On Isomorphism Theorems and Characterizing Properties of Groups

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Introduction

- Isomorphism literally means "equal form".
- In group theory, isomorphisms are used to determine if two or more groups have similar properties to one another.
- Isomorphisms allow us to learn more about the properties of a group G if we can find other groups isomorphic to G.

Example

If G is isomorphic to H (denoted $G \cong H$) and G is abelian, then H is also abelian.

Relevant Definitions for Isomorphism Theorems

- Let G and H be groups:
- **Quotient Group:** The set of all right cosets of N in G, denoted by G/N.
- **Normal**: A subgroup N of a group G is normal if $Na = aN$ for every $a \in G$.
- **Homomorphism:** Let G and H be equipped with operation $*$. A function $f: G \to H$ is a homomorphism if and only if $f(a * b) = f(a) * f(b)$.
- **Kernel:** Let $f: G \to H$ be a homomorphism of groups. The kernel of f is the set $\{a \in G \mid f(a) = e_H\}$, denoted $Ker(f)$.
- **Isomorphism:** G is isomorphic to H, denoted $G \cong H$, if and only if $f: G \to H$ is a homomorphism of groups, f is surjective, and f is injective.

Isomorphism Theorems

- **First Isomorphism Theorem**: Let $f: G \to H$ be a surjective homomorphism of groups with kernel K. Then the quotient group G/H is isomorphic to H.
- **Second Isomorphism Theorem**: Let K and N be subgroups of G and let N be normal in G. The following statements are true:
  1. NK is a subgroup of G.
  2. N is normal in NK.
  3. $NK/N \cong K/(N \cap K)$.
- **Third Isomorphism Theorem**: Let K and N be normal subgroups of a group G with $N \subseteq K \subseteq G$. Then K/N is a normal subgroup of G/N, and the quotient group (G/N)/(K/N) is isomorphic to G/K.

Relevant Theorems for Proving Isomorphism Theorems

- **Theorem 1**: Let $f : G \to H$ be a homomorphism of groups with kernel K. Let $a, b \in G$. Then $f(a) = f(b)$ if and only if $Ka = Kb$.
- **Theorem 2**: Let K be a subgroup of G and let $a, c \in G$. Then $a \equiv c (modK)$ if and only if $Ka = Kc$.
- **Theorem 3**: The following statements of a subgroup N of a group G are equivalent:
  1. N is a normal subgroup of G.
  2. $a^{-1}Na \subseteq N$ for every $a \in G$, where $a^{-1}Na = \{a^{-1}na \mid n \in N\}$.
  3. $aNa^{-1} \subseteq N$ for every $a \in G$, where $aNa^{-1} = \{ana^{-1} \mid n \in N\}$.
  4. $a^{-1}Na = N$ for every $a \in G$.
  5. $aNa^{-1} = N$ for every $a \in G$.

First Isomorphism Theorem

Let $f\colon G \to H$ be a surjective homomorphism of groups with kernel K. Then the quotient group G/H is isomorphic to H.

Proof

Suppose $f\colon G \to H$ is a surjective homomorphism of groups with kernel K. Let $\varphi\colon G/K \to H$ be defined by $\varphi(Ka) = f(a)$. Suppose $Ka = Kb$. By Theorem 1, then $f(a) = f(b)$. It follows that $\varphi(f(a)) = \varphi(f(b))$, so $\varphi(Ka) = \varphi(Kb)$. Hence, φ is a well-defined function. Suppose $h \in H$. Since f is surjective, there exists $c \in G$ such that $h = f(c)$, $c \in G$, so $\varphi(Kc) = f(c) = h$. Thus φ is surjective.

First Isomorphism Theorem Cont.

Proof Cont.

We show that φ is injective. Suppose $\varphi(Ka) = \varphi(Kb)$, where $a, b \in G$. Since f is a homomorphism, it follows by Theorem 1 that $Ka = Kb$. Thus, φ is injective.

Finally, we show that $\varphi\colon G/K \to H$ is a homomorphism.

$\varphi(KaKb) = \varphi(Kab) = f(ab)$. Since f is a homomorphism, $f(ab) = f(a)f(b) = \varphi(Ka)\varphi(Kb)$. Thus $\varphi\colon G/K \to H$ is a homomorphism.

We have shown that $\varphi\colon G/K \to H$ is bijective and homomorphic.

Therefore, $\varphi\colon G/K \to H$ is an isomorphism, so $G/K \cong H$. QED.

Second Isomorphism Theorem

Let K and N be subgroups of G and let N be normal in G. The following statements are true:

1. NK is a subgroup of G.
2. N is normal in NK.
3. $NK/N \cong K/(N \cap K)$.

Proof

1. Suppose K and N are subgroups of G and N is normal in G. Let $n \in N$ and $k \in K$. Thus, $nk \in NK$. Since N is normal in G, by Theorem 3 it follows that $k^{-1}nk \in N$. Let $n_1, n_2 \in N$ and $k_1, k_2 \in K$. Thus $n_1k_1, n_2k_2 \in NK$.
$(n_1k_1)(n_2k_2) = n_1(k_1n_2k_1^{-1}k_1)k_2 = n_1(k_1n_2k_1^{-1})(k_1k_2) \in NK$.
Also,
$(nk)^{-1} = k^{-1}n^{-1}) = (k^{-1}n^{-1})(kk^{-1}) = (k^{-1}n^{-1}k)k^{-1} \in NK$.
Thus, NK is a subgroup of G.

Second Isomorphism Theorem Cont.

Proof Cont.

2. Since N is normal in G, $aNa^{-1} \subseteq N$, for all $a \in G$. Let $b \in G$ such that $b = nk$, where $n \in N$ and $k \in K$. We showed in (1) that $NK \subseteq G$. Thus $b = nk \in NK \subseteq G$. It follows that $(nk)N(nk)^{-1} \subseteq N$, so $bNb^{-1} \subseteq N$. Therefore, N is normal in NK.
3. Let $f\colon K \to NK/N$ be given by $f(k) = Nk$. Let $nk \in NK$. Note that $k \in K$ and with how f is defined, $nk = f(k)$. Thus f is surjective. Let $k_1, k_2 \in K$.
$f(k_1)f(k_2) = Nk_1Nk_2 = Nk_1k_2 = f(k_1k_2)$. Thus f is a homomorphism. Let $p \in K$ and $p \in Ker(f)$. Then $f(p) = Np = Ne_{NK/N} = N$, so $NP = N$. Therefore, there are $n_1, n_2 \in N$ such that $n_1p = n_2$. Hence, $p \in N$, but $p \in K$, so $p \in N \cap K$ and $Ker(f) = N \cap K$. It follows by the First Isomorphism Theorem that $NK/N \cong K/(N \cap K)$. QED.

Visualization of Second Isomorphism Theorem

Diamond Theorem

$$\begin{tikzcd}[arrows=-]
& G \arrow[d] & \\
& NK \arrow[dl] \arrow[dr] & \\
N \arrow[dr] & & K \arrow[dl] \\
& N \cap K \arrow[d] & \\
& \{e\} &
\end{tikzcd}$$

Third Isomorphism Theorem

Let K and N be normal subgroups of a group G with $N \subseteq K \subseteq G$. Then K/N is a normal subgroup of G/N, and the quotient group $(G/N)/(K/N)$ is isomorphic to G/K.

Proof

Suppose K and N be normal subgroups of a group G with $N \subseteq K \subseteq G$. Let $a, c \in G$. By Theorem 2, if $Na = Nc$ in G/N, then $a/equivc(modN)$, and $ac^{-1} \in N$, by definition of congruency. Since N is a subgroup of K, $ac^{-1} \in K$. By Theorem 2, it follows that $Ka = Kc$ is in G/K. If we define $f: G/N \to G/K$ by $f(Na) = Ka$, f is well-defined.
If we can show that f is a surjective homomorphism with $Ker(f) = K/N$, we can utilize the First Isomoprhism to obtain $(G/N)/(K/N) \cong G/K$.

Third Isomorphism Theorem Cont.

Proof Cont

First, we show that f is surjective. Let $Kc \in G/K$. With how f is defined, there exists $n \in N$ such that $Kc = f(nc)$.
$nc \in Nc \in G/N$, thus $Kc = f(Nc)$. Hence, f is surjective.
Next, we show that f is a homomorphism. Since $NaNb = Nab$, it follows that $f(NaNb) = f(Nab) = Kab = KaKb = f(Na)f(Nb)$.
Thus, f is a homomorphism.
Finally, we show that $Ker(f) = K/N$. Ke is the identity element of G/K. If $Na \in Ker(f)$, then $f(Na) = Ka = Ke$, so it follows that $ae^{-1} = a \in K$. Thus, $Ker(f) = \{Na | a \in K\} = K/N$.
Thus, by the First Isomorphism Theorem, $(G/N)/(K/N) \cong G/K$.
QED.

Conclusion

- Questions?

References

Hungerford, T. W. (2013) *Abstract Algebra: An Introduction* (3rd edition). Cengage. ISBN: 9788131525722.