot intersect A at more than one point because then B would contain points which are incomparable to each other. So, the intersection of B and A is at most one point.

Now, by way of contradiction, assume that $B \cap A = \emptyset$. We will show that this implies the existence of an element not in A which is incomparable to any element of A, a contradiction for A is maximal. For every $q \in A$ define $B_q = \{x \in B : x \not\leq q\}$. B_q is well ordered and nonempty, so it has a least element. Let b_q be that least element of B_q. For every $q \in A$, $q \not< b_q$, otherwise $q \in B$. Since A is finite, the set $\{b_q : q \in A\}$ is finite and has a greatest element, call it b_p, where $p \in A$. It must be that $b_p \not< q$, otherwise $b_q \leq b_p < q$, and $b_q \not< q$ by definition. Then $b_p \not< q$ and $q \not< b_p$ for all $q \in A$, so A is not maximal. Thus it must be that $B \cap A \neq \emptyset$. □

Definition 3.15. We denote $\mathcal{B}[T]$ as the branch space of a tree T, that is, the set of all branches of T. We will also denote $\mathcal{B}(x)$ as the set of all branches which contain the point $x \in T$.

Note that if x_0 is the least element of T, then $\mathcal{B}[T] = \mathcal{B}(x_0)$. If $A \subset T$, then we let $\mathcal{B}[A]$ be the set of all branches in T which contain a vertex in A. It is also the case that $\mathcal{B}(x_0) = \mathcal{B}[L_n(T)]$ for any level $L_n(T)$. We now must define a topology on $\mathcal{B}[T]$, for any general tree T.

Definition 3.16. Let T be a tree. We will let the $branch\ topology$ on $\mathcal{B}[T]$ be the topology generated by the basis $\{\mathcal{B}(x) : x \in T\}$.

Lemma 3.17. *The collection of* $\{\mathcal{B}(x) : x \in T\}$ *is a basis for a topology on* $\mathcal{B}[T]$.

Proof. First, as per the definition of a basis, we need to show that if $A \in \mathcal{B}[T]$, then there is $x \in T$ such that $A \in \mathcal{B}(x) \in \{\mathcal{B}(x) : x \in T\}$. Since $A \in T$, there is some point $x \in T$ which is contained within A. Thus $A \in \mathcal{B}(x) \in \{\mathcal{B}(x) : x \in T\}$.

Now let $A \in \mathcal{B}(x) \cap \mathcal{B}(y)$ for $x, y \in T$. We need to find a subset of $\mathcal{B}(x) \cap \mathcal{B}(y)$ which contains A. It must be that $x < y$ or $x > y$. Without loss of generality, suppose that $x < y$. Then any branch which contains y must also contain x, otherwise, T would not be linearly ordered. Then, $\mathcal{B}(y) \subset \mathcal{B}(x)$. So $A \in \mathcal{B}(y) \subseteq \mathcal{B}(x) \cap \mathcal{B}(y)$.

Thus, $\{\mathcal{B}(x) : x \in T\}$ is a basis for a topology on $\mathcal{B}[T]$. □

Lemma 3.18. $\mathcal{B}[T]$ *is Hausdorff.*

Proof. Let $X, Y \in \mathcal{B}[T]$ with $X \neq Y$. There exists a level $L_k(T)$ with $x \in L_k(T)$ with $x \neq y$ such that $x \in X$ and $y \in Y$. By Lemma 3.7, x and y are incomparable, so $x \notin Y$ and $y \notin X$. Then $X \in \mathcal{B}(x)$ and $Y \in \mathcal{B}(y)$. $\mathcal{B}(x) \cap \mathcal{B}(y) = \emptyset$, otherwise a branch would contain two incomparable vertices. Thus $\mathcal{B}[T]$ is Hausdorff. □

Lemma 3.19. $\mathcal{B}[T]$ *is zero-dimensional.*

Proof. Let $x \in T$ and $n \in \omega$ such that $x \in L_n(T)$.

$$\begin{aligned}
\mathcal{B}[T] - \mathcal{B}(x) &= \mathcal{B}[L_n(T)] - \mathcal{B}(x)) \\
&= \mathcal{B}[L_n(T) - \{x\}] \\
&= \bigcup_{y \in L_n(T) - \{x\}} \mathcal{B}(y).
\end{aligned}$$

Thus $\mathcal{B}[T] - \mathcal{B}(x)$ is a union of open sets in the branch topology on $\mathcal{B}[T]$, so $\mathcal{B}[T] - \mathcal{B}(x)$ is open and $\mathcal{B}(x)$ is closed for all $x \in T$. Therefore the basis $\{\mathcal{B}(x) : x \in T\}$ consists of sets which are both open and closed in the branch topology on $\mathcal{B}[T]$, so $\mathcal{B}[T]$ is zero-dimensional. $\square$

Definition 3.20. Define the collection of all $finite\ split\ trees$ as $\mathcal{T}_f$ such that if $F \in \mathcal{T}_f$, then each element in F has a finite number of successors. That is, $|s(x)| \in \omega$ for for all $x \in F$. A special example of a finite split tree is the $two\ split\ tree$, D. If $x \in D$, then $|s(x)| = 2$.

The two split tree D is the tree which was pictured above in Figure 2. In a finite split tree, each level will consist of a finite number of elements.

Lemma 3.21. *Every level of a finite split tree F is a finite maximal antichain.*

Proof. We know that since F is a finite split tree, $|L_n(T)| \in \omega$. Let $L_n(F)$ be the n^{th} level of F. First we show that $L_n(F)$ is an antichain. By way of contradiction, suppose that it is not an antichain. Then, there exists $x, y \in L_n(F)$ such that $x < y$. Then $|\{p \in F : p < x\}| = n$ so $|\{p \in F : p < y\}| > n$, a contradiction because $y \in L_n(F)$.

Now we show that $L_n(F)$ is maximal. Let $x \in F - L_n(F)$. Then, there is $m \in \omega, m \neq n$ such that $x \in L_m(F)$. First, suppose $m < n$. Then there is $y \in\ L_n(F)$ such that $x \in \{p \in F : p < y\}$, so $x < y$. If $m > n$, then there exists $y \in L_y(F)$ such that $y \in \{p \in F : p < x\}$, so $y < x$. In either case, $x \in F -\ L_n(F)$ is comparable to some point y in $L_n(F)$, so $L_n(F)$ is a finite maximal antichain. $\square$

So by Lemma 3.14, every branch of F must intersect each level at exactly one point.

Now, we notice that the two split tree D and $\{0,1\}^\omega$ naturally share similar properties. In fact, we will be able to define a homeomorphism between $\mathcal{B}[D]$ and $\{0,1\}^\omega$. However, we must first establish a labeling method for the vertices in D.

Wen can label each vertex in D using finite binary sequences. First, we let the least element of D be $v^\emptyset$. In the first level of D, $v^\emptyset$ splits into two new vertices. The vertex on the left will be v^0 and the vertex on the right will be v^1. In the second level, both v^0 and v^1 split into two new vertices, a left and right vertex. We let v^0 split into v^{00} and v^{01}. We let v^1 split into v^{10} and v^{11}. Notice that $L_1(D) = \{v^{a_1} : a_1 \in \{0,1\}^1\}$, and $L_2(D) = \{v^{a_2} : a_2 \in \{0,1\}^2\}$. An induction argument will show that $L_n(D) = \{v^{a_n} : a_n \in \{0,1\}^n\}$. Another induction argument will show that if $a \in \{0,1\}^\omega$, then $\{v^{a|_n} : n \in \omega\}$ is a branch in D, because $v^{a|_k} < v^{a|_{k+1}}$ for all k. We can now define a homeomorphism between $\{0,1\}^\omega$ and $\mathcal{B}[D]$.

Theorem 3.22. *The function $\beta : \{0,1\}^\omega \to \mathcal{B}[D]$, defined as $\beta(a) = \{v^{a|_n} : n \in \omega\}$, is a homeomorphism between $\{0,1\}^\omega$ and $\mathcal{B}[D]$.*

As was the case for α, we begin by showing β is a bijection.

Lemma 3.23. *β is one-to-one.*

Proof. Let $a, b \in \{0,1\}^\omega, a \neq b$. There is $k \in \omega$ such that $a(k) \neq b(k)$. So $v^{a|_k} \neq v^{b|_k}$ and are both in $L_k(D)$. By Lemma 3.14, a branch can only intersect $L_k(D)$ at one point, so $\{v^{a|_n} : n \in \omega\} \neq \{v^{b|_n} : n \in \omega\}$. Thus, $\beta(a) \neq \beta(b)$ and β is one-to-one. $\square$

Lemma 3.24. *β is onto.*

Proof. Let $B \in \mathcal{B}[D]$. We need to show that there exists $a \in \{0,1\}^\omega$ such that $\beta(a) = B$. It must be that B intersects each level of D at exactly one point by Lemma 3.14. We will recursively define $a \in \{0,1\}^\omega$. We have $a|_0 = \emptyset$. Let $a|_1$ be such that $v^{a|_1} = B \cap L_1(D)$. Now assume that we have defined $a|_k$ for $k \leq n$. Then, similarly, define $a|_{n+1}$ to be such that $v^{a|_{k+1}} = B \cap L_{n+1}(D)$. Thus, we have defined $a|_n$ for all $n \in \omega$, and therefore, we have found $a \in \{0,1\}^\omega$ such that $\beta(a) = B$, so β is onto. □

We know what open sets in both $\{0,1\}^\omega$ and $\mathcal{B}[D]$ look like from Lemma 2.8 and Lemma 3.17. Since C is compact and is homeomorphic to $\{0,1\}^\omega$, by Theorem 1.47, $\{0,1\}^\omega$ is compact. By Lemma 3.18, $\mathcal{B}[D]$ is Hausdorff. So, if β is shown to be continuous, then by Corollary 1.49, β is a homeomorphism.

Lemma 3.25. *β is continuous.*

Proof. Let $x \in D$. Then $x = v^{a_k}$ for some $a_k \in \mathcal{A}_f$ of length k. Also, we know that $\mathcal{B}(x)$ is open in the branch topology on D.

$$\begin{aligned}\beta^{-1}[\mathcal{B}(x)] &= \beta^{-1}[\mathcal{B}(v^{a_k})] \\ &= \{a \in \{0,1\}^\omega : a|_k = a_k\} \\ &= \{a \in \{0,1\}^\omega : \forall i, a(i) = a_k(i)\}\end{aligned}$$

Thus, the inverse image of an open set in $\mathcal{B}[D]$ is an open set in the binary sequence topology, so β is continuous. □

We have proven Theorem 3.22. We now have two homeomorphisms, $\alpha : \{0,1\}^\omega \to C$ and $\beta : \{0,1\}^\omega \to \mathcal{B}[D]$. Since we know that $\alpha^{-1} : C \to \{0,1\}^\omega$ is also a homeomorphism, then by Lemma 1.31, $\alpha^{-1} \circ \beta : C \to \mathcal{B}[D]$ is a homeomorphism between C and $\mathcal{B}[D]$. Thus, the Cantor Set is homeomorphic to both $\{0,1\}^\omega$ and the branch space of the two split tree, $\mathcal{B}[D]$.

Lemma 3.26. *The Cantor Set is zero-dimensional.*

Proof. By Lemma 3.19, $\mathcal{B}[D]$ is zero-dimensional, so by Lemma 1.38, C is also zero-dimensional. □

We have shown that the Cantor Set is homeomorphic to the branch space of the two split tree. As we know, this tree is a just a special kind of finite split tree. So now we will establish a function γ mapping the the branch space of a finite split tree, $\mathcal{B}[F]$, into the branch space of the two split tree, $\mathcal{B}[D]$.

First, we inductively define a different function, $\Gamma : F \to D$.

Definition 3.27. If u_0 is the least element of F then $\Gamma(u_0)$ is the least element of D. Recall the labeling method we used for vertices in D. Then, $\Gamma(u_0) = v^\emptyset$. Now we assume that Γ has been defined on the first n levels of F. We must now define $\Gamma[L_{n+1}(F)]$.

Let $u \in L_n(F)$. There is $k \in \omega$ such that $|s(u)| = k$. We label u's successors from left to right beginning with 1 and ending with k. So, $s(u) = \{u^m : 1 \leq m \leq k\}$. Since we have defined $\Gamma(u)$, let v^a be such that $v^a = \Gamma(u)$, where $a \in \{0,1\}^j$ for some $j \in \omega$. We define $\Gamma(u^m)$ as follows:

If $m = 1$, then $\Gamma(u^m) = v^{a_m}$ where

$$a_m(i) = \begin{cases} a(i) & \forall i \leq j \\ 0 & i = j+1. \end{cases}$$

If $2 \leq m < k$, then $\Gamma(u^m) = v^{a_m}$ where

$$a_m(i) = \begin{cases} a(i) & \forall i \leq j \\ 1 & j < i < j+m \\ 0 & i = j+m. \end{cases}$$

If $m = k$, then $\Gamma(u^m) = v^{a_m}$ where

$$a_m(i) = \begin{cases} a(i) & \forall i \leq j \\ 1 & j < i \leq j+m. \end{cases}$$

Thus we have defined $\Gamma[s(u)]$ and since u was an arbitrary point in $L_n(F)$, we have defined $\Gamma[S(L_n(F))] = \Gamma[L_{n+1}(F)]$. This function has many conditional factors that determine its mapping. A visual representation of this mapping can help in understanding how the function works. Depicted below in Figure 3 is how Γ maps an arbitrary point's successors into D.

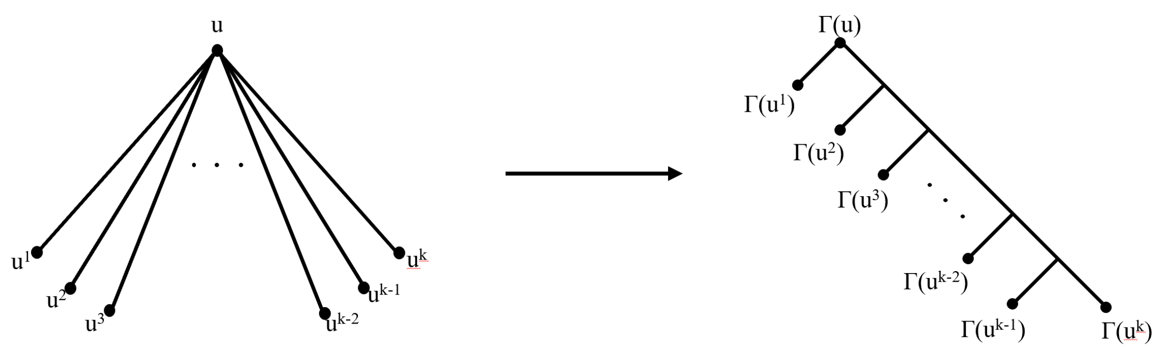

FIGURE 3. Visual representation of applying Γ to the successors of an arbitrary vertex u in the finite split tree

Lemma 3.28. *$\Gamma : F \to D$ preserves the linear order of F.*

Proof. Let $u \in F$. There is a some $n \in \omega$ such that $u \in L_n(F)$. Let $p \in F$, with $p > u$. Then $p \in L_m(F)$ where $m > n$. First suppose that $m = n+1$. Then, $p \in s(u)$ and therefore $\Gamma(p) > \Gamma(u)$. Now assume that $m = n+2$. Since $p > u$ there exists $q \in L_{n+1}(F)$ such that $p > q > u$. Thus, $q \in s(u)$ and $p \in s(q)$, so by definition $\Gamma(p) > \Gamma(q) > \Gamma(u)$. An inductive process will show that for any $m > n$, $\Gamma(p) > \Gamma(u)$ and therefore, $\Gamma : F \to D$ preserves then linear order of F. $\square$

Corollary 3.29. *If B is a branch of F, then $\Gamma[B]$ is a tall chain in D.*

Proof. First we show that $\Gamma[B]$ is a chain. For all $x, y \in B$, either $x < y$ or $x > y$. Then, by the previous Lemma 3.28, we have $\Gamma(x) < \Gamma(y)$ or $\Gamma(x) > \Gamma(y)$. So, any two vertices in $\Gamma[B]$ are comparable and $\Gamma[B]$ is a chain. Now we show that $\Gamma[B]$ is tall. Let $n \in \omega$. Let

$x \in L_n(F)$. Since Γ preserves order, $|\{\Gamma(p) : p < x\}| = n$. So, there are at least n vertices in D which are below $\Gamma(X)$. Let $y \in B \cap L_{n+1}(F)$. Then, $\Gamma(y) > \Gamma(x)$ and so it must be that $|\{\Gamma(p) : p < y\}| > n$. Then $\Gamma(Y) \in L_m(D)$ for some $m > n$ and there exists some $v^{a_n} \in L_n(D)$ with $\Gamma(y) > v^{a_n}$. So, for any $n \in \omega$, there is $\Gamma(x) \in \Gamma[B]$ such that $\Gamma(x)$ is greater than some vertex in $L_n(D)$. Therefore, $\Gamma[B]$ is a tall chain in D. □

As we have shown in Lemma 3.10, if A is a tall chain in a tree T, then $\downarrow A$ is a branch in T. Therefore, if B is a branch in F, then $\downarrow \Gamma[B]$ is a branch in D. We can now define our function between $\mathcal{B}[F]$ and $\mathcal{B}[D]$.

Theorem 3.30. *The function $\gamma : \mathcal{B}[F] \to \mathcal{B}[D]$, defined as $\gamma(\sigma) =\downarrow \Gamma[\sigma]$, where Γ is the function in Definition 3.27, is a homeomorphism between $\mathcal{B}[F]$ and $\mathcal{B}[D]$.*

First we show that γ is a bijection.

Lemma 3.31. *γ is one-to-one*

Proof. Let B_x and B_y be two distinct branches in F. Then there exists $n \in \omega$ and $x, y \in L_n(F)$, with $x \neq y$, such that $x \in B_x$ and $y \in B_y$. So, x and y are incomparable in F. By Lemma 3.28, $\Gamma(x)$ is also incomparable to $\Gamma(y)$. Then $\downarrow \Gamma[B_x]$ contains an element incomparable to an element of $\downarrow \Gamma[B_y]$, so $\downarrow \Gamma[B_x] \neq \downarrow \Gamma[B_y]$. Thus, $\gamma(B_x) \neq \gamma(B_y)$ and γ is one-to-one. □

Lemma 3.32. *γ is onto*

Proof. Let $a \in \{0,1\}^\omega$, so $\{v^{a|_n} : n \in \omega\}$ is a branch in D. Our goal is to construct a branch B in F such that $\gamma(B) = \{v^{a|_n} : n \in \omega\}$.

For this argument, we will let $B_n = B \cap L_n(F)$. That is, B_n is the vertex in F in which our constructed branch intersects the n^{th} level. Then, $B = \{B_n : n \in \omega\}$. To define our branch, we need to define B_n for all $n \in \omega$, and we will do so recursively. B_0 is the least element of F so $\Gamma(B_0) = v^\emptyset$ is the least element of $\{v^{a|_n} : n \in \omega\}$.

Now assume we have defined B_j for all $j \leq n$ such that $\Gamma(B_j) \in \{v^{a|_n} : n \in \omega\}$. So, there is some $m \in \omega$ such that $\Gamma(B_n) = v^{a|_m}$.

Now, we know that $|s(B_n)| = k$ for some $k \in \omega$. Using a labeling method similar to that used in Definition 3.27, we say that $s(B_n) = \{B_n^i : 1 \leq i \leq k\}$, where the successors of B_n are labeled from left to right in $L_{n+1}(F)$. $\Gamma[s(B_n)]$ will look very similar to the mapping depicted in Figure 3. We will define B_{n+1} to be one of these successors.

By definition, there exists $j \in \omega$ such that $v^{a|_j} > v^{a|_m}$ and $v^{a|_j} \in \Gamma[s(B_n)] = \{\Gamma(B_n^i) : 1 \leq i \leq k\}$. Then choose B_{n+1} such that $\Gamma(B_{n+1}) = v^{a|_j}$.

Thus, we have now defined B_{n+1} such that $\Gamma(B_{n+1}) \in \{v^{a|_n} : n \in \omega\}$. So we have recursively constructed a branch $B = \{B_n : n \in \omega\}$ in F such that $\Gamma[B] \subseteq \{v^{a|_n} : n \in \omega\}$.

Since $\Gamma[B]$ is a tall chain and a subset of the branch $\{v^{a|_n} : n \in \omega\}$ in D, by Lemma 3.11, $\gamma(B) =\downarrow \Gamma[B] = \{v^{a|_n} : n \in \omega\}$. So, for an $a \in \{0,1\}^\omega$, we can find a branch B in F such that $\gamma(B) = \{v^{a|_n} : n \in \omega\}$, so γ is onto. □

We know that $\mathcal{B}[T]$ is Hausdorff for any tree T and we have shown $\mathcal{B}[D]$ and C to be homeomorphic, so $\mathcal{B}[D]$ is a compact space. By Corollary 1.49, if we show that $\gamma^{-1} : \mathcal{B}[D] \to \mathcal{B}[F]$ is continuous, then we have shown γ^{-1}, and therefore also γ, to be a homeomorphism.

Lemma 3.33. *γ^{-1} is continuous.*

Proof. Let $x \in F$.

$$\begin{aligned}\gamma[\mathcal{B}(x)] &= \{\gamma(B) : x \in B\} \\ &= \{\downarrow \Gamma[\sigma] : x \in \sigma\} \\ &= \{B \in \mathcal{B}[D] : \Gamma(x) \in \Gamma[\sigma] \subseteq B\} \\ &= \{B \in \mathcal{B}[D] : \Gamma(x) \in B\} \\ &= \{\mathcal{B}(\Gamma(x)) : \Gamma(x) \in D\}.\end{aligned}$$

Thus, the inverse image of an open set in $\mathcal{B}[F]$ is an open set in $\mathcal{B}[D]$, both with the branch topology, so γ^{-1} is continuous. □

We have now shown that γ is a homeomorphism between the branches of a finite split tree F and the branches of the two split tree D, proving Theorem 3.30. As we have previously established a homeomorphism between $\mathcal{B}[D]$ and the Cantor Set, we know that the Cantor Set is also homeomorphic to $\mathcal{B}[F]$.

4. THE CANTOR SET AND COMPACT METRIC SPACES

Now our goal is to establish a homeomorphism between the Cantor set and a more general compact metric space. To do this, we will find a bridging connection to a finite split tree, which we now know to be homeomorphic to the Cantor Set. To define a function to a tree, we will look at sequences of nested subsets of our original metric space, which will act as our branches.

We will add two constraints on our set M. Along with being a compact metric space, we will require M to be zero-dimensional and have no isolated points, two properties which we have shown C to have.

Lemma 4.1. *If M is a compact metric space which is zero-dimensional and contains no isolated points, then M can be partitioned into a finite number of compact subsets, all with diameter less than or equal to $\frac{1}{2}\operatorname{diam}(M)$.*

Proof. Let $\mathcal{U}$ be an open cover of M consisting of open balls of radius $r_1 = \frac{1}{4}\operatorname{diam}(M)$. We know that we can find open subsets with arbitrarily small diameters consisting of more than one point since there are no isolated points in M. So, $\mathcal{U}$ consists of open sets which are half the diameter of M. Every set in $\mathcal{U}$ is of the form $B(x, r_1)$ for $x \in M$. Since M is zero-dimensional, around each x we can find a closed and open set V such that $V \subseteq B(x, r_1)$. Let $\mathcal{V}^1$ be the collection of such sets. $\mathcal{V}^1$ covers M, so there is a finite subcover of $\mathcal{V}^1$, call it $\mathcal{V}^1_n$, and let $\mathcal{V}^1_n = \{V^1_1, V^1_2, ..., V^1_n\}$.

Now we would like for our finite subcover to partition M rather than just be a finite cover, so we will define a new collection $\mathcal{W}^1$. Let $W^1_1 = V^1_1$. Now, let $W^1_k = V^1_k - \bigcup_{i=1}^{k-1} V^1_i$ for all $k \in \omega$ with $1 \leq k \leq n$. Let $\mathcal{W}^1 = \{W^1_1, W^1_2, ..., W^1_n\}$. Then $\mathcal{W}^1$ is a finite partition of M, consisting of sets which are both open and closed. Since each W^1_k is a closed subset of M, by Theorem 1.45, each W^1_k is compact. The diameters of each set in $\mathcal{W}^1$ are less than or equal to $\frac{1}{2}\operatorname{diam}(M)$. □

We will define our sequence of nested disjoint subsets through recursion. So, we assume that $\mathcal{W}^n$ has been defined for $n \in \omega$, with $1 \leq n$, such that $\mathcal{W}^n$ covers M with a finite partition of open, closed, and therefore also compact sets with diameters less than or equal to $\frac{1}{2^n}\operatorname{diam}(M)$.

Let W_j^n be some set in $\mathcal{W}^n$. W_j^n is also zero-dimensional and contains no isolated points by Lemmas 1.37 and 1.40, so by Lemma 4.1 above we can find a finite partition of W_j^n consisting of open, closed, and compact sets such that the diameter of each set is less than or equal to $\frac{1}{2}\operatorname{diam} W_j^n = \frac{1}{2^{n+1}}\operatorname{diam}(M)$. Let $\mathcal{X}_m^j$ be this partition.

Since W_j^n was an arbitrary set in $\mathcal{W}^n$, so we can find such $\mathcal{X}_m^j$ for all $j \in \omega$ where $W_j^n \in \mathcal{W}^n$. Then, we define $\mathcal{W}^{n+1} = \bigcup_{j\in\{j:W_j^n\in\mathcal{W}^n\}} \mathcal{X}_m^j$.Then $\mathcal{W}^{n+1}$ partitions M into finitely many open, closed, and compact sets, all with diameters less than or equal to $\frac{1}{2^{n+1}}\operatorname{diam}(M)$. This splitting process can continue indefinitely because M has no isolated points.

To establish the function between M and the finite split tree, we must give an order to M, so we will give M the subset order as established in Lemma 1.2. That is, if $X, Y \subseteq M$, we say $X < Y$ if and only if $Y \subset X$. In this order, M is the smallest set.

Now we have $W_j^n < X_1^j$, $W_j^n < X_2^j$,...,$W_j^n < X_m^j$. That is, W_j^n has a finite number, m, of successors. Since W_j^n was an arbitrary set in the collection $\mathcal{W}^n$, and n was an arbitrary index of $\mathcal{W}$, it has been shown that each set in this partitioning process of M has a finite number of successor sets. We also know that no two sets in a partition, $\mathcal{W}^n$ share a successor because they do not intersect and therefore cannot share a subset.

We will let a set in the partitioning process be a vertex and give the $<$ relation that we have established on M to the vertices. M is the first vertex of the tree as it is the least set in the order. The tree created with these vertices and edges will be called T_M. We define T_M to be the set of all such W_j^n as described above. To better understand T_M, we would like to describe what the levels and branches of T_M look like. An inductive argument will show that $\mathcal{W}^n = L_n(T_M)$, the n^{th} level of T_M. Each $W_j^n \in \mathcal{W}^n$ is a subset of, and therefore greater than, one distinct $W_i^{n-1} \in \mathcal{W}^{n-1}$. Recall that a branch of a tree is a maximal chain, or a maximal set of comparable vertices. In T_M, two vertices are comparable if and only if one is a subset of the other, so a chain in T_M is a sequence of nested subsets from different levels. Thus, a branch of T_M is a sequence of nested subsets from every level.

The end goal is to show that the Cantor Set is homeomorphic to M. Since we have already established a homeomorphism between C and a the branch space of the finite split tree F, and T_M is a finite split tree, we will establish a homeomorphism between $\mathcal{B}[T_M]$ and M.

As we increase in the level of T_M, the vertex sets of the level become smaller in diameter. On the n^{th} level, which is $\mathcal{W}^n$, we know that each vertex set has a diameter less than or equal to $\frac{1}{2^n}\operatorname{diam}(M)$. Thus, as n becomes arbitrarily large, the diameters of the sets in the n^{th} level tend to zero. From Corollary 1.52, the intersection of an infinite sequence of nested compact sets is nonempty. A nonempty set with diameter zero is just a single point. Specifically, the intersection of all vertex sets of a branch in T_M is a single point of M. We will use this fact to define a function $\delta : \mathcal{B}[T_M] \to M$ which we will show is a homeomorphism. If $B \in \mathcal{B}[T_M]$ then we will let W_B^n denote the vertex set of $\mathcal{W}^n$ which is in the branch B.

Theorem 4.2. *The function $\delta : \mathcal{B}[T_M] \to M$, defined as $\delta(B) = \bigcap_{n\in\omega} W_B^n$, is a homeomorphism between $\mathcal{B}[T_M]$ and M.*

To show δ is a homeomorphism, we show δ^{-1} is a homeomorphism. We first need to show that δ, and therefore also δ^{-1}, is a bijection.

Lemma 4.3. *δ is one-to-one*

Proof. Let $A, B \in \mathcal{B}[T_M]$ with $A \neq B$. Then there is some level, $\mathcal{W}^k$, of T_M for which $W_A^k, W_B^k \in \mathcal{W}^k$, with $W_A^k \neq W_B^k$, and $W_A^k \in A, W_B^K \in B$. Since $W_A^k \neq W_A^k$ in a partition, $W_A^k \cap W_B^k = \emptyset$. Then,

$$\bigcap_{n\in\omega} W_A^n = W_A^k \cap (\bigcap_{n\in\omega, n\neq k} W_A^n) \neq W_B^k \cap (\bigcap_{n\in\omega, n\neq k} W_B^n) = \bigcap_{n\in\omega} W_B^n$$

Thus, δ is one-to-one. □

Lemma 4.4. *δ is onto.*

Proof. Let $x \in M$. We need to show that there exists $B \in T_M$ such that $\delta(B) = \bigcap_{n\in\omega} W_B^n = x$. We can define a branch B to be the branch for which $x \in W_B^n$ for all $n \in \omega$. Then $\bigcap_{n\in\omega} W_B^n = x$, otherwise there would have to be some $k \in \omega$ such that $x \notin W_B^k$. Thus, δ is onto. □

Therefore, δ^{-1} is also one-to-one and onto. Since $\delta^{-1} : M \to \mathcal{B}[T_M]$ is a mapping from a compact space to a Hausdorff space, if δ^{-1} is shown to be continuous, by Corollary 1.49, δ^{-1} is a homeomorphism.

Lemma 4.5. *δ^{-1} is continuous.*

Proof. Let $V \in T_M$, so V is an open compact subset of M as well. Then $\mathcal{B}[V]$ is the set of all branches of V and is an open set in $\mathcal{B}[T]$ with the branch topology. We would like to show that $\delta[\mathcal{B}[V]] = V$, because $\mathcal{B}[V]$ is open in $\mathcal{B}[T_M]$ and V is open in M. Suppose that $x \in \delta[\mathcal{B}[V]]$. Then, $x = \bigcap_{n\in\omega} W_B^n$, where V is in the branch B. V is on some level of T_M, so let $V \in \mathcal{W}^k$. Then $x = \bigcap_{n\in\omega, n\neq k} W_B^n \cap V$. So $x \in V$ and $\delta[\mathcal{B}[V]] \subseteq V$. Now we suppose that $x \in V$. Then, $x \in M$ since $V \subseteq M$, and since δ is onto, there exists $B \in \mathcal{B}[T_M]$ such that $\delta(B) = x$. Since $x \in V$, $V = \mathcal{W}^k \cap B$, for some level $\mathcal{W}^k$. Thus $V \in B$ and so $B \in \mathcal{B}[V]$, so $x = \delta(B) \in \delta[\mathcal{B}[V]]$ and $V \subseteq \delta[\mathcal{B}[V]]$. Therefore, it must be that $\delta[\mathcal{B}[V]] = V$. Since V is an open subset of M, δ^{-1} is continuous. □

Therefore we have proved Theorem 4.2, showing that $\delta : \mathcal{B}[T_M] \to M$ is a homeomorphism. We are now ready to state and prove the main Theorem of this paper.

Theorem 4.6. *The Cantor Set, C, is homeomorphic to any compact metric space, M, which is zero-dimensional and contains no isolated points.*

Proof. By Theorem 2.10, $\alpha^{-1} : C \to \{0,1\}^\omega$ is a homeomorphism. By Theorem 3.22, $\beta : \{0,1\}^\omega \to \mathcal{B}[D]$ is a homeomorphism. By Theorem 3.30, $\gamma^{-1} : \mathcal{B}[D] \to \mathcal{B}[F]$ is a homeomorphism. The function γ^{-1} is a homemorphism between D and any general finite split tree F, so it is also the case that $\gamma^{-1} : \mathcal{B}[D] \to \mathcal{B}[T_M]$ is a homeomorphism, as T_M is a finite split tree. Lastly, by Theorem 4.2, $\delta : \mathcal{B}[T_M] \to M$ is a homeomorphism. Let $\Phi = \delta \circ \gamma^{-1} \circ \beta \circ \alpha^{-1}$. Then, by Lemma 1.31, $\Phi : C \to M$ must also be a homeomorphism. □