4-17-2013

Research exercise: Simulation of Nonlinear Waves Using Sinc Collocation-Interpolation

Simulation of Nonlinear Waves Using Sinc Collocation-Interpolation

By: Jessica Steve and Eric Gerwin
Advised by Dr. Muhammad Usman

INTRODUCTION.

The study of nonlinear wave phenomena is one of the relevant topics in fluid mechanics. Historical research and numerical work have found that partial differential equations are able to model and give numerical results to these waves. Our project will be simulating nonlinear waves that can be obtained by collocation interpolation methods based on the Sinc functions.

We will focus on the Dirichlet and Neumann boundary conditions as well as two classical nonlinear wave equations: the generalized third--order Korteweg-de Vries equations and the sine-Gordon equation.

NUMERICAL METHOD.

Consider nonlinear partial differential wave equations of the type

$$\frac{\partial u}{\partial t} = f\left(t, x, \frac{\partial u}{\partial x}, \dots, \frac{\partial^r u}{\partial x^r}\right),$$

where t is the time, x is the space, and u is the dependent variable. The numerical simulation of the above initial boundary value problem can be developed through the following steps:

STEP 1.

The space variable is discretized into the collocation(discrete model in space)

$$i = 1, \dots, n: \{x_1 = a, \dots, x_i = a + (i-1)h, \dots, x_n = b\},$$
$$h = \frac{|b-a|}{n-1},$$

so that the independent variable $u = u(t,x)$ is interpolated and approximated by means of the Sinc function as follows:

$$u(t,x) \cong u^n(t,x) \equiv \sum_{j=1}^{n} S_j(x;h) u_j(t),$$

where $u_j(t) = u(t, x_j)$ denotes the values of u in the nodal points of the chosen collocation and

$$S_j(x:h) = \frac{\sin z_j}{z_j}, \quad z_j = \frac{\pi}{h}(x - (j-1)h - a).$$

In this way, the spatial behavior and the time evolution of the solution are decoupled, meaning they no longer depend on each other.

STEP 2.

According to the collocation-interpolation defined in the previous step, the partial derivatives of the variable u with respect to space are approximated as follows:

$$\frac{\partial^r u}{\partial x^r}(t; x_i) \cong \frac{\partial^r u^n}{\partial x^r}(t; x_i) = \sum_{j=1}^{n} a_{ji}^{(r)} u_j(t),$$

$$\text{where } a_{ji}^{(r)} = \frac{d^r s_j}{dx^r}(x_i).$$

STEP 3.

Substitute partial derivatives into **Step 1**:

$$\frac{du_i}{dt} = f(t, x_i, u, a), \text{for } i = 1, \dots, n$$

where $u(t) = \{u_i(t)\}$ and $a = \left\{a_{ji}, a_{ji}^{(1)}, \dots\right\}$.

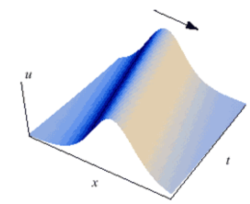
EXAMPLE 1.

GENERALIZED THIRD-ORDER KORTEWEG-DE VRIES EQUATION. This example often arises in the field of fluid mechanics.

$$u_t + u^m u_x + \mu u_{xxx} = 0,$$

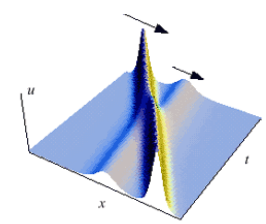
EXAMPLE 1.1

SINGLE SOLITION



EXAMPLE 1.2

INTERACTION BETWEEN SOLITIONS.



EXAMPLE 2.

SINE-GORDON EQUATION.

$$u_{tt} - u_{xx} + \sin u = 0$$