Picard's Successive Approximation vs. Banach's Contraction Principle

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Abstract

The existence of a unique solution to the initial value problem:

$$x^{'}(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

can be obtained employing Banach's contraction principle or Picard's successive approximation method. Generally, the norm used in the contraction principle is the supremum norm, which requires a condition that is not needed in the successive approximation method. Therefore, it seems that the successive approximation method is a superior method. The objective of this project is to show that these two methods are equally efficient. This is due to the fact that the extra condition needed in the contraction principle can be eliminated by using a different norm that is equivalent to the supremum norm.

Successive Approximations

We want to find a unique solution to the initial value problem:

$$x^{'}(t) = f(t, x(t)), \; x(t_0) = x_0 \qquad (1)$$

This is equivalent to finding the solution to the integral equation

$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$

Let f be continuous and locally Lipschitz on the domain of f, i.e. there exists a region $R = \{(t,x)\colon t_0 - a \le t \le \; t_0 + a, \; x_0 - b \le x \le \; x_0 + b\}$ for positive constants a and b such that f is continuous and Lipschitz on R . The function f is Lipschitz on R means there is a positive constant k such that

$|f(t,x_1) - f(t,x_2)| \le k|x_1 - x_2| \quad$ for $(t,x_1),(t,x_2) \, \epsilon \, R$.

Suppose $\sup_{(t,x)\in R} |f(t,x)| = M$.

Let $\alpha = \min\{a, \frac{b}{M}\}$ and let $I_\alpha = [t_0 - \alpha, t_0 + \alpha]$.

Theorem 1

Suppose $f(t,x)$ is continuous and locally Lipschitz, on its domain. Then the Initial Value Problem (1) has a solution $\phi(t)$ on the interval I_α containing t_0.

We prove this using successive approximations method.

Let

$\phi_0(t) = x_0,$

$\phi_{j+1} = x_0 + \int_{t_0}^{t} f(s, \phi_j(s))\, ds; \quad j = 0,1,2 \ldots$

We then construct a sequence $\{\phi_j\}$ and show it converges to some function $\phi(t)$, on I_α which is a solution to the Initial Value Problem (1).

Gronwall inequality – Let n be a non-negative constant and let f and g be continuous non-negative function on some interval $u \le x \le v$ satisfying: $f(x) \le n + \int_u^x f(s)g(s)ds$.

Then, $f(x) \le ne^{\int_u^x g(s)ds} \quad$ for $\quad u \le x \le v$

Theorem 2

Under the same conditions as Theorem 1, the Initial Value Problem (1) has a unique solution.

To prove this we use the Gronwall inequality stated above to show that $\phi(t)$ is a unique solution to the Initial Value Problem(1).

Contraction Principle with Supremum norm

Let B be the set of all real-valued, continuous, and bounded functions x on I_α.

Then B is a Banach space with the supremum norm:

$\|x\| = \sup_{t \in I_\alpha} |x(t)|.$

Let $A = \{x \in B : |x(t) - x_0| \le M\alpha\}$. It is easy to show that A is closed.

Define $T\colon A \to A$ as, follows. For each ϕ in A,

$(T\phi)(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))ds \quad$ for t in I_α

Theorem 3

Suppose $f(t,x)$ is continuous and locally Lipschitz, i.e. equation (3) holds, on its domain. Choose $\bar{\alpha} \le \alpha$ such that $k\,\bar{\alpha} < 1$, and where k is the Lipschitz constant. Then the Initial Value Problem (1) has a unique solution $\phi(t)$ on $I_{\bar{\alpha}}$ containing t_0.

To prove this we show $\|T\phi_1 - T\phi_2\| \le k\,\bar{\alpha}\|\phi_1 - \phi_2\|$. Which implies T is a contraction since $k\,\bar{\alpha} < 1$.

So, by the Contraction Principle, there exists a unique fixed point ϕ in A such that $T(\phi) = \phi$ meaning:

$\phi(t) = \phi_0 + \int_{t_0}^{t} f(s, \phi(s)\, ds, \;\; t \in I_{\bar{\alpha}}.$

which is the unique solution to the Initial Value Problem (1).

We have shown using the Contraction Principle that there exists a unique solution to the Initial Value Problem (1). Note that this proof required that $k\,\bar{\alpha} < 1$, which was not required when the Successive Approximations method was employed.

Contraction Principle using Alternate norm

We can eliminate the need for this condition by using an alternate norm.

Let $\|x\| = \sup_{t_0 \le t \le t_0 + \alpha} \{e^{-2k(t-t_0)}|x(t)|\}$ where k is the Lipschitz constant.

Note that this norm is equivalent to the supremum norm defined earlier. This means the set B defined above is a Banach space under this norm as well.

Let A be the set defined above.

Let T be defined on A as follows, for each ϕ in A,

$(T\phi)(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))ds \quad$ for t in $t_0 \le t \le t_0 + \alpha$.

Theorem 4

Suppose $f(t,x)$ is continuous and locally Lipschitz, i.e. equation (3) holds, on its domain. Then the Initial Value Problem (1) has a unique solution $\phi(t)$ on I_α containing t_0.

To prove this we show $\|T\phi_1 - T\phi_2\| \le \frac{1}{2}(1 - e^{-2k(t-t_0)})\|\phi_1 - \phi_2\|$

Since the quantity $\frac{1}{2}(1 - e^{-2k(t-t_0)}) < 1$, the operator T is a contraction and thus by the contraction principle there exists a unique fixed point in A such that $(T\phi)(t) = \phi(t)$. This means:

$\phi(t) = \phi_0 + \int_{t_0}^{t} f(s, \phi(s)\, ds,$

which is a solution to the Initial Value problem (1) for all t in $t_0 \le t \le t_0 + \alpha$.

We can show there exists a unique solution $\phi(t)$ on $t_0 - \alpha \le t \le t_0$ similarly using

$\|x\| = \sup_{t_0 - \alpha \le t \le t_0} \{e^{2k(t-t_0)}|x(t)|\}.$

Thus using this alternate norm we can find a unique solution, $\phi(t)$ to the Initial Value problem (1) for all t in I_α without the condition that $k\bar{\alpha} < 1$.

We have shown that both the contraction principle and the method of successive approximations are equally efficient. Both methods obtain a unique solution to the Initial Value Problem (1) under the same conditions.

References

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