A Business Application of Markov Chains

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Abstract

The purpose of this project is to look at the economic applications of Markov Chains and stochastic matrices in a real-world problem. For this project, I studied a specific case for a rental car company which has a fleet of vehicles for rent in a number of locations. It is very important for the company to know the number of vehicles that will be available on a typical day at each location. This project will show that, on a typical day, the number of available vehicles at the various locations can be known by calculating what is called the steady-state vector of the stochastic matrix associated with the problem. This will help the company to better manage the customer demands. It can then be used to meet any increased demand at a particular location which can occur due to various reasons.

Explanation

Markov Chains are used as mathematical models on a variety of different situations. It can be used for several real-world applications in areas such as science, business, engineering, and elsewhere. In each case, the model is used to describe an experiment or measurement that is performed many times in the same way. The outcome of each trial of an experiment will be one of several possible outcomes where the outcome of one trial depends only on the trial right before it.

In order to better understand Markov Chains and its applications to real world problems, there are first several vocabulary terms that should be defined. A vector with nonnegative entries that add up to one is called a probability vector. A stochastic matrix is a square matrix whose columns are probability vectors. A Markov Chain is a sequence of probability vectors

$$\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$$

together with a stochastic matrix P, such that

$$\boldsymbol{x}_1 = P\boldsymbol{x}_0,\ \boldsymbol{x}_2 = P\boldsymbol{x}_1,\ \boldsymbol{x}_3 = P\boldsymbol{x}_2,\ \ldots$$

If matrix P, of the above equation, is an nxn regular stochastic matrix, then it is known that P has a unique steady-state vector (or equilibrium vector) $\boldsymbol{q}$ (see Theorem 1 below). If $\boldsymbol{x}_0$ is any initial state vector and

$$\boldsymbol{x}_{k+1} = P\boldsymbol{x}_k, \qquad k = 0,1,2,3,\ldots$$

then the Markov Chain $\{\boldsymbol{x}_k\}$ converges to a steady-state vector $\boldsymbol{q}$, as k approaches infinity, which implies

$$P\boldsymbol{q} = \boldsymbol{q}. \tag{1}$$

Definition 1: Regular Stochastic Matrix – a stochastic matrix P is regular if some matrix power P^k contains only strictly positive entries.

Theorem 1: If P is a regular stochastic matrix, then P has a unique steady-state vector $\boldsymbol{q}$. If $\boldsymbol{x}_0$ is any initial state vector and $\boldsymbol{x}_{k+1} = P\boldsymbol{x}_k$ for $k = 0,1,2,\ldots$, then the Markov chain $\{\boldsymbol{x}_k\}$ converges to $\boldsymbol{q}$ as $k \to \infty$.

One of the interesting aspects of Markov Chain is the ability to use it to study long-term behavior. Long-term behaviors can be determined from steady-state vectors. From equation (**1**), we see that the steady-state vector of P can be obtained by solving the matrix equation,

$$P\boldsymbol{q} = \boldsymbol{q}.$$

To solve this equation, you must set the matrix equation equal to zero.

$$P\boldsymbol{q} - \boldsymbol{q} = 0$$

Then by using the fact that $I\boldsymbol{x} = \boldsymbol{x}$ the equation becomes

$$P\boldsymbol{q} - I\boldsymbol{q} = 0$$

$$(P - I)\boldsymbol{q} = 0.$$

If this equation cannot be solved exactly a numerical approach can be applied to compute the sequence $\{\boldsymbol{x}_k\}$ for $k = 1,2,3,\ldots n$. Since $\{\boldsymbol{x}_k\}$ converges, a good estimate of the steady-state vector can be obtained after a number of steps.

A Business Application

Now we will look at a business application of Markov Chain on a car rental company. For this problem, we will look at a specific company that has about 2,000 cars. There are three locations where the cars can be rented out and returned to; the city airport (CA), the metro airport (MA) and the downtown location (DL). The pattern of rental and return locations is given by the fractions in the matrix below.

$$\textit{Cars rented from}$$

$$P = \begin{array}{c} \begin{array}{ccc} CA & DL & MA \end{array} \\ \begin{bmatrix} .90 & .01 & .09 \\ .01 & .90 & .01 \\ .09 & .09 & .90 \end{bmatrix} \end{array} \begin{array}{c} \\ CA \\ DL \\ MA \end{array} \qquad \textit{Cars returned to}$$

Since all the entries of the above matrix P are strictly positive, P is a regular stochastic matrix. Therefore, there exist a steady-state vector $\boldsymbol{q}$ such that $P\boldsymbol{q} = \boldsymbol{q}$. The steady-state vector $\boldsymbol{q}$ will provide us an estimate of how many cars will be available to rent from and be returned to each location. Obtaining this information will help the car rental company plan for appropriate number of cars they should have at each location. Solving $P\boldsymbol{q} = \boldsymbol{q}$, yields

$$\begin{bmatrix} .90 & .01 & .09 \\ .01 & .90 & .01 \\ .09 & .09 & .90 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -.10 & .01 & .09 \\ .01 & -.10 & .01 \\ .09 & .09 & -.10 \end{bmatrix},$$

by using the rules of row reduction, we reduce the matrix to its echelon form:

$$\begin{bmatrix} 1 & 0 & -.9192 \\ 0 & 1 & -.1919 \\ 0 & 0 & 0 \end{bmatrix}$$

From this matrix we get the following:

$$\boldsymbol{x}_1 = .9192\boldsymbol{x}_3 \quad , \quad \boldsymbol{x}_2 = .1919\boldsymbol{x}_3 \quad , \quad \boldsymbol{x}_3 \text{ is free}$$

This can be written as the following:

$$\boldsymbol{x}_3 \begin{bmatrix} .9192 \\ .1919 \\ 1 \end{bmatrix}$$

We can then set $\boldsymbol{x}_3$ equal to $\frac{1}{2.1111}$ (we get 2.1111 from $0.9192 + 0.1919 + 1$). When we apply $\boldsymbol{x}_3 = \frac{1}{2.1111}$ to the above matrix we obtain the steady-state vector

$$\boldsymbol{x}_3 \begin{bmatrix} .9192 \\ .1919 \\ 1 \end{bmatrix} \sim \begin{bmatrix} {}^{.9192}/_{2.1111} \\ {}^{.1919}/_{2.1111} \\ {}^{1}/_{2.1111} \end{bmatrix} \sim \begin{bmatrix} .435 \\ .091 \\ .474 \end{bmatrix} = \boldsymbol{q}.$$

Now we can use $\boldsymbol{q}$ to find the number of cars that will be rented or ready to rent from all locations. For that, we need to multiply the entries of $\boldsymbol{q}$ by 2,000, the total number of vehicles. This gives us the number of vehicles that will be available on a typical day. Therefore, on a typical day, the number of vehicles that will be available at each location is as follows:

City airport: $0.435 * 2000 = 870$

Metro airport: $0.474 * 2000 = 948$

Downtown location: $0.091 * 2000 = 182$