ω_1, the first uncountable ordinal

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By contrast, an uncountable set is an infinite set that is not countable, and has strictly more elements than a countable set.

Every ordinal number is a well-ordered set, meaning that every nonempty subset has a least element. The elements of an ordinal number are themselves ordinals and every ordinal is the set of ordinals smaller than it. The empty set, denoted 0, is the smallest ordinal. Then $1 = \{0\}$, $2 = \{0, 1\}$, and so on.

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The set ω_1 is the set of all countable ordinals.

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4. All of the increasing sequences contained in ω_1 converge to an element of ω_1.
5. Every countable subset of ω_1 has a least upper bound in ω_1.

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There is a one to one function from ω_1 into $\mathbb{R}$.

Proof: $\mathbb{R}$ is uncountable. Define $f(0) \in \mathbb{R}$. Then, suppose we have defined $f(\alpha)$ for every $\alpha < \beta$. Then, $\mathbb{R} - \{f(\alpha) : \alpha < \beta\} \neq \emptyset$ because $\{f(\alpha) : \alpha < \beta\}$ is countable. Let $f(\beta) \in \mathbb{R} - \{f(\alpha) : \alpha < \beta\}$. This process defines a function $f : \omega_1 \to \mathbb{R}$. By the definition of the function, $f(\beta)$ cannot equal $f(\alpha)$ when $\alpha < \beta$, so *f* is one-to-one.

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ω_1 is obviously a cub set. Another example of a cub set is the set L of limit ordinals in ω_1.

For every $\alpha \in \omega_1$, the set $\{\beta \in \omega_1 : \alpha < \beta\}$ is nonempty so it must have a least element. We will denote this element as $\alpha + 1$. Note that there is no $\beta \in \omega_1$ such that $\alpha < \beta < \alpha + 1$.

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A limit ordinal is an ordinal that is not a successor.

The ordinal 0 is not a successor ordinal because it is the smallest ordinal so there is no $\alpha \in \omega_1$ such that $0 = \alpha + 1$. The set ω of all finite ordinals is also a limit ordinal because if α is finite, so is $\alpha + 1$.

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Proof.

Let σ be a strictly increasing sequence in ω_1 and let $\beta = \lim_{n \to \infty} \sigma(n)$. Then for every $\alpha < \beta$ there must be $n \in \omega$ such that $\alpha < \sigma(n) < \beta$. If β is a successor ordinal then there is $\alpha < \beta$ such that $\alpha + 1 = \beta$. But then there cannot be an $n \in \omega$ such that $\alpha < \sigma(n) < \beta$. Thus β cannot be successor and must be a limit ordinal. □

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The set L of limit ordinals in ω_1 is a cub set.

Proof.

That L is closed is a consequence of the previous lemma since the limit of an increasing sequence of limit ordinals in ω_1 will again be a limit ordinal in ω_1.

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To see that L is unbounded, let $\alpha \in \omega_1$. Recursively define a sequence σ in ω_1 by setting $\sigma(0) = \alpha$ and $\sigma(n+1) = \sigma(n) + 1$ for every $n \in \omega$. Then σ is a strictly increasing sequence in $\omega + 1$ and must converge to an element β of ω_1. Then β is a limit ordinal and $\alpha < \beta$ so L is unbounded. □

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Proof.

Let $\alpha \in \omega$ and define sequences σ in C and τ in D recursively as follows. Choose $\sigma(0)$ to be any element of C with $\alpha < \sigma(0)$. Since D is a cub set there we can choose $\tau(0) \in D$ such that $\sigma(0) < \tau(0)$. Let $n \in \omega$ and assume that $\sigma(n) \in C$ and $\tau(n) \in D$. Since C and D are cub sets we can choose $\sigma(n+1) \in C$ and $\tau(n+1) \in D$ such that $\tau(n) < \sigma(n+1) < \tau(n+1)$. This recursively defines sequences σ in C and τ in D such that $\lim_{n\to\infty} \sigma(n) = \lim_{n\to\infty} \tau(n) = \lambda$. Since C and D are closed, $\lambda \in C \cap D$ and $\alpha < \lambda$. Thus $C \cap D$ is unbounded.

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That $C \cap D$ is closed follows from the fact that any increasing sequence in $C \cap D$ is a sequence in both C and D, so its limit will be in both. □

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If C is a cub set in ω_1 then C contains a strictly increasing sequence. Since C is closed, the limit of this sequence will be an element of C, so $C \cap L \neq \emptyset$.

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Stationary sets and cub sets in some ways look the same. They are both “big”. Every cub set is stationary and the intersection of a stationary set and a cub set is again stationary. We will see two ways in which they are very different.

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Let $A \subseteq \omega_1$. A function $f : A \to \mathbb{R}$ is continuous if and only if f preserves the limits of sequences in A that converge to an element of A. That is, if σ is a sequence in A that converges to an element α of A then $\lim_{n\to\infty} f(\sigma(n)) = f(\alpha)$.

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Let C be a cub set in ω_1 and $f : C \to \mathbb{R}$. If f is continuous then $f[C]$ is bounded and there is a $\alpha \in C$ such that $f(\beta) = f(\alpha)$ for all $\beta \in C$ with $\beta > \alpha$.

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However, there is a stationary set S and a function $f : X \to \mathbb{R}$ such that $f[S]$ is unbounded.

Let S be the set of all limit ordinals larger than ω together with all the finite ordinals. The sequence $\sigma(n) = n$ does not converge in S because it converges to ω which is not in S. So we can define $f : S \to \mathbb{R}$ by $f(n) = n$ for every finite ordinal n and $f(\alpha) = 0$ for every $\alpha \in S$ with $\alpha > \omega$.

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The following theorem shows that not only are their disjoint stationary sets, but there are many of them.

THEOREM (Ulam Matrix Theorem)

There is an uncountable collection of pairwise disjoint stationary subsets of ω_1.