Use of LU Factorization in Solving a Real-World Problem

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Abstract

We applied the LU factorization to find the temperature distribution in a two-dimensional flat rectangular metallic plate. The temperatures on the boundary are known and the interior temperature distribution will be determined. The problem will require solving a linear system: $A\boldsymbol{x} = \boldsymbol{b}$
This system can also be solved by finding the inverse of A. However, we will show that for this problem, the LU factorization technique is more suitable.

Objective

We consider a two-dimensional flat rectangular metallic plate. Our objective is to determine the interior temperature when the temperature on the boundary will be known.

Process of LU factorization method

In this project we will apply the LU factorization which is an equation that illustrates *A* as a product of two matrices $A = LU$, where *L* is a lower triangular matrix and *U* is an upper triangular matrix of *A*.
When $A = LU$ we can write the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ as $L(Ux) = b$ where $U\boldsymbol{x} = y$
Therefore, we can find $\boldsymbol{x}$ by solving the two systems, $Ly = \boldsymbol{b}$ and $U\boldsymbol{x} = y$
The following figure 1 illustrate a two-dimensional flat rectangular metallic plate. We divide this into 8 temperature points: We want to find the interior temperatures $T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8$ at these points.

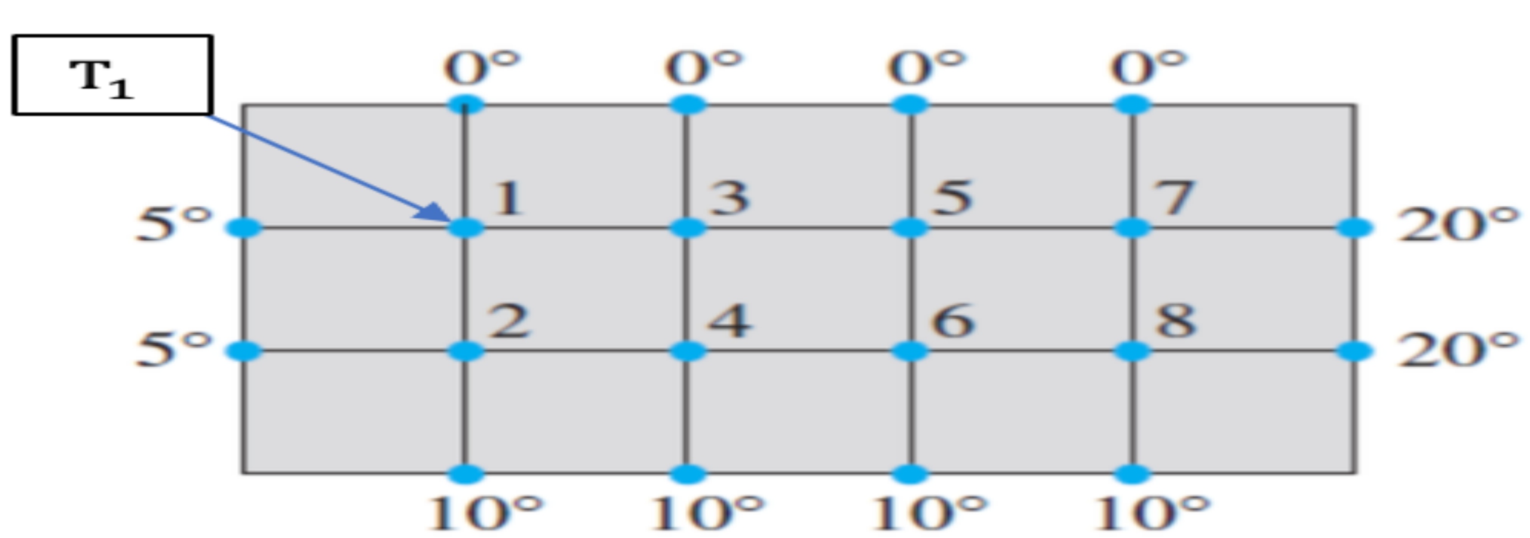

Figure 1: Metallic plate with temperature

The temperature T_1 is approximately the average of the temperatures of the four nearest nodes. Similar rule determines the temperatures at the other nodes. So, for T_1 we get

$$T_1 = \frac{1}{4}(0 + 5 + T_2 + T_3)$$

From which we write the first equation

$$4T_1 - T_2 - T_3 + 0 + 0 + 0 + 0 + 0 = 5$$

Similarly other seven equations are

$$-T_1 + 4T_2 + 0 - T_4 + 0 + 0 + 0 + 0 = 15$$
$$-T_1 + 0 + 4T_3 - T_4 - T_5 + 0 + 0 + 0 = 0$$
$$0 - T_2 - T_3 + 4T_4 + 0 - T_6 + 0 + 0 = 10$$
$$0 + 0 - T_3 + 0 + 4T_5 - T_6 - T_7 + 0 = 0$$
$$0 + 0 + 0 - T_4 - T_5 + 4T_6 + 0 - T_8 = 10$$
$$0 + 0 + 0 + 0 - T_5 + 0 + 4T_7 - T_8 = 20$$
$$0 + 0 + 0 + 0 + 0 - T_6 - T_7 + 4T_8 = 30$$

From these equations we write the following matrix equation

$$\begin{bmatrix} 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 4 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \\ 0 \\ 10 \\ 0 \\ 10 \\ 20 \\ 30 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 4 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 5 \\ 15 \\ 0 \\ 10 \\ 0 \\ 10 \\ 20 \\ 30 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{bmatrix}$$

Then the system becomes the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$.
To find the values of $T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8$, we need to solve the system $A\boldsymbol{x} = \boldsymbol{b}$. We solve it by employing LU factorization method first. Then we solve this system using inverse method. We apply MATLAB in the computations. We obtain *L* and *U* as given below.

```
L =

    1.0000         0         0         0         0         0         0         0
   -0.2500    1.0000         0         0         0         0         0         0
   -0.2500   -0.0667    1.0000         0         0         0         0         0
         0   -0.2667   -0.2857    1.0000         0         0         0         0
         0         0   -0.2679   -0.0833    1.0000         0         0         0
         0         0         0   -0.2917   -0.2921    1.0000         0         0
         0         0         0         0   -0.2697   -0.0861    1.0000         0
         0         0         0         0         0   -0.2948   -0.2931    1.0000


U =

    4.0000   -1.0000   -1.0000         0         0         0         0         0
         0    3.7500   -0.2500   -1.0000         0         0         0         0
         0         0    3.7333   -1.0667   -1.0000         0         0         0
         0         0         0    3.4286   -0.2857   -1.0000         0         0
         0         0         0         0    3.7083   -1.0833   -1.0000         0
         0         0         0         0         0    3.3919   -0.2921   -1.0000
         0         0         0         0         0         0    3.7052   -1.0861
         0         0         0         0         0         0         0    3.3868
```

Now, we solve for $\boldsymbol{x}$ using $Ly = \boldsymbol{b}$ and $U\boldsymbol{x} = y$

$$[L \quad \boldsymbol{b}] = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ -0.2500 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 15 \\ -0.2500 & -0.0667 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0667 & -0.2857 & 1.0000 & 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & -0.2679 & -0.0833 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.2917 & -0.2921 & 1.0000 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & -0.2697 & -0.0861 & 1.0000 & 0 & 20 \\ 0 & 0 & 0 & 0 & 0 & -0.2948 & -0.2931 & 1.0000 & 30 \end{bmatrix}$$

$$\sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 16.25 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2.333875 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 11.7506630875 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1.60407534768875 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 13.8962188316836 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 21.6290835626796 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 40.4360897038017 \end{bmatrix} = [I \quad \boldsymbol{y}]$$

$$[U \quad \boldsymbol{y}] = \begin{bmatrix} 4.0000 & -1.0000 & -1.0000 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 3.7500 & -0.2500 & -1.0000 & 0 & 0 & 0 & 0 & 16.25 \\ 0 & 0 & 3.7333 & -1.0667 & -1.0000 & 0 & 0 & 0 & 2.333875 \\ 0 & 0 & 0 & 3.4286 & -0.2857 & -1.0000 & 0 & 0 & 11.7506630875 \\ 0 & 0 & 0 & 0 & 3.7083 & -1.0833 & -1.0000 & 0 & 1.60407534768875 \\ 0 & 0 & 0 & 0 & 0 & 3.3919 & -0.2921 & -1.0000 & 13.8962188316836 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3.7052 & -1.0861 & 21.6290835626796 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.3868 & 40.4360897038017 \end{bmatrix}$$

$$\sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.7914093944188 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 6.2813973194661 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3.8842402582090 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 6.3341798834456 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 5.41048947430058 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 8.42092921807389 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 9.33725014792064 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 11.9393202148936 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} 3.7914 \\ 6.2814 \\ 3.8842 \\ 6.3342 \\ 5.4105 \\ 8.4209 \\ 9.3372 \\ 11.9393 \end{bmatrix}$$

The inverse method

Since *A* is invertible, we can compute $A^{-1}b$ to find $\boldsymbol{x}$.
We applied equation $\boldsymbol{x} = A^{-1}b$ in MATLAB to solve the system using the invers of *A*.

$$\boldsymbol{x} = \begin{bmatrix} 3.9575 \\ 6.5895 \\ 4.237 \\ 7.396 \\ 5.6005 \\ 8.7595 \\ 9.413 \\ 12.045 \end{bmatrix}$$

LU factorization method vs. Inverse

By comparing the two methods we find that LU factorization method is more efficient and accurate as the invers method results are rounded up. Also, A^{-1} is a dense matrix hence the number of computations is more than the corresponding number when LU factorization technique is used. This is due to the fact that the matrices *L* and *U* are more sparse.

Reference

Lay, David C., et al. Linear Algebra and Its Applications. Fifth ed., David C. Lay, 2016.