Truncation Error for a Finite Difference Scheme for the Black-Scholes Model

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ABSTRACT

Finite difference methods are simplest and oldest methods among all the numerical techniques to approximate the solution of partial differential equations (PDEs). The derivatives in the partial differential equation are approximated by finite difference formulas. The error between the numerical solution and the exact solution is determined by the error between a differential operator to a difference operator. This error is called the discretization error or truncation error. The term truncation error reflects the fact that a finite part of a Taylor series is used in the approximation. In this work we will analyze the truncation error for a finite difference scheme for the Black-Scholes PDE for the valuation of an option.

1 Introduction

The Black-Scholes (BS) Option Pricing Model is one of the prominent models in finance. It was published by Fisher Black and Myron Scholes in their 1973 paper, *The Pricing of Options and Corporate Liabilities*. Their work involved calculating a derivative to measure how the discount rate of a warrant varies with time and stock price. The result was a partial differential equation similar to the heat transfer equation. The standard BS-PDE is

$$\frac{\partial C}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-\delta)S\frac{\partial C}{\partial S} - rC = 0.$$

Analytically, the solution to the BS-PDE is called the Black-Scholes Formula given by:

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

- $C(S,t)$ is the price of a European call option,
- S is the underlying asset price (e.g. stock price, an index),
- $N(.)$ the cumulative distribution function of the standard normal distribution,
- $T-t$ is the time to expiration,
- K is the strike price,
- r is the continuous compounding risk free rate (annual),
- σ is the volatility of return of the underlying asset.

The **figure 1** below is an example of a simple option pricing method calculated using the analytical solution of the Black-Scholes formula with changing stock prices and time.

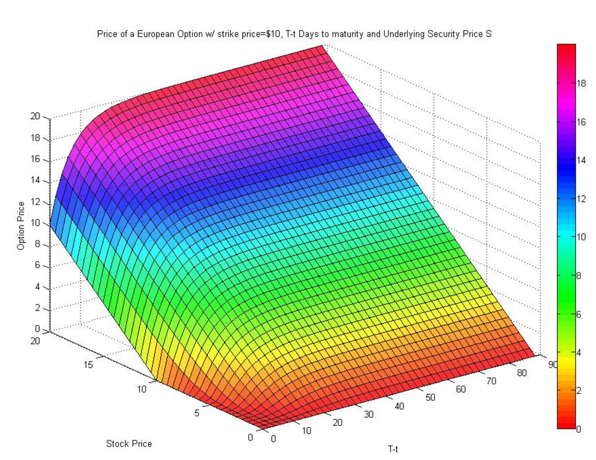

Figure 1: Price of an option as a function of time to maturity and the stock price due to the analytic BS formula. (Source: http://www3.math.tu-berlin.de/Vorlesungen/SS10/PWRpy)

This formula is used to price plain vanilla European call options. The formula does not work well in pricing more complex exotic options. Most of these options are path dependent and there is no closed form solution for the price. Hence, the BS formula is not useful in pricing complex options.

Therefore, our motivation is to solve the BS-PDE numerically with different boundary conditions as most exotic options have different conditions to those used in solving the BS-PDE for simple European call options. First step towards numerical solution of partial differential equations is the discretization of PDE. By discretization we mean any method of reducing continuous PDE to a discrete set of difference equations that can be solved on a computer. We discretize the PDE and obtain the truncation error of the finite difference schemes. **Truncation error** is the error made by cutting off an infinite sum and approximating it by a finite sum. We take a finite number of steps to approximate the solution to the BS-PDE.

2 Black-Scholes Partial Differential Equation

We begin by writing our general heat equation

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\frac{\partial^2 U}{\partial x^2} + \frac{r-\sigma^2/2}{\sigma^2}\frac{\partial U}{\partial x} - \frac{r}{\sigma^2}U.$$

We rewrite the coefficients of the differential operators by replacing them with letters A, B and D

$$\frac{\partial U}{\partial \tau} = A\frac{\partial^2 U}{\partial x^2} + B\frac{\partial U}{\partial x} + DU. \tag{1}$$

To solve the PDE numerically we need to discretize the PDE (1). We construct a weighted finite difference approximation to each of the derivative terms in the PDE. We use weighted finite difference scheme with α as temporal and ω the spatial weight.

$$\begin{aligned}\frac{U_i^{n+1}-U_i^n}{\Delta\tau} =& A\left[\omega\frac{U_{i+1}^{n+1}-2U_i^{n+1}+U_{i-1}^{n+1}}{\Delta x^2} + (1-\omega)\frac{U_{i+1}^n-2U_i^n+U_{i-1}^n}{\Delta x^2}\right]\\ &+ B\left[\omega\left(\frac{(1-\alpha)U_i^{n+1}+\alpha U_{i+1}^{n+1}-(1-\alpha)U_{i-1}^{n+1}-\alpha U_i^{n+1}}{\Delta x}\right)\right.\\ &\left.+ (1-\omega)\left(\frac{(1-\alpha)U_i^n+\alpha U_{i+1}^n-(1-\alpha)U_{i-1}^n-\alpha U_i^n}{\Delta x}\right)\right]\\ &- D\left[\omega U_i^{n+1}+(1-\omega)U_i^n\right].\end{aligned} \tag{2}$$

This is a general case scheme, but if we set $\omega = 0, 0.5$ or 1 and set $\alpha = 0$ we get an explicit/forward, Crank-Nicolson (CN) or implicit backward finite difference in time, respectively.

Now we expand the terms U_{i+1}^n, U_{i-1}^n, U_i^{n+1}, U_{i+1}^{n+1} and U_{i-1}^{n+1} using Taylor expansion, where h is a spatial step and k is a temporal step. We drop terms that have partial derivatives of order higher than 2. For example, we drop any term involving $\frac{\partial}{\partial x^3}$, $\frac{\partial}{\partial x^4}$, e.t.c. Since we only consider a finite part of the Taylor's Series we generate truncation errors in our numerical scheme.

$$U_{i+1}^n \approx U_i^n + h\frac{\partial U}{\partial x} + \frac{h^2}{2}\frac{\partial^2 U}{\partial x^2}. \tag{3}$$

$$U_{i-1}^n \approx U_i^n - h\frac{\partial U}{\partial x} + \frac{h^2}{2}\frac{\partial^2 U}{\partial x^2}. \tag{4}$$

Taylor expansion with respect to time derivatives:

$$U_i^{n+1} \approx U_i^n + \sum_{n=1}^{\infty}\left(\frac{k^n}{n!}\frac{\partial^n U}{\partial \tau^n}\right), \tag{5}$$

$$U_{i+1}^{n+1} \approx U_i^{n+1} + h\frac{\partial U^{n+1}}{\partial x} + \frac{h^2}{2}\frac{\partial^2 U^{n+1}}{\partial x^2}, \tag{6}$$

$$U_{i-1}^{n+1} \approx U_i^{n+1} - h\frac{\partial U^{n+1}}{\partial x} + \frac{h^2}{2}\frac{\partial^2 U^{n+1}}{\partial x^2}. \tag{7}$$

Taking higher order partial derivative of equation (1), we get,

$$\begin{aligned}\frac{\partial^2 U}{\partial \tau^2} &= A\frac{\partial^2 U}{\partial x^2}\left(\frac{\partial U}{\partial \tau}\right) + B\frac{\partial U}{\partial x}\left(\frac{\partial U}{\partial \tau}\right) + DU,\\ &= A\frac{\partial^2}{\partial x^2}\left(A\frac{\partial^2 U}{\partial x^2} + B\frac{\partial U}{\partial x} + DU\right) + B\frac{\partial}{\partial x}\left(A\frac{\partial^2 U}{\partial x^2} + B\frac{\partial U}{\partial x} + DU\right) + D\left(A\frac{\partial^2 U}{\partial x^2} + B\frac{\partial U}{\partial x} + DU\right).\end{aligned}$$

We ignore the third or higher order spatial derivatives to get:

$$\frac{\partial^2 U}{\partial \tau^2} \approx (B^2+2AD)\frac{\partial^2 U}{\partial x^2} + 2BD\frac{\partial U}{\partial x} + D^2U.$$

Next, we take the third and fourth partial derivatives with respect to time:

$$\begin{aligned}\frac{\partial^3 U}{\partial \tau^3} &\approx \left(3B^3D + 3AD^2\right)\frac{\partial^2 U}{\partial x^2} + 3BD^2\frac{\partial U}{\partial x} + D^3U,\\ \frac{\partial^4 U}{\partial \tau^4} &\approx \left(6B^2D^2 + 4AD^3\right)\frac{\partial^2 U}{\partial x^2} + 4BD^3\frac{\partial U}{\partial x} + D^4U.\end{aligned}$$

We notice a pattern and if we continue taking the derivatives and leave out the 3rd or higher order derivatives we get, for $n \geq 2$, the generalized form

$$\frac{\partial^n U}{\partial \tau^n} \approx (-1)^n\left(\frac{n(n-1)}{2}(-D)^{n-2}B^2 - n(-D)^{n-1}A\right)\frac{\partial^2 U}{\partial x^2} - (-1)^n n(-D)^{n-1}B\frac{\partial U}{\partial x} + (-1)^n n(-D)^n U.$$

If substitute this into equation (5) we obtain

$$\begin{aligned}U_i^{n+1} \approx U_i^n + k\frac{\partial U}{\partial \tau} + \sum_{n=2}^{\infty}\frac{k^n}{n!}&\left[(-1)^n\left(\frac{n(n-1)}{2}(-D)^{n-2}B^2 - n(-D)^{n-1}A\right)\frac{\partial^2 U}{\partial x^2}\right.\\ &\left. - (-1)^n n(-D)^{n-1}B\frac{\partial U}{\partial x} + (-1)^n n(-D)^n U\right].\end{aligned}$$

We again substitute this and using the following cross partial derivatives in (6) and (7)

$$\frac{\partial^2 U}{\partial x\partial \tau} = A\frac{\partial^3 U}{\partial x^3} + B\frac{\partial^2 U}{\partial x^2} + D\frac{\partial U}{\partial x},$$

$$\frac{\partial^3 U}{\partial x^2\partial \tau} = A\frac{\partial^4 U}{\partial x^4} + B\frac{\partial^3 U}{\partial x^3} + D\frac{\partial^2 U}{\partial x^2},$$

we get a long form of U_{i+1}^{n+1} and U_{i-1}^{n+1} which we then substitute in equation (2) to get a discretized PDE

$$\begin{aligned}\frac{\partial U}{\partial \tau} = &\left\{A + 2\omega ADk + \left(\alpha - \frac{1}{2}\right)\omega BDhk + \left(\alpha - \frac{1}{2}\right)Bh + \omega kB^2\right.\\ &+ \sum_{n=2}^{\infty}\left[\frac{k^{n-1}}{(n-1)!}(-1)^n\left(\frac{n-1}{2}B^2(-D)^{n-2} - A(-D)^{n-2}\right)\right](-1+\omega Dk)\\ &+ \sum_{n=2}^{\infty}\left[\frac{k^n}{(n-1)!}(-1)^n(-D)^{n-1}(-B)\right]\omega B\\ &\left.+ \sum_{n=2}^{\infty}\left[\frac{k^n}{n!}(-1)^n(-D)^n\right]\left(\omega A + \omega\alpha Bh - \omega\frac{hB}{2}\right)\right\}\frac{\partial^2 U}{\partial x^2}\\ &+ \left\{B + 2\omega ABDk + \sum_{n=2}^{\infty}\left[\frac{k^{n-1}}{(n-1)!}(-1)^n(-D)^{n-1}B\right)\right](1-\omega Dk)\\ &\left.+ \sum_{n=2}^{\infty}\left[\frac{k^n}{n!}(-1)^n(-D)^{n-1}\right]\omega B\right\}\frac{\partial U}{\partial x}\\ &+ \left\{D + \omega D^2k - \sum_{n=2}^{\infty}\left[\frac{k^{n-1}}{n!}(-1)^n(-D)^n\right](1-\omega kD)\right\}U.\end{aligned} \tag{8}$$

Now, we compare equations (8) and (1). In (8) the coefficients of the partial derivatives correspond to the exact coefficients in (1). We call these *numerical coefficients* and the difference between these coefficients and the exact ones is the truncation error. The form of PDE (1) with numerical coefficients is

$$\frac{\partial U}{\partial \tau} = A_{num}\frac{\partial^2 U}{\partial x^2} + B_{num}\frac{\partial U}{\partial x} + D_{num}U$$

where "num" stands for numerical coefficients.

- $\tau_A = A_{num} - A$ is the second order truncation error in A,
- $\tau_B = B_{num} - B$ is the first order truncation error in B,
- $\tau_D = D_{num} - D$ is zero-order truncation error in D.

3 Truncation Errors

If we substitute $\omega = 0.5$ and $\alpha = 0.5$ in equation (8) we obtain the Crank-Nicolson approximation of the truncation errors in A, B and D:

$$\begin{aligned}\tau_A = &\left\{ADk + 0.5kB^2 + \sum_{n=2}^{\infty}\left[\frac{k^{n-1}}{(n-1)!}(-1)^n\left(\frac{n-1}{2}B^2(-D)^{n-2} - A(-D)^{n-2}\right)\right](-1+0.5Dk)\right.\\ &\left.+ \sum_{n=2}^{\infty}\left[\frac{k^n}{(n-1)!}(-1)^n(-D)^{n-1}(-B)\right](0.5)B + \sum_{n=2}^{\infty}\left[\frac{k^n}{n!}(-1)^n(-D)^n\right](0.5)A\right\}.\end{aligned}$$

$$\tau_B = \left\{ABDk + \sum_{n=2}^{\infty}\left[\frac{k^{n-1}}{(n-1)!}(-1)^n(-D)^{n-1}B\right](1-0.5k) + \sum_{n=2}^{\infty}\left[\frac{k^n}{n!}(-1)^n(-D)^{n-1}\right](0.5)B\right\}.$$

$$\tau_D = \left\{0.5D^2k - \sum_{n=2}^{\infty}\left[\frac{k^{n-1}}{n!}(-1)^n(-D)^n\right](1-0.5kD)\right\}.$$

4 Conclusion

We have demonstrated how to apply the finite difference method to approximate the solution of the BS-PDE. Apart from the truncation error analysis, we have found the truncation error formulas in the numerical coefficients. Using this approach we can improve in approximation of the the BS-PDE by carefully selecting boundary conditions that will minimize the errors. The finite difference method can be used to price more complex, exotic, options which we could not solve exactly. In our future work we plan to use the corrected coefficients instead of A_{num}, B_{num} and D_{num} and simulate the numerical solution of the scheme for the valuation of different options with appropriate boundary conditions.

References

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