Properties of the Line Graph and Total Graph Operators

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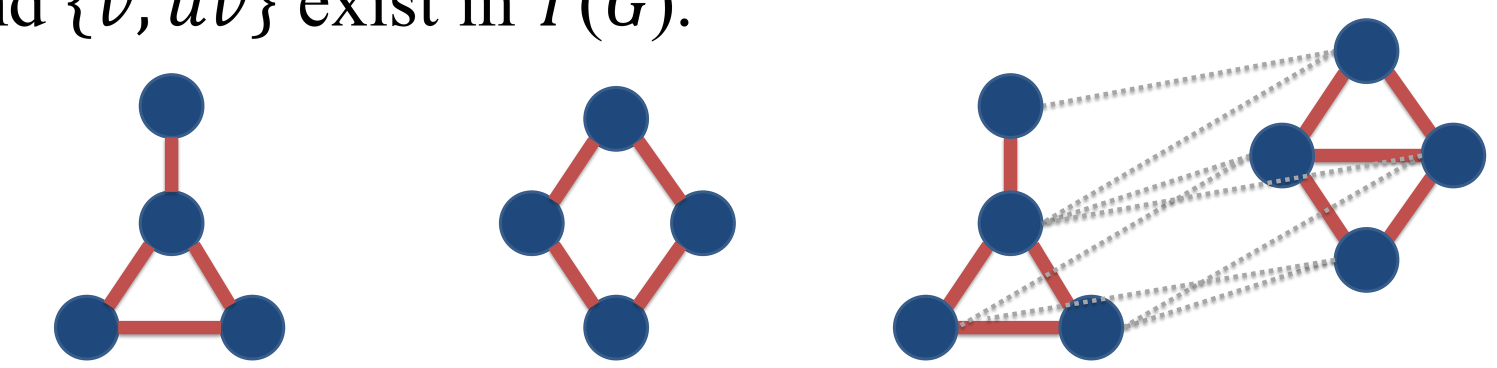
Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The **line graph** of G, denoted $L(G)$, is the graph whose vertex set is $E(G)$ and in which two vertices are adjacent if they are adjacent in G. The **total graph** of G, denoted $T(G)$, is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if they are adjacent or incident in G. Note that G and $L(G)$ are both subgraphs of $T(G)$, and for each vertex $uv \in V(L(G))$, where $u, v \in V(G)$, the edges $\{u, uv\}$ and $\{v, uv\}$ exist in $T(G)$.



G $L(G)$ $T(G)$

Example graph with its line graph and total graph

What properties of a graph *G* are preserved under the line graph and total graph operators?

Regular Graphs

A graph G is r-**regular** if each of its vertices has degree r.

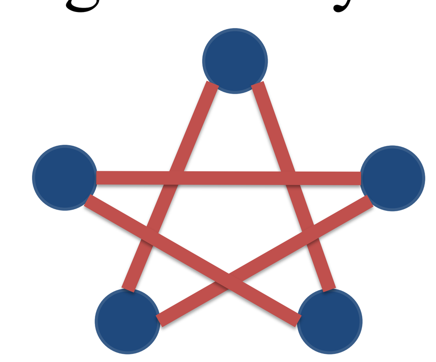
It is a simple argument to show that if G is r-regular, then $L(G)$ is $2(r-1)$-regular and $T(G)$ is $2r$-regular.

References

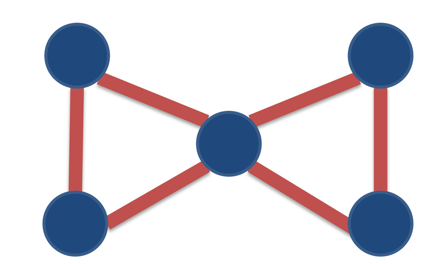
- Behzad, M., & Chartrand, G. (1966). Total Graphs and Traversability. *Proceedings of the Edinburgh Mathematical Society*, *15*(2), 117.
- Behzad, M., & Radjavi, H. (1968). The Total Group of a Graph. *Proceedings of the American Mathematical Society*, *19*(1), 158.
- Hoover, C. (1991). The Behavior of Properties of Graphs under the Line Graph Operation. Honors Thesis, UD.

Hamiltonian and Eulerian Graphs

A graph G is **Hamiltonian** if it contains a cycle of length n, where n is the number of vertices in G. A graph G is **Eulerian** if it contains a circuit that visits every edge exactly once.



A Hamiltonian Graph



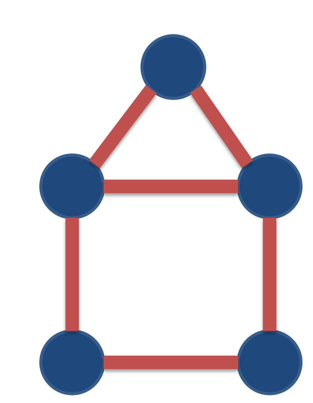
An Eulerian Graph

Theorem (Behzad and Chartrand, 1966). If a graph G is Hamiltonian, then $T(G)$ is Hamiltonian.

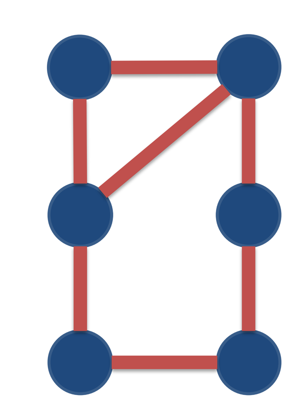
Theorem (Behzad and Chartrand, 1966). If a graph G is Eulerian, then $T(G)$ is Eulerian and $T(G)$ is Hamiltonian.

Pancyclic Graphs

A graph G is **pancyclic** if it contains cycles of all possible lengths from three to n, where n is the number of vertices in G.



A pancyclic graph



Not a pancyclic graph (missing a 4-cycle)

Lemma (Hoover, 1991). If G is pancyclic, then $L(G)$ is pancyclic.

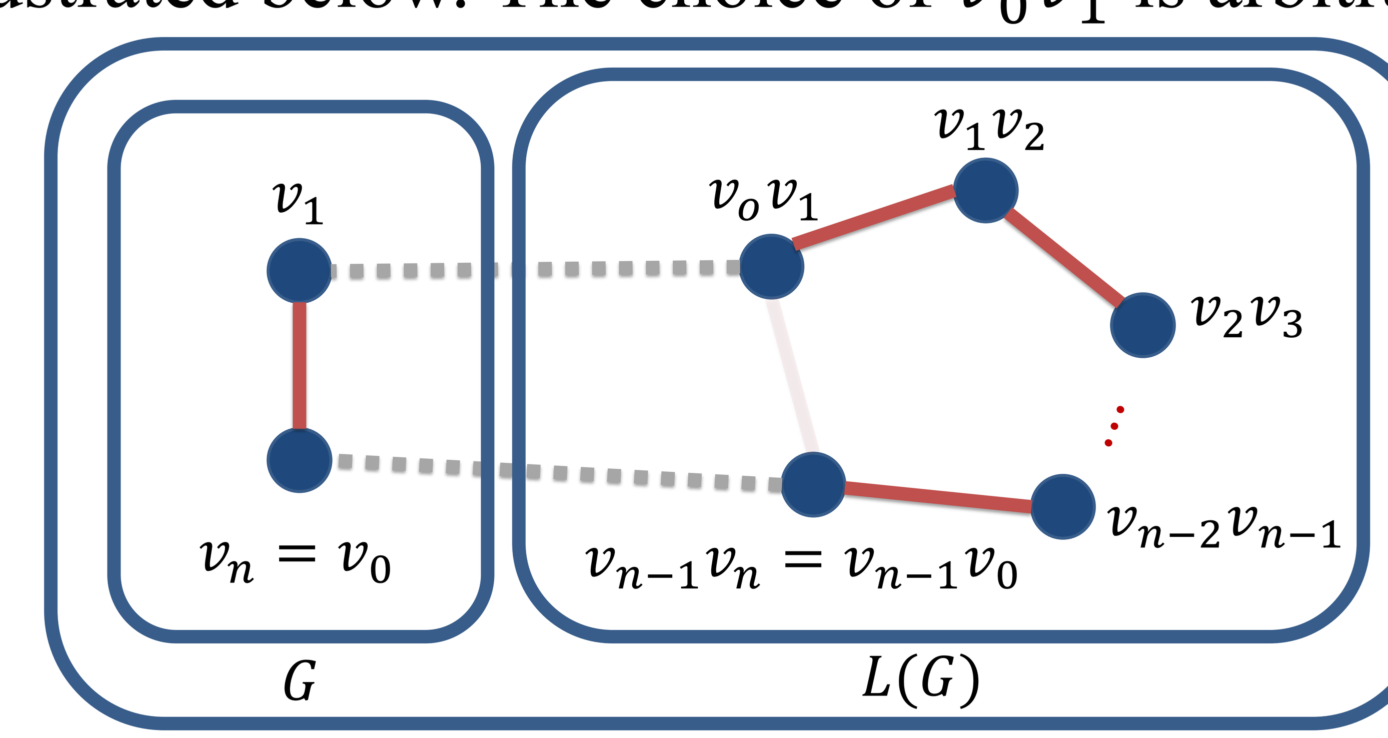
Lemma. If G is pancyclic, then $|V(G)| \leq |V(L(G))|$.

Theorem (New)

If G is pancyclic, then $T(G)$ is pancyclic.

"Proof." Let G be a pancyclic graph with n vertices and m edges. For each $k \in \{3, 4, \dots, |V(T(G))|\}$, we find a k-cycle in $T(G)$. For any $k \leq m$, a k-cycle can be found in the pancyclic subgraph $L(G)$.

For $k \geq m+3$, a k-cycle can be constructed by "breaking" a Hamiltonian cycle in $L(G)$ in order to pick up a $(k-m)$-cycle in G. In doing so, we remove an edge from both the Hamiltonian m-cycle and the $(k-m)$-cycle, and we gain two edges by traversing to and from the $(k-m)$-cycle in G through subsequent vertices in the Hamiltonian cycle in $L(G)$. This is possible since G and $L(G)$ are pancyclic and by application of the definition of total graph. For $k = m+1$ or $k = m+2$, a k-cycle can similarly be constructed by "breaking" a Hamiltonian cycle in $L(G)$ in order to pick up an extra one or two edges. Construction of a $k = (m+2)$-cycle is illustrated below. The choice of v_0v_1 is arbitrary.



Construction of $k = (m+2)$-cycle in $T(G)$

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