Rainbow Ramsey Numbers

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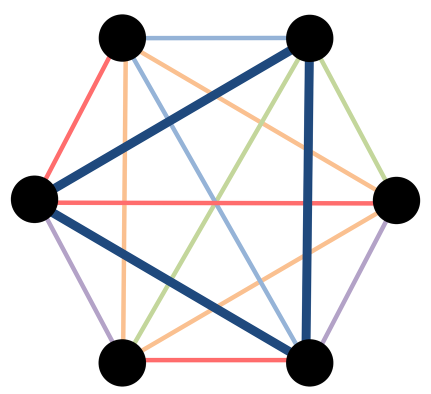
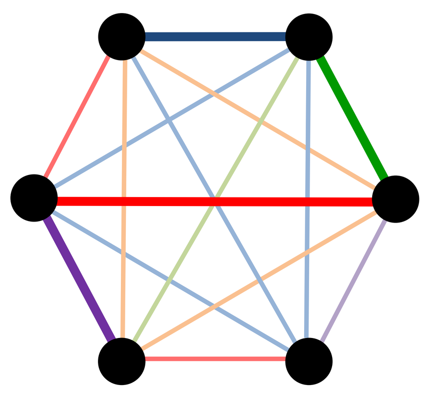
Definitions

An ***edge coloring*** of a graph is the assignment of a color to each of its edges.
A subgraph is ***monochromatic*** if each color is the same and ***rainbow*** if every color is distinct.



A ***monochromatic coloring*** *in complete graph* K_6



A ***rainbow coloring*** in complete graph K_6

Let G be a graph with vertex set $\{v_1, v_2, \dots, v_n\}$
In a ***minimum coloring*** of G, edge $v_i v_j$ is color $min\{i, j\}$
In a ***maximum coloring*** of G, edge $v_i v_j$ is color $max\{i, j\}$

Ramsey's Theorem:
There exists a smallest positive integer $r(K_i, K_j) = n$ for which every red-blue coloring of K_n contains a red K_i or a blue K_j

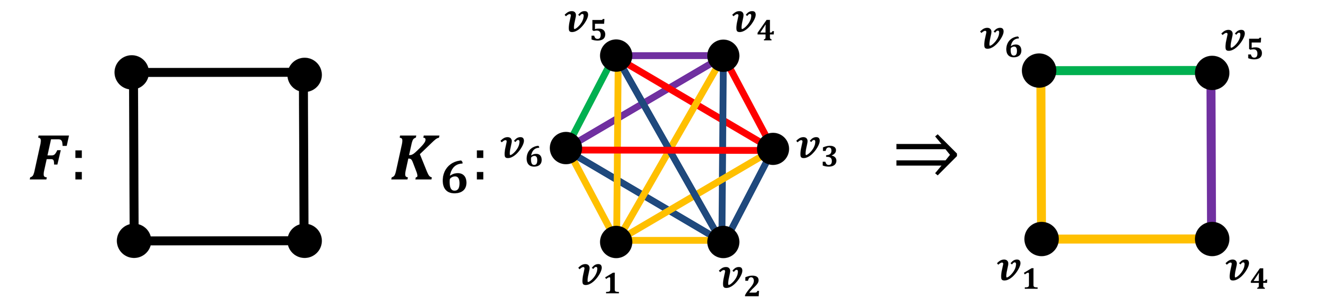
Rainbow Ramsey Number:

1. The smallest positive integer $RR(F) = n$ for which every coloring of K_n contains a monochromatic F or a rainbow F
2. The smallest positive integer $RR(F_1, F_2) = n$ for which every coloring of K_n contains either a monochromatic F_1 or a rainbow F_2

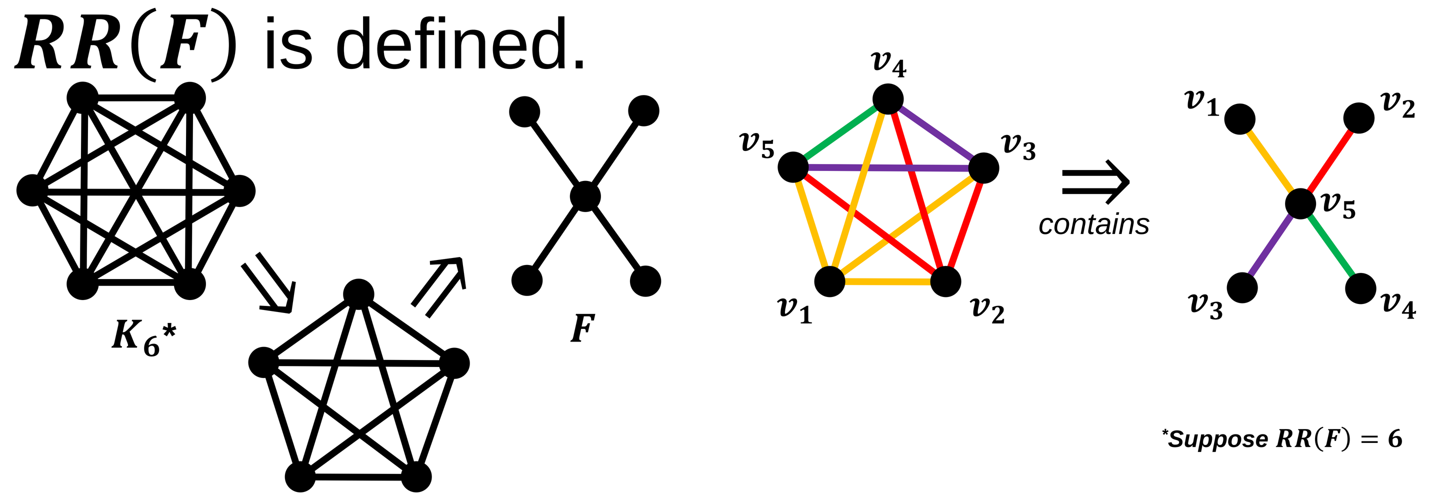
Theorem

$RR(F)$ *exists if and only if* F *is a forest*

Let $RR(F) = n$.
Assume F contains cycle $\{v_1, v_2, \dots, v_k, v_1\}$.
Create a minimum coloring of K_n.
As a result, v_1v_2 and v_1v_k have the same color and v_2v_3 has a different color.
Therefore, no cycle in a min. coloring can be monochromatic or rainbow, so $RR(F)$ is defined only if F is a forest.



For the converse, let F be a forest of order k.
By Theorem 11.20[1], there is some positive integer n such that any coloring of K_n contains a complete graph of order k that is monochromatic or rainbow or has a min. or max. coloring.
Call this complete subgraph G with vertex set $\{v_1, v_2, \dots, v_k\}$.
Suppose G has a min. coloring.
Produce tree T from F with root v_k, and label remaining vertices according to their distance from the root.
As this produces a rainbow T, there is a rainbow F, so F being a forest implies that $RR(F)$ is defined.

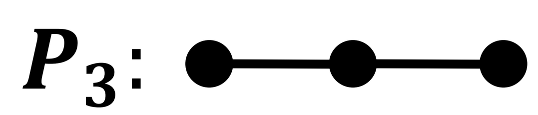


Examples

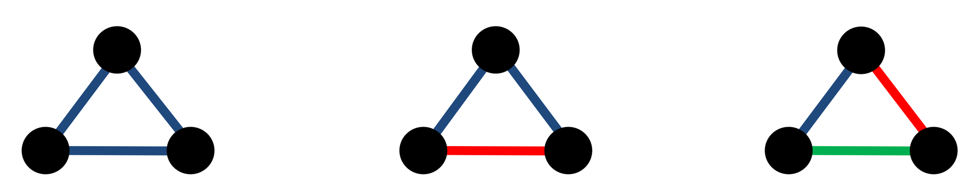
Determine $RR(C_n, P_2)$*:*
P_2 is a graph of two vertices connected by one edge, so it is both monochromatic and rainbow

Therefore, $RR(C_n, P_2) = 2$

Determine $RR(C_n, P_3)$*:*
P_3 is the following graph:



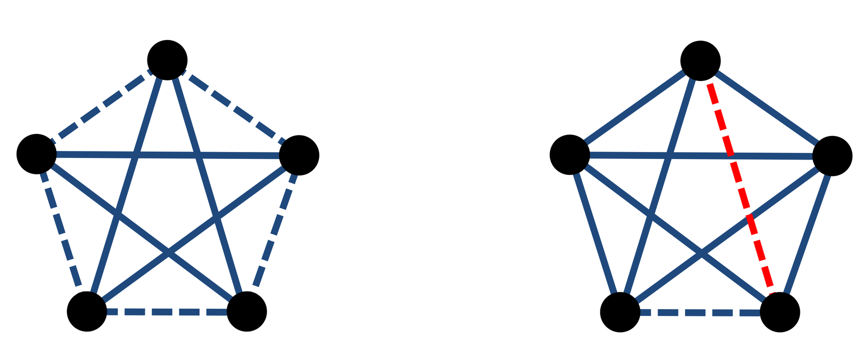
So $RR(C_n, P_3)$ must be at least 3.
For K_3, there are three possible colorings:



So there is either a monochromatic C_3 or a rainbow P_3.
Extend this logic to K_n:

- $\#\, colors = 1 \Rightarrow$ monochromatic C_n
- $\#\, colors \geq 1 \Rightarrow$ rainbow P_3

Therefore, $RR(C_n, P_3) = n$



contains monochromatic C_5 — *contains rainbow* P_3

References

1. Chartrand, Gary, and Ping Zhang. *A First Course in Graph Theory*. Dover Publications, 2012.
2. Eroh, Linda, "Rainbow Ramsey Numbers" (2000). *Dissertations*. 1448. https://scholarworks.wmich.edu/dissertations/1448